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RICAN STATISTICAL ASSOCIATION

No. 1

THE ANNALS
of
ATHEMATICAL STATISTICS

FEBRUARY - 1930

THE ANNALS OF MATHEMATICAL STATISTICS

By

WILLFORD I. KING

For ninety-one years the American Statistical Association has held the van in matters statistical in the United States. At the time when our Association was founded, statistical method was an extremely simple science. In recent years, the technique has, however, been growing more and more complex. The *Journal of the American Statistical Association* has served all the members of the Association and an attempt has been made to cover, in its pages, all phases of statistical method. For some time past, however, it has been evident that the membership of our organization is tending to become divided into two groups — those familiar with advanced mathematics, and those who have not devoted themselves to this field. The mathematicians are, of course, interested in articles of a type which are not intelligible to the non-mathematical readers of our Journal. The Editor of our Journal has, then, found it a puzzling problem to satisfy both classes of readers.

Now a happy solution has appeared. The Association at this time has the pleasure of presenting to its mathematically inclined members the first issue of the ANNALS OF MATHEMATICAL STATISTICS, edited by Prof. Harry C. Carver of the University of Michigan. This Journal will deal, not only with the mathematical technique of statistics, but also with the applications of such technique to the fields of astronomy, physics, psychology, biology, medicine, education, business, and economics. At present, mathematical articles along these lines are scattered through a great variety of publications. It is hoped that in the future they will be gathered together in the ANNALS.

The editorial policy will be to select articles that will best meet the needs of the time. There can be no questioning the statement that at the present time there are in this country many more who need stimulation in the fundamentals of mathematical statistics than there are individuals whose prime interest is in the advancement of modern statistical theory. Therefore particular stress will be laid on articles of a fundamental nature during the first few years of the life of the ANNALS. The officers, after due deliberation, have chosen a new method of printing in order to facilitate the composition of original articles and the obtaining of reprints. A photographic process is employed, which will permit the Association at any point in the future to furnish reprints or back numbers. The advantages of this to libraries and classes in statistics is apparent. A particular effort will be made to insert from time to time tables that must be constantly referred to by statisticians. Nevertheless, the chronicling of research will in no sense be neglected.

My personal opinion is that the advent of the ANNALS constitutes an important milestone in the history of our Association. I am sure that this new publication will be welcomed heartily, not only by the mathematically trained section of our membership, but also by the non-mathematical group, for the latter recognize that the more advanced phases of mathematics are rendering extremely valuable service in furthering the progress of statistical technique, thus aiding in the solution of problems of the greatest moment.

REMARKS ON REGRESSION

By

S. D. WICKSELL

1. In a paper published twelve years ago[1] I derived a set of formulae for bivariate regression which were found to give good results on unimodal materials of a fairly general nature and which, in the case of moderately skew distributions, were reduced to very simple and easily applicable forms. Two years later I extended the theory also to the case of multiple correlations of similar types.[2] These formulae were deduced on the assumption that the correlation surface could be expressed by a so-called series of type A[3], i. e. that the deviations from the best fitting normal surface could be expressed as a series, developed according to the derivatives of different orders of the Bravais function, expressing that normal surface.

When, after the lapse of so many years, I find that this theory has not received the attention which it seems to me it merits in view of the very simple, and on a fairly large class of curved regressions readily applicable results, I attribute this in part at least to the apparent (not actual) speciality of the assumptions made with regard to the mathematical expression for the correlation surface, and in part also to the rather repellent show of mathematics involved in the deductions. In the hope to give the theory a better chance of coming to the attention of statisticians, I propose here to deduce some of my main results in an entirely different way, bringing the theory back on more simple principles. I believe that by this method of deduction, it will be more easy for the reader to see exactly where assumptions come in, and also the nature of the restrictions caused by these assumptions.

2. Let x and y be a pair of correlated variates, our material

---

[1] The correlation function of Type A, and the regression of its characteristics. Kungl. Svenska Vetenskapsakademiens Handlingar Bd. 58, Nr. 3, 1917. Also "Meddelanden fran Lunds Astronomiska Observatorium" Ser. II, Nr. 17.

[2] Multiple correlation and non-linear regression. Arkiv for Matematik, Fysik och Astronomi. Bd. 14 Nr. 10, 1919. Also "Meddelanden fran Lunds Astronomiska Observatorium." Ser. I, Nr. 91.

[3] Charlier. Contributions to the mathematical theory of statistics. 6. The correlation function of type A. Arkiv for Matematik, Fysik och Astronomi. Bd 9, Nr. 26, 1914. Also "Meddelanden fran Lunds Astronomiska Observatorium" Ser. I, Nr. 58.

consisting of N such pairs. Computing the means and central moments, we have

$$M_x=\frac{1}{N}\sum x\ ;\quad M_y=\frac{1}{N}\sum y\ ;\quad \mu_{ij}=\frac{1}{N}\sum(x-M_x)^i(y-M_y)^j$$

The standard deviations of x and y and the coefficient of correlation are then defined by

$$\sigma_x=\sqrt{\mu_{20}}\ ,\qquad \sigma_y=\sqrt{\mu_{02}}\qquad r=\frac{\mu_{11}}{\sigma_x\,\sigma_y}$$

Following Yule[1] and Pearson[2] we now treat the problem of regression as a simple problem of graduation, defining the regression of y on x as a parabola of a given degree, which, with x as argument, is fitted to the y's by the method of least squares. The regression may then be written in the form

$$y_x-M_y=a_0+a_1(x-M_x)+a_2(x-M_x)^2+\cdots+a_p(x-M_x)^p,$$

and the least squares normal equations for determining the parameters $a_0,\ a_1,\ a_2,\cdots a_p$ assume the form (Pearson Op. Cit. p. 25).

$$(1)\quad\left\{\begin{array}{lllllll}
0 &= a_0 & & +a_2\mu_{20} & +a_3\mu_{30} & +\cdots & +a_p\mu_{p,0}\\
\mu_{11} &= & a_1\mu_{20} & +a_2\mu_{30} & +a_3\mu_{40} & +\cdots & +a_p\mu_{p+1,0}\\
\mu_{21} &= a_0\mu_{20} & +a_1\mu_{30} & +a_2\mu_{40} & +a_3\mu_{50} & +\cdots & +a_p\mu_{p+2,0}\\
\mu_{31} &= a_0\mu_{30} & +a_1\mu_{40} & +a_2\mu_{50} & +a_3\mu_{60} & +\cdots & +a_p\mu_{p+3,0}\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\mu_{p,1} &= a_0\mu_{p,0} & +a_1\mu_{p+1,0} & +a_2\mu_{p+2,0} & +a_3\mu_{p+3,0} & +\cdots & +a_p\mu_{2p,0}
\end{array}\right.$$

---

[1] On the Theory of Correlation. Jour. Roy. Stat. Soc., Vol. 60, 1897, and On the Theory of Correlation for any number of Variables treated by a new System of Notation. Proc. Roy. Soc., Ser. A, Vol. 79, 1907.

[2] Mathematical Contributions to the Theory of Evolution XIV. On the General Theory of Skew Correlation and non-Linear Regression. Drapers Co. Research Memoirs Biometric Series II. Cambridge Univ. Press, 1905.

Writing the solution in the form of determinants, we have

$$a_{i-1} = \frac{1}{\Delta} \cdot \Delta_i ,$$

where

$$(3) \qquad \Delta = \begin{vmatrix} 1, & 0, & \mu_{20}, & \mu_{30}, & \cdots & \mu_{p,0} \\ 0 & \mu_{20}, & \mu_{30}, & \mu_{40}, & \cdots & \mu_{p+1,0} \\ \mu_{20}, & \mu_{30}, & \mu_{40}, & \mu_{50}, & \cdots & \mu_{p+2,0} \\ \mu_{30}, & \mu_{40}, & \mu_{50}, & \mu_{60}, & \cdots & \mu_{p+3,0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{p,0}, & \mu_{p+1,0}, & \mu_{p+2,0}, & \mu_{p+3,0}, & \cdot & \mu_{2p,0} \end{vmatrix}$$

and Δ_i is obtained when the i'th row in Δ is exchanged for the left membra of equations (1), i e for the series of elements:

$$0, \ \mu_{11}, \ \mu_{21}, \ \mu_{31}, \cdots \mu_{p,1}.$$

3 Some important general conclusions may at once be derived from this system. Defining as *non-regression of the p'th order* the case that all the coefficients $a_1, a_2, a_3, \cdots a_p$ turn out to be practically equal to zero, i e. that a horizontal straight line is the best parabola of the p'th degree that can be fitted to the series of y's, it is first seen, from the first of equations (1), that then also $a_0 = 0$. Secondly we can draw the conclusion that this can take place only if all the elements $\mu_{11}, \ \mu_{21}, \ \mu_{31}, \cdots \mu_{p,1}$, are equal to zero. Hence the condition for non-regression of the p'th order of y on x is that we have

$$(4) \qquad \mu_{i,0} = 0 \quad for \ i = 1, 2, 3, \cdots p$$

This clearly involves also that the coefficient of correlation, r equals zero.

Defining further as *linear regression of the p'th order* the case that the coefficients $a_2, a_3, \cdots a_p$ are equal to zero, i e that a non-horizontal straight line is the best parabola of the p'th degree that can be fitted to the series of y's, we immediately see, from the two first of equations (1), that then we must have

$$(5) \qquad a_0 = 0 ; \qquad a_1 = \frac{\mu_{11}}{\mu_{20}}$$

Referring here to the well-known theorem that any determinant will disappear, when the elements of two rows are proportional (the elements of any one row being obtained by multiplying the corresponding elements of another row by a constant factor) it is easily seen that all the determinants Δ_i except Δ_2, and hence by (2) all the coefficients $a_0, \cdots a_p$; except a_1, will disappear if the quantities $0, \mu_{11}, \mu_{21}, \mu_{31}, \cdot \mu_{p,1}$, in the left membra of (1) are proportional to the elements $0, \mu_{20}, \mu_{30}, \cdots \cdot \mu_{p+1,0}$ in the second row of the determinant Δ. Hence the condition for linear regression of the p'th order of y on x is that we have

$$(6) \qquad \mu_{11} \, \mu_{i+1,0} = \mu_{20} \, \mu_{i,1} \qquad \text{for } i = 1, 2, 3, \cdots p.$$

A few considerations will show that this condition is not only sufficient but also necessary For $p = 3$ these criteria were demonstrated by Pearson

4 Thus far there are no other assumptions involved than the principle of least squares, and that the regression of y on x may be described by a whole rational function. The chief difficulty in the application of this theory of regression is that, as seen from equation (1), in order to determine a regression of the p'th degree we must compute and use moments (of the series of x's up to the order $2p$) Now, as justly remarked by Pearson, moments of high orders are, on account of their large standard errors, very little to be relied upon, at least in the case of ordinary materials (N not very large) . Besides this, the numerical labor involved in computing higher moments is comparatively very great Hence, Pearson's theory of regression will be practically applicable only in cases when the regression is at the most parabolic of the second degree Indeed, this is a very serious restriction, because curved regressions often have at least one inflection. Thus in order to meet fairly frequent cases of regression we must needs have recourse at least to cubic parabolas But this should require the computation of all the moments of x up to the sixth order.

In order to remove, as far as possible, this difficulty, I take refuge in a golden rule expressed by Thiele[1]. Thiele introduces, instead of the moments, a system of coefficients called the semi-invariants. These semi-invariants (here denoted by $\lambda_{i,0}$) are defined in terms of the moments by the identity:

---

[1] Theory of Observations London 1903, p 49

$$\lambda_{20}\frac{x^2}{2!}+\lambda_{30}\frac{x^3}{3!}+\lambda_{40}\frac{x^4}{4!}+\cdots$$
$$=\log_e\left(1+\mu_{20}\frac{x^2}{2!}+\mu_{30}\frac{x^3}{3!}+\mu_{40}\frac{x^4}{4!}+\cdots\right.$$

Developing, we find

$$\text{(7)}\quad \begin{aligned}&\lambda_{20}=\mu_{20};\ \lambda_{30}=\mu_{30};\ \lambda_{40}=\mu_{40}-3\mu_{20}^2;\\ &\lambda_{50}=\mu_{50}-10\mu_{30}\mu_{20};\ \lambda_{60}=\mu_{60}-15\mu_{40}\mu_{20}+30\mu_{20}^3-10\mu_{30}^2\end{aligned}$$

Now, the rule indicated by Thiele is the following:

To obtain the first semi-invariants rely entirely on computations. To obtain the intermediate semi-invariants rely partly on computations, partly on theoretical considerations. But to obtain the higher semi-invariants rely entirely on theoretical considerations.

Of course, this rule is just as well applicable to the determination of moments, as any moment may be expressed in terms of the semi-invariants of the same and lower order. In particular we have

$$\text{(8)}\quad \begin{aligned}&\mu_{20}=\lambda_{20},\ \mu_{30}=\lambda_{30};\ \mu_{40}=\lambda_{40}+3\lambda_{20}^2;\\ &\mu_{50}=\lambda_{50}-10\lambda_{30}\lambda_{20};\ \mu_{60}=\lambda_{60}+15\lambda_{40}\lambda_{20}+15\lambda_{20}^3+10\lambda_{30}^2\end{aligned}$$

5. A most natural way of applying the rule is afforded by Pearson's celebrated theory of frequency-functions. The moments $\mu_{i,0}$ are the moments of one of the marginal distributions (here the distribution of the x's). Computing μ_{20}, μ_{30} and μ_{40} in the ordinary way from the observations, criteria can be formed[1] showing to which of the Pearson Types the frequency curve of x belongs. This being decided, the parameters of the curve may be determined by the aid of the same moments. As the moments of higher order are easily expressed in terms of the parameters we get, in this way, μ_{50} and μ_{60} expressed in terms of μ_{20}, μ_{30} and μ_{40}.

To state the matter in a more general way, we may use the formulae given by Pearson in his memoir on regression, loc. cit. pp. 5 and 6.

---

[1] See W. Palin Elderton, Frequency Curves and Correlation. London 1927 Table VI

Pearson starts from a differential equation of the form

$$(9)\quad f'(x)(b_0 + b_1 x + b_2 x^2 + b_3' x^3 + \cdots) = (x+a)f(x)$$

where $f(x)$ is the frequency function of x.

Multiplying on both sides by x and integrating by parts, he finds the following formulae[1] (placing the origin in the mean)

$$(10)\quad n b_0 \mu_{n-1,0} + (n+1) b_1 \mu_{n,0} + (n+2) b_2 \mu_{n+1,0} + \cdots$$
$$= -\mu_{n+1,0} - a\mu_{n,0}$$

Now, Pearson remarks that experience shows that for the great bulk of frequency distributions the higher terms, multiplied by b_3, b_4, etc., may be neglected. In fact, Pearson's system of frequency curves is obtained as a result of putting $b_i = 0$ for $i \geqq 3$.

Following Pearson's example, we get the recursion formula,

$$(11)\quad n b_0 \mu_{n-1,0} + \left[(n+1) b_1 + a\right] \mu_{n,0} = -\left[(n-2)\ b_2 - 1\right] \mu_{n+1,0}$$

Putting here $n = 0, 1, 2, 3$, we get four equations to determine a, b_0, b_1, and b_2 in terms of the moments μ_{20}, μ_{30}, and μ_{40}. This being done, we get μ_{50} and μ_{60} on putting $n = 4$ and 5.

The procedure indicated above leads, in fact, to the theory of skew regression which is the natural consequence of Pearson's theory of skew frequency curves

6. As the theory just indicated above is at present at my request being worked out in detail by one of my pupils, Mr. Walter Anderson, I refrain from proceeding further into the matter

It remains, however, to show how the special formulae for cubic regression, given by me twelve years ago, arise out of a somewhat similar procedure.

Instead of starting from Pearson's theory of frequency functions, I now start from Thiele's theory of frequency functions. Just as in the preceding section the coefficients b_3, b_4 etc. were neglected in the equation (10), given by Pearson, I now neglect the semi-invariants λ_{50} and λ_{60} in the equations (8), given by Thiele. There is no doubt that the former approximation is of far more general validity than the latter; still the latter may be justified by the following considerations.

---

[1] See also Palin Elderton, Op cit, p 39

Assuming the variate x to be generated as the sum of a large number of independent, elementary increments, each of which has its own frequency distribution and its own set of semi-invariants, it follows from the theory of Thiele that any semi-invariant $\lambda_{r,0}$ of x is the sum of the elementary semi-invariants of the same order. Supposing the elementary increments to be s in number and denoting by λ'_r the mean value of the r elementary semi-invariants of order r we consequently have

$$\lambda_{r,0} = s\lambda'_r$$

Hence we get

$$\gamma_{r,0} = \frac{\lambda_{r,0}}{\lambda_{20}^{r/2}} = \frac{\lambda'_r}{\lambda_2'^{r/2}}\ \frac{1}{s^{\frac{r-2}{2}}}$$

Except under rather special conditions, which it is not necessary to dwell on here, the ratios $\lambda'_r / \lambda_2'^{r/2}$ are not extensively great. Thus if s is a large number we see that the "standardized" semi-invariants $\gamma_{r,0}$ of x are small of the order of magnitude of $\left(\frac{1}{\sqrt{s}}\right)^{r-2}$ In particular we have.

γ_{30} of the order $\frac{1}{\sqrt{s}}$
γ_{40} " " " $\frac{1}{s}$
γ_{50} " " " $\frac{1}{s\sqrt{s}}$
γ_{60} " " " $\frac{1}{s^2}$

We now have, denoting by

$$\alpha_{r,0} = \frac{\mu_{r,0}}{\mu_{20}^{r/2}}$$

the "standardized" moment of x , by a simple transformation of equation (8).

$$(8') \quad \alpha_{20} = 1\,; \quad \alpha_{30} = \gamma_{30}\,; \quad \alpha_{40} = \gamma_{40} + 3\,;$$

$$\alpha_{50} = \gamma_{50} + 10\gamma_{30}\,; \quad \alpha_{60} = \gamma_{60} + 15\gamma_{40} + 10\gamma_{30}^2 + 15$$

Stopping with quantities of the order $\frac{1}{s}$ we get

$$(13) \quad \alpha_{50} = 10\gamma_{30}\,; \quad \alpha_{60} = 15\gamma_{40} + 10\gamma_{30}^2 + 15$$

In practice we can, of course, not very well know if the hypothesis of elementary increments is valid, but if we have, on computing the moments up to the fourth order, found that γ_{30} and γ_{40} are rather small, and that γ_{40} is of the order of magnitude of γ_{30}^2, there is a certain plausibility in assuming that γ_{50} and γ_{60} are still smaller and that they may be neglected as compared to γ_{40} and γ_{30}^2.

The curve of cubic regression of y on x we may write in the form

$$t_y = c_0 + c_1 t_x + c_2 t_x^2 + c_3 t_x^3$$

where we have put

$$t_x = \frac{x - M_x}{\sqrt{\mu_{20}}} \quad ; \qquad t_y = \frac{y_x - M_y}{\sqrt{\mu_{20}}}$$

and it is evident that equation (1) now takes the form

$$\begin{aligned} 0 &= c_0 & & + c_2 & & + c_3 \alpha_{30} \\ r &= & + c_1 & + c_2 \alpha_{30} & & + c_3 \alpha_{40} \\ \alpha_{21} &= c_0 & + c_1 \alpha_{30} & + c_2 \alpha_{40} & & + c_3 \alpha_{50} \\ \alpha_{31} &= c_0 \alpha_{30} & + c_1 \alpha_{40} & + c_2 \alpha_{50} & & + c_3 \alpha_{60} \end{aligned}$$

We get

$$(14) \Delta = \alpha_{60}(\alpha_{40} - \alpha_{30}^2 - 1) - \alpha_{50}(\alpha_{50} - 2\alpha_{30}\alpha_{40} - 2\alpha_{30}) \\ - \alpha_{40}(\alpha_{40}^2 - \alpha_{40} + 3\alpha_{30}^2) + \alpha_{30}^4$$

$$\Delta_1 = r(\alpha_{30}\alpha_{60} - \alpha_{50}\alpha_{40}) - \alpha_{21}(\alpha_{60} - \alpha_{40}^2) + \alpha_{31}(\alpha_{50} - \alpha_{40}\alpha_{30}) \\ - r\alpha_{30}(\alpha_{30}\alpha_{50} - \alpha_{40}^2) + \alpha_{21}\alpha_{30}(\alpha_{50} - \alpha_{30}\alpha_{40}) - \alpha_{31}\alpha_{30}(\alpha_{40} - \alpha_{30}^2)$$

$$\Delta_2 = r(\alpha_{40}\alpha_{60} - \alpha_{50}^2 - \alpha_{60} + 2\alpha_{30}\alpha_{50} - \alpha_{30}^2\alpha_{40}) \\ - \alpha_{21}(\alpha_{30}\alpha_{60} - \alpha_{40}\alpha_{50} + \alpha_{30}\alpha_{40} - \alpha_{30}^3) + \alpha_{31}(\alpha_{30}\alpha_{50} - \alpha_{40}^2 + \alpha_{40} + \alpha_{30}^2)$$

$$\Delta_3 = -r(\alpha_{30}\alpha_{60} - \alpha_{50}\alpha_{40} + \alpha_{40}\alpha_{30} - \alpha_{30}^3) + \alpha_{21}(\alpha_{60} - \alpha_{40}^2 - \alpha_{30}^2) \\ - \alpha_{31}(\alpha_{50} - \alpha_{30}\alpha_{40} - \alpha_{30})$$

$$\Delta_4 = r(\alpha_{50}\alpha_{30} - \alpha_{40}^2 + \alpha_{40} - \alpha_{30}^2) - \alpha_{21}(\alpha_{60} - \alpha_{30}\alpha_{40} - \alpha_{30})$$
$$+ \alpha_{31}(\alpha_{40} - \alpha_{30}^2 - 1)$$

And the coefficients are

$$c_0 = \frac{\Delta_1}{\Delta} \qquad c_1 = \frac{\Delta_2}{\Delta} \qquad c_2 = \frac{\Delta_3}{\Delta} \qquad c_4 = \frac{\Delta_4}{\Delta}$$

We now introduce the semi-invariants by (8'), taking for α_{50} and α_{60} the approximate formulae (13). For α_{21} and α_{31} we put

$$(15) \qquad \alpha_{21} = \gamma_{21} \quad ; \quad \alpha_{31} = \gamma_{31} + 3r$$

The coefficients γ_{21} and γ_{31} are then the standardized correlation semi-invariants, according to a generalized theory of semi-invariants for bi-variate distributions.

It is now a consequence of our principle of approximation that all powers and products $\gamma_{ij} \cdot \gamma_{k,l} \cdot \gamma_{m,n} \cdot \ldots$, of which the sum $i+j+k+l+m+n+\ldots$ of the indices exceeds 6, shall be neglected as compared to powers and products of lower order. Observing this, the determinants reduce to the following:

$$\Delta = 12(1 - 2\gamma_{30}^2 + 2\gamma_{40}),$$

or $$\frac{1}{\Delta} - \frac{1}{12}(1 + 2\gamma_{30}^2 - 2\gamma_{40}),$$

$$\Delta_1 = 6(r\gamma_{30} - \gamma_{21}),$$

$$\Delta_2 = 12r + 6(r\gamma_{40} - \gamma_{31}) + 24r\gamma_{40} - 24r\gamma_{30}^2$$
$$-12\gamma_{30}(r\gamma_{30} - \gamma_{21}),$$

$$\Delta_3 = -6(r\gamma_{30} - \gamma_{21}),$$

$$\Delta_4 = -2(r\gamma_{40} - \gamma_{31}) + 6\gamma_{30}(r\gamma_{30} - \gamma_{21}).$$

Using the same rule of approximation on multiplying by $\frac{1}{\Delta}$, we finally get

$$
\begin{aligned}
c_0 &= \frac{1}{2}(r\gamma_{30}-\gamma_{21}), \\
c_1 &= r+\frac{1}{2}(r\gamma_{40}-\gamma_{31})-\gamma_{30}(r\gamma_{30}-\gamma_{21}), \\
c_2 &= -\frac{1}{2}(r\gamma_{30}-\gamma_{21}), \\
c_3 &= -\frac{1}{6}(r\gamma_{40}-\gamma_{31})+\frac{1}{2}\gamma_{30}(r\gamma_{30}-\gamma_{21}).
\end{aligned}
\tag{16}
$$

In my cited memoir of twelve years ago I put[1]

$$r_{30}=\frac{1}{2}(r\gamma_{30}-\gamma_{21}); \quad r_{40}=-\frac{1}{6}(r\gamma_{40}-\gamma_{31}),$$

Using this notation, we get

$$
\begin{aligned}
c_0 &= r_{30}, \\
c_1 &= r-3r_{40}-2\gamma_{30}\,r_{30} \\
c_2 &= -c_0=-r_{30}, \\
c_3 &= r_{40}+\gamma_{30}r_{30}.
\end{aligned}
\tag{17}
$$

These coefficients are exactly the same as in equation (34*, II) of my former memoir. As shown in that memoir on several numerical examples, the regression formula in question applies very well in cases of moderately skew correlations.

It is seen that the coefficients r_{30} and r_{40} determine the curvature of the regression. If $r_{30}=r_{40}=0$ the regression is linear (of the third order). I have called these coefficients the correlation coefficients of higher order If the correlation surface is approximately normal we have the following formulae for the standard errors of the coefficients involved.

---

[1] In Pearson's notation we have $r_{30}=\frac{1}{2}\bar{\epsilon}$ and $r_{40}=\frac{1}{6}\bar{\zeta}$.

$$\sigma_{(\gamma_{30})} = \sqrt{\frac{6}{N}}\,; \quad \sigma_{(\gamma_{40})} = \sqrt{\frac{24}{N}}\,; \quad \sigma_{(\gamma_{21})} = \sqrt{\frac{2+4r^2}{N}}\,; \quad \sigma_{(\gamma_{31})} = \sqrt{\frac{6+18r^2}{N}}$$

$$(18) \quad \sigma_{(r)} = \frac{1-r^2}{\sqrt{N}}\,; \quad \sigma_{(r_{20})} = \sqrt{\frac{1-r^2}{2N}}\,; \quad \sigma_{(r_{40})} = \sqrt{\frac{1-r^2}{6N}}\,; \quad \sigma_{(c_0)} = \sqrt{\frac{1-r^2}{2N}}\,;$$

$$\sigma_{(c_1)} = \sqrt{\frac{(5-2r^2)(1-r^2)}{2N}}\,; \quad \sigma_{(c_2)} = \sqrt{\frac{1-r^2}{2N}}\,; \quad \sigma_{(c_3)} = \sqrt{\frac{1-r^2}{6N}}$$

Lund (Sweden).

SYNOPSIS OF ELEMENTARY MATHEMATICAL STATISTICS*

By

B. L. SHOOK

SECTION I. ELEMENTARY STATISTICAL FUNCTIONS

1 *Variates.* Practically all statistical data[1] is obtained as the result of observations that endeavor to establish the magnitudes of certain variables. The individual magnitudes that are recorded are termed variates. Thus in computing the average annual rainfall of a region, the variable is rainfall, and the amount of rainfall for any single year is a variate Likewise, if the bank clearings for the City of New York be under consideration, then the variable is bank clearings, and the clearings for any specified interval is a variate.

2 The *arithmetic mean* of a series of variates is equal to the sum of the variates divided by the number of variates in the series. If M_v designates the arithmetic mean of the N variates v_1, v_2, v_3, $\cdot\cdot\cdot$, v_N,

(1) $$M_v=\frac{1}{N}(v_1+v_2+\cdots+v_N)=\frac{1}{N}\sum v$$

3. The nth moment of a series of variates is defined as the arithmetic mean of the nth powers of these variates and is represented by the symbol $\mu'_{n,v}$. Thus

(2) $$\mu_{n,v}=\frac{1}{N}(v_1^n+v_2^n+v_3^n+\cdots+v_N^n)=\frac{1}{N}\sum v^n$$

That is

$$\mu'_{1v}=\frac{1}{N}\sum v$$

$$\mu'_{2,v}=\frac{1}{N}\sum v^2$$

$$\mu'_{3,v}=\frac{1}{N}\sum v^3$$

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* An abstract of a series of lectures on elementary statistics given by the mathematical statistical staff at the University of Michigan.

[1] Observe that the number of variates in a series is denoted by N, whereas the smaller italic n is employed as an ordinal number

Obviously, by definitions (1) and (2)

$$\mu'_{1,v} = M_v \tag{3}$$

4. *The deviation of a variate from the arithmetic mean* will be designated by the symbol $\bar{v}$, i. e.

$$\bar{v}_i = v_i - M_v \tag{4}$$

5. The *nth moment about the mean** is defined as the arithmetic mean of the nth powers of the deviations of the variates from the mean, and is represented symbolically by $\mu_{n,v}$. Thus

$$\mu_{n,v} = \frac{1}{N}\sum \bar{v}^n \tag{5}$$ so that

$$\mu_{1,v} = \frac{1}{N}\sum \bar{v} = 0 \tag{5a}$$

$$\mu_{2,v} = \frac{1}{N}\sum \bar{v}^2 \tag{5b}$$

$$\mu_{3,v} = \frac{1}{N}\sum \bar{v}^3 \tag{5c}$$

The fact that $\mu_{1,v} = 0$, is demonstrated as follows:

$$\begin{aligned}
\bar{v}_1 &= v_1 - M_v \\
\bar{v}_2 &= v_2 - M_v \\
&\vdots \\
\bar{v}_N &= v_N - M_v \\
\hline
\sum \bar{v} &= \sum v - NM_v
\end{aligned}$$

$$\mu_{1,v} = \frac{\sum \bar{v}}{N} = \frac{\sum v}{N} - M_v = M_v - M_v = 0 \qquad Q.\ E.\ D.$$

The numerical example of Table I illustrates the definitions of the preceding paragraphs. The data consists of thirteen variates, which represent the number of even numbers found in consecutive blocks of 100 numbers, drawn to determine the order of call for drafting United States soldiers in 1918. These variates were obtained from the *first* 1300 drawings made.

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* For convenience the arithmetic mean is frequently referred to as *the* mean. When referring to geometric or harmonic means, the adjectives geometric or harmonic must therefore be specified.

The most obvious conclusion to be drawn from Table I is that the use of fractions in determining the values of $\mu_{n\,v}$ is cumbersome. if M_v is a whole number, then the values of $\bar{v}$, $\bar{v}^2$ and $\bar{v}^3$ are integers, and the procedure is simple Generally, however, M_v will be fractional, and consequently awkward expressions for $\bar{v}$, $\bar{v}^2$ and $\bar{v}^3$ will result. On the other hand, the computation of values of $\mu'_{n\,v}$ is relatively easy, and hence it is expedient to express $\mu_{2\,v}$ and $\mu_{3,v}$ in terms of the moments $\mu'_{n\,v}$. This may be done as follows:

Since by definition,

$$\bar{v}_1 = v_1 - M_v, \qquad \text{it follows that}$$

$$\bar{v}_1^2 = v_1^2 - 2v_1 M_v + M_v^2 \qquad \text{, and}$$

$$\bar{v}_1^3 = v_1^3 - 3v_1^2 M_v + 3v_1 M_v^2 - M_v^3$$

Consequently

$$\begin{array}{ll}
\bar{v}_1^2 = v_1^2 - 2M_v v_1 + M_v^2 & \bar{v}_1^3 = v_1^3 - 3v_1^2 M_v + 3v_1 M_v^2 - M_v^3 \\
\bar{v}_2^2 = v_2^2 - 2M_v v_2 + M_v^2 & \bar{v}_2^3 = v_2^3 - 3v_2^2 M_v + 3v_2 M_v^2 - M_v^3 \\
\bar{v}_3^2 = v_3^2 - 2M_v v_3 + M_v^2 & \bar{v}_3^3 = v_3^3 - 3v_3^2 M_v + 3v_3 M_v^2 - M_v^3 \\
\vdots & \vdots \\
v_N^2 = v_N^2 - 2M_v v_N + M_v^2 & \bar{v}_N^3 = v_N^3 - 3v_N^2 M_v + 3v_N M_v^2 - M_v^3 \\
\hline
\sum \bar{v}^2 = \sum v^2 - 2M_v \sum v + NM_v^2 & \sum \bar{v}^3 = \sum v^3 - 3M_v \sum v^2 + 3M_v^2 \sum v - NM_v^3
\end{array}$$

Dividing both sides of these equations through by N yields, respectively

$$\frac{\sum \bar{v}^2}{N} = \frac{\sum v^2}{N} - 2M_v \cdot M_v + M_v^2$$

$$\frac{\sum \bar{v}^3}{N} = \frac{\sum v^3}{N} - 3M_v \frac{\sum v^2}{N} + 3M_v^2 \cdot M_v - M_v^3$$

Hence

$$(6) \quad \begin{cases} \mu_{2\cdot v} = \mu'_{2:v} - M_v^2 \\ \mu_{3\cdot v} = \mu'_{3\cdot v} - 3M_v \mu'_{2:v} + 2M_v^3 \end{cases}$$

TABLE I

i	V_1	V_1^2	V_1^3	$\bar{V}_1 = V_1 - M_v$	$\bar{V}_1^2$	$\bar{V}_1^3$
1	51	2601	132651	− 3/13	9/169	− 27/2197
2	49	2401	117649	− 29/13	841/169	− 24389/2197
3	60	3600	216000	114/13	12996/169	1481544/2197
4	53	2809	148877	23/13	529/169	12167/2197
5	48	2304	110592	− 42/13	1764/169	− 74088/2197
6	51	2601	132651	− 3/13	9/169	− 27/2197
7	42	1764	74088	− 120/13	14400/169	− 1728000/2197
8	50	2500	125000	− 16/13	256/169	− 4096/2197
9	51	2601	132651	− 3/13	9/169	− 27/2197
10	52	2704	140608	10/13	100/169	1000/2197
11	54	2916	157464	36/13	1296/169	46656/2197
12	53	2809	148877	23/13	529/169	12167/2197
13	52	2704	140608	10/13	100/169	1000/2197
Total	666	34314	1777716	0	32838/169	− 276120/2197

$$M_v = \mu'_{1\cdot v} = \frac{666}{13} \qquad \mu_{1\cdot v} = 0$$

$$\mu'_{2\cdot v} = \frac{34314}{13} \qquad \mu_{2\cdot v} = \frac{32838}{13\cdot 169} = \frac{2526}{169}$$

$$\mu'_{3\cdot v} = \frac{177716}{13} \qquad \mu_{3\cdot v} = \frac{-276120}{13\cdot 2197} = \frac{-21240}{2197}$$

These formulae are perhaps the most important in our work, since they enable us to obtain the moments about the mean without requiring that we actually determine the deviations. Applying these formulae to the numerical example of Table I,

$$\mu_{2v} = \frac{34314}{13} - \left(\frac{666}{13}\right)^2 = \frac{2526}{169}$$

$$\mu_{3v} = \frac{177716}{13} - 3\left(\frac{34314}{13}\right)\left(\frac{666}{13}\right) + 2\left(\frac{666}{13}\right)^3 = -\frac{21240}{2197}$$

The results thus obtained by this *indirect* method are identical with the results obtained in Table I by employing the *direct* method.

7 *Standard Deviation.* The second moment about the mean, μ_{2v}, is a function of the variability of the data, since its essential elements are the deviations of the variates from the mean. But if the original variates happen to be measured in *inches*, then since $\mu_{2\cdot v}$ is the average of the squares of the deviations, it follows that the unit of $\mu_{2:v}$ is square inch Nevertheless, by extracting the square root of μ_{2v} we would obtain a function which would in general measure the variability of, and possess the same unit as the original data. This function is known as the *standard deviation* and is denoted by the symbol σ_v. Thus

$$\sigma_v = \sqrt{\mu_{2:v}} \tag{7}$$

Verbally we may say that the standard deviation is defined as the square root of the mean of the squared deviations of the variates from their mean.

Actually σ_v is rarely computed directly from the squared deviations, but rather by employing the relationship given in formula (6). For the data of Table I

$$\sigma_v = \sqrt{\frac{2526}{169}} = \frac{50.2593}{13} = 3.78918$$

8. *Standard Units.* If we assume that the arithmetic mean and the standard deviation of the weights of adult males are 150 lbs. and 20 lbs. respectively, then we may say that a man weighing 190 lbs. is

40 lbs or 2 *standard units* above the average in weight. Likewise an individual weighing 120 lbs. may be considered as being 30 lbs. or 1.5 *standard units* under average weight. Conversely, if the arithmetic mean and the standard deviation for heights be 67 inches and 2.5 inches respectively, then an individual who is 2 standard units above the average height must be five inches above the average stature, or in other words must be 72 inches tall. The magnitude of an observation expressed in standard units is therefore defined as follows:

$$t_1 = \frac{V_1 - M_v}{\sigma_v} = \frac{\bar{V}_1}{\sigma_v} \tag{8}$$

It will be observed that these *standard* variates, t_1, are abstract numbers For example, if the original variates be expressed in the unit inch then the unit of M_v, $\bar{V}$ and σ_v is also inch, and it follows that if both the numerator and denominator of a fraction be expressed as *inches* the quotient must be an abstract number, *independent of the unit employed in the measurements.* For instance, one series of variates would result if the height of each of a group of individuals were recorded in *inches* However, if their heights had been recorded in *centimeters*, each of the resulting set of variates would be numerically about 2 54 times as large as the corresponding variate expressed in inches. Nevertheless, the *standard variates* obtained by both methods would agree in the case of each individual. Thus, if

$$M_v = 67 \text{ ins} = 67(2\ 54) \text{ cms.},$$

and

$$\sigma_v = 2.5 \text{ ins.} = 2.5(2.54) \text{ cms.},$$

then for an individual 6 feet tall

$$V = 72 \text{ ins} = 72(2.54) \text{ cms.},$$

$$\bar{V} = 5 \text{ ins.} = 5(2.54) \text{ cms.},$$

$$t = \frac{5 \text{ ins.}}{2\ 5 \text{ ins.}} = \frac{5(2.54) \text{ cms.}}{2.5(2.54) \text{ cms.}}, \text{ or}$$

$$t = 2 = 2.$$

With the aid of a computing machine, the series of standard variates corresponding to any observed series of variates may be completed very rapidly by means of a so-called continuous process. To illustrate, we found that for the data of Table I, page 17,

$$M_v = 51.230769$$

$$\sigma_v = 3.86610$$

By formula (8), then

$$t_i = \frac{v_i - 51.230769}{3.86610} = -13.2513 + .258659\ v_i$$

In using this equation one should first subtract out 13.2513 from the machine, and then set up 258659 as a multiplier. The product of this multiplier by 51 will cause the value $t = -.059691$ to appear on the machine. By merely subtracting the multiplier two times, the value $t = -.577009$, corresponding to $t = 49$, appears. Continuing this "build-over" method, the following set of standard variates is readily obtained.

TABLE II

i	v_i	t_i
1	51	- .06
2	49	- 58
3	60	2.27
4	53	.46
5	48	- .84
6	51	- .06
7	42	- 2.39
8	50	- .32
9	51	- .06
10	52	.20
11	54	.72
12	53	.46
13	52	.20
Total	666	0.00

It is scarcely an exaggeration to state that the theory of mathematical statistics hinges on standard units. Although in many problems this might not appear on the surface, yet we shall see that the fact is nevertheless true

9. The properties of the *moments* of standard variates are both

interesting and important Thus

(9) $\mu_{1t} = M_t = 0$

since

$$M_t = \frac{\Sigma t}{N} = \frac{1}{N}\sum \frac{V_i - M_v}{\sigma_v} = \frac{1}{N\sigma_v}\sum \bar{V}_i = 0$$

(see formula 5a)

Referring to formula (6) we see that

$$\mu_2 = \mu_2' - M^2$$

$$\mu_3 = \mu_3' - 3\mu_2' M + 2M^3$$

But since M_t has already been proven equal to 0,

$$\mu'_{2\cdot t} = \mu_{2t}$$

$$\mu'_{3t} = \mu_{3t}$$

Which is an important simplification in the moments of the standard variates.

(10) $\mu_{2t} = 1$

for $$\mu_{2t} = \frac{\Sigma t^2}{N} = \frac{1}{N}\sum\left(\frac{V_i - M_v}{\sigma_v}\right)^2 = \frac{1}{\sigma_v^2}\sum\frac{\bar{V}^2}{N}$$

$$= \frac{\mu_{2v}}{\sigma_v^2} = 1$$

(see formula 7)

(11) $$\mu_{3,t} = \frac{\mu_{3,v}}{\sigma_v^3} = \frac{\mu_{3v}}{\mu_{2,v}\sigma_v}$$

for

$$\mu_{3t} = \frac{\Sigma t^3}{N} = \frac{1}{N}\sum\left(\frac{V_i - M_v}{\sigma_v}\right)^3 = \frac{1}{\sigma_v^3}\sum\frac{\bar{V}^3}{N} = \frac{\mu_{3v}}{\sigma_v^3}$$

We see, therefore, that although the values of μ_{1t} and μ_{2t} are always 0 and 1 respectively, the value of μ_{3t} will possess an abstract value depending, nevertheless, upon the variates themselves. The expression, μ_{3t}, is known as the coefficient of *skewness* and is denoted by the symbol $\alpha_{3,v}$, i. e.

$$(12) \qquad \alpha_{3v} = \frac{\mu_{3v}}{\sigma_v^3} = \frac{\mu_{3v}}{\mu_{2v}\ \sigma_v}$$

Summary of Section I From the viewpoint of *Elementary Mathematical Statistics,* we characterize a series of variates by its

(a) number, N,
(b) mean, M_v ,
(c) standard deviation, σ_v , and
(d) skewness, $\alpha_{3,v}$

The *moments about the mean,* μ_{nv} , are introduced solely to facilitate the determination of σ_v and $\alpha_{3,v}$. Other moments, $\mu'_{n,v}$, are used to simplify the numerical calculation of the moments about the mean, $\mu_{n,v}$.

Verbally, we may state that the mean serves as a convenient average, and the standard deviation measures the concentration of the variates about their mean.

A thorough discussion of the significance of the coefficient of skewness must be slightly deferred. We may say at this time merely that the value of $\alpha_{3,v}$ depends obviously upon the value of $\mu_{3,v}$ and that a glance at the last column of Table I will lend weight to the statement that a positive or negative skewness indicates a weighted preponderance of those variates which are considerably greater than, or less than the mean, respectively.

Finally, the operations of mathematical statistics, and even certain comparisons in descriptive statistics, require that we introduce the notion of a standard variate, defined as follows:

$$t_i = \frac{v_i - M_v}{\sigma_v}$$

SECTION II.

INDIRECT METHOD OF OBTAINING ELEMENTARY FUNCTIONS

10. One of the fundamental theorems of moments states that if a constant be added to, or subtracted from each variate of a series, the moments computed about the mean for the revised series will be

identical with the corresponding moments of the original series. By way of a simple example:

The mean of the following five variates is 138, consequently the values of $\bar{V}$ are as given below:

i	V_i	$\bar{V}_i$
1	133	-5
2	142	4
3	138	0
4	141	3
5	136	-2
Total	690	0

If we subtract, say, 130 from each of the variates, then for the revised series x_1, x_2, x_3, x_4 and x_5,

i	$M_o = 130$ x_i	$\bar{x}_i = x_i - M_x$
1	3	− 5
2	12	4
3	8	0
4	11	3
5	6	− 2
Total	40	0

$$M_x = \frac{40}{5} = 8, \quad M_v = 130 + 8 = 138$$

The value subtracted, 130, is termed the *provisional mean*, and in general is designated by the symbol, M_o. It follows, therefore, that

(13) $$x_i = V_i - M_o$$

(14) $$M_V = M_o + M_x$$

(15) $$\mu'_{n\,x} = \frac{\sum x^n}{N}$$

(16) $$\mu_{n\,v} = \mu_{n\,x}$$

It is understood that the functions of x are defined in precisely the same manner as corresponding functions of v, that is

$$M_x = \frac{\sum x}{N}$$

$$\bar{x}_i = x_i - M_x$$

$$\mu'_{n\,x} = \frac{\sum x^n}{N}$$

$$\mu_{n\,\bar{x}} = \frac{\sum \bar{x}^n}{N}$$

etc.

11. Formula (13) follows from definition, although (14)—seemingly self-evident—needs proof. Thus by (13)

$$\begin{aligned} v_1 &= M_0 + x_1 \\ v_2 &= M_0 + x_2 \\ v_3 &= M_0 + x_3 \\ \cdot &\quad \cdot \quad \cdot \\ v_N &= M_0 + x_N \\ \hline \sum v &= NM_0 + \sum x \end{aligned}$$

Dividing both sides through by N yields, by definition,

$$M_v = M_0 + M_x \qquad Q.\,E.\,D.$$

Formula (15) is proved by means of (13) and (14) as follows:

$$\begin{aligned} \bar{x}_i &= x_i - M_x && \text{(Definition)} \\ &= (v_i - M_0) - (M_v - M_0) && \text{(Formulae 13 and 14)} \\ &= v_i - M_v \\ &= \bar{v}_i && Q.\,E.\,D. \end{aligned}$$

Since

$$\mu_{n\,v} = \frac{\sum \bar{v}^n}{N} \quad \text{and} \quad \mu_{n\,x} = \frac{\sum \bar{x}^n}{N}$$

and we have just shown that always for corresponding values

$$\bar{V}_i = \bar{x}_i$$

the truth of (16) is apparent.

12. A comparison of tables III and I will reveal an advantage of the indirect over the direct method of calculation.

TABLE III

i	V_i	$M_0=50$ x_i	x_i^2	x_i^3
1	51	1	1	1
2	49	− 1	1	− 1
3	60	10	100	1000
4	53	3	9	27
5	48	− 2	4	− 8
6	51	1	1	1
7	42	− 8	64	− 512
8	50	0	0	0
9	51	1	1	1
10	52	2	4	8
11	54	4	16	64
12	53	3	9	27
13	52	2	4	8
Total		16	214	616

$$M_x = \frac{16}{13}$$

$$\mu'_{2,x} = \frac{214}{13} \qquad \mu_{2,x} = \mu'_{2,x} - M_x^2 = \frac{2526}{13^2}$$

$$\mu'_{3,x} = \frac{616}{13} \qquad \mu_{3,x} = \mu'_{3,x} - 3\mu_{2,x}M_x + 2M_x^3 = -\frac{21240}{13^3}$$

$$\sigma_x = \sqrt{\frac{2526}{13^2}} = 3.78918$$

$$\alpha_{3,x} = \frac{\mu_{3,x}}{\sigma_x \, \mu_{2,x}} = -\frac{21240}{2526\sqrt{2526}} = -.167303$$

$$M_V = 50 + \frac{16}{13} = 51\ 3/13$$

$$\sigma_V = \sigma_x = 3.78918$$

$$\alpha_{3,V} = \alpha_{3,x} = -.167303$$

It will be observed that the values

$$\mu_{2\cdot x} = \frac{2526}{13^2} \quad \text{and} \quad \mu_{3\cdot x} = \frac{-21240}{13^3}$$

agree exactly with those of Table I, namely

$$\mu_{2\cdot v} = \frac{2526}{169} \quad \text{and} \quad \mu_{3\cdot v} = \frac{-21240}{2197}$$

The following will illustrate an important advantage of the indirect method of determining the moments, $\mu_{n\cdot v}$. Let us suppose that after computing the values of M_v, σ_v and $\alpha_{3\cdot v}$ for the 13 variates of Table I we desire to delete the 13th variate, $v_{13} = 52$, and compute the values of M, σ and α_3 for the remaining twelve variates.

By the direct method of Table I, the revision would be quite laborious, but by the indirect method of Table III, revisions are made easily, as follows:

$$N = 13 - 1 = 12, \quad \sum x = 16 - 2 = 14, \quad \sum x^2 = 214 - 4 = 210;$$
$$\sum x^3 = 616 - 8 = 608$$

Consequently

$$M_x = \frac{14}{12}$$

$$\mu'_{2\cdot x} = \frac{210}{12} \quad \mu_{2\cdot x} = \mu'_{2\cdot x} - M_x^2 = \frac{581}{6^2}$$

$$\mu'_{3\cdot x} = \frac{608}{12} \quad \mu_{3\cdot x} = \mu'_{3\cdot x} - 3\mu'_{2\cdot x}M_x + 2M_x^3 = -\frac{1600}{6^3}$$

$$\sigma_x = \frac{1}{6}\sqrt{581} = 4.01732$$

$$\alpha_{3\cdot x} = \frac{-1600}{581\sqrt{581}} = -.114250$$

$$M_v = 50 + \frac{14}{12} = 51\ 1/6$$

$$\sigma_v = \sigma_x = 4.01732$$

$$\alpha_{3\cdot v} = \alpha_{3\cdot x} = -.114250$$

13. In a word, revisions of series arising from

(a) increasing or decreasing the number of variates,
(b) combining two or more series, or
(c) correcting the original variates

together with the resulting smaller numbers that result by employing the indirect method, lead us ordinarily to avoid using the direct method of section I in computing the fundamental functions, *mean, standard deviation* and *skewness.*

In practice, one continually faces the problem of revision. Thus, in business statistics, publications serving as sources of data frequently are obliged to present revisions for estimates made in previous issues. Moreover, monthly and annual endeavors to bring statistics up to date require the addition of variates to series. In problems arising in the field of psychology and education, it may develop after preliminary calculations have been made that one or more observations of the original series must be deleted due to the presence of factors such as unusual physical or mental impairment at the time of examination, cheating, etc. Again, we may desire to combine the statistics for several distinct intervals, for several classes, or for various schools of a city or state, etc

In the numerical examples above, calculations were made in terms of fractions, rather than decimals, in order to emphasize the fact that the direct and indirect methods will yield identical results. Ordinarily, decimals are employed, and the results will consequently differ slightly.

SECTION III

FREQUENCY DISTRIBUTIONS

14. In dealing with *large* groups of quantitative data, the computation of the elementary statistical functions and an appreciation of the variation in the magnitudes of the series of measurements is greatly facilitated by systematically presenting the data in the form of a *frequency distribution.* Such a distribution may present in tabular form

(a) each *different* variate observed, and
(b) the number of times that each different variate was observed in the investigation

It is evident at the very outset, therefore, that if a frequency distribution merely reproduces precisely the same data that might otherwise have been listed serially, the values of M, σ and α_3 computed from such a frequency distribution must correspond exactly with the values of M, σ and α_3 that would have been obtained by the serial method. This *serial method* has been considered in the two preceding sections.

15. As an illustration, suppose that we consider the complete table from which the 13 variates, used in earlier computations, were taken. Since, according to the regulations, 17,000 numbers were withdrawn, we shall have 170 groups of one hundred numbers each, consequently 170 variates. These are listed below

We shall see that one can compute the fundamental functions from the frequency distribution more readily than from Table IV. Again, certain phenomena are apparent at a glance at Table V, though by no means evident from a short inspection of Table IV. Thus the *range* of the variates is immediately observed in Table V, and the degree of symmetry in the distribution can be guessed rather accurately by one accustomed to computing the coefficient of skewness from distributions.

TABLE IV

Number of even numbers in 170 samples of 100 numbers each.

U. S Order of Call, 1918

51	42	49	53	49	46	47	51	57	48
49	51	55	50	46	53	46	47	46	54
60	59	42	42	58	43	53	49	54	53
53	46	47	50	55	50	48	47	44	51
48	57	49	52	57	56	45	64	37	58
51	53	51	49	39	54	51	56	44	41
42	46	50	56	42	54	50	45	47	58
50	52	53	55	52	48	50	53	45	48
51	55	47	45	55	51	47	54	48	46
52	60	52	53	49	52	46	62	43	48
54	50	51	50	50	53	44	54	51	45
53	47	44	48	55	45	55	45	55	50
52	55	54	56	42	49	45	55	45	55
44	37	44	53	52	50	51	47	56	44
54	56	50	53	49	52	60	48	50	51
56	45	50	51	53	44	47	54	46	54
42	44	49	43	57	46	48	48	49	48

The frequency distribution for Table IV may be obtained readily by means of the "cross-five" method as follows.

TABLE V

Frequency Distribution for Data of Table IV

V	Tabulation	f
37	||	2
38		0
39	|	1
40		0
41	|	1
42	𝍸 ||	7
43	|||	3
44	𝍸 ||||	9
45	𝍸 𝍸	10
46	𝍸 𝍸	10
47	𝍸 𝍸	10
48	𝍸 𝍸 ||	12
49	𝍸 𝍸 |	11
50	𝍸 𝍸 𝍸	15
51	𝍸 𝍸 ||||	14
52	𝍸 ||||	9
53	𝍸 𝍸 ||||	14
54	𝍸 𝍸 |	11
55	𝍸 𝍸 |	11
56	𝍸 ||	7
57	||||	4
58	|||	3
59	|	1
60	|||	3
61		0
62	|	1
63		0
64	|	1
Total		170

16. The above type of distribution should be differentiated from others in which it has been found advantageous to combine the variates

into *classes* and likewise to group together the corresponding frequencies. A distribution of grades will serve to illustrate this second type of distribution

TABLE VI

Distribution of Examination
Grades of 168 Students

Class	Frequency
0- 10	0
11- 20	2
21- 30	3
31- 40	5
41- 50	7
51- 60	16
61- 70	39
71- 80	45
81- 90	41
91-100	10
Total	168

Such a table does not represent *exactly* the original data in which the grades were recorded for each student as an integral number of per cents; nevertheless, it gives a very good idea of the general form of the distribution and enables us to compute the fundamental functions with a considerable degree of accuracy

17 *Discrete Variates* The distribution of Table V is obviously one in which the variates can, from their very nature, be expressed only as integers. A distribution of this type is termed one of *discrete variates*, or one of a *discrete variable* Common illustrations of this type are to be found in distributions of the number of individuals in a family, the number of petals on a flower, the number of coins turning up heads, etc.

18. *Continuous Variates.* In the majority of distributions the variates by their nature may differ by infinitesimals, and the observed values, as recorded, are merely more or less accurate estimates of the *true values*, which never can be established with ***absolute*** accuracy by any method of measurement. Thus the variates in the case of heights may be correct to the nearest inch, one-hundredth of an inch, or even the one millionth part of an inch, etc., but theoretically it can be shown that the chances that any measurement of a continuous variable is exact is about one in infinity. A frequency table for the distribution of continuous variates must always, therefore, be one of *grouped frequencies.*

19. The fundamental differences between distributions which may be classified as

(a) discrete
(b) grouped discrete, and
(c) continuous

are of vital importance whenever the accurate determination of the mean, standard deviation, or skewness, is concerned. We shall now illustrate in detail and by numerical examples the procedure which should be followed in each case.

20. *Frequency Distributions of Discrete Variates.*

If 180 dice were thrown, and a throw of a six spot counted a success, then the *expected* frequencies of successes that would be obtained in one thousand such trials are as follows:

TABLE VII

V	f	$M_0 = 30$ x	x^2	x^3
15	1	−15	225	−3375
16	1	−14	196	−2744
17	2	−13	169	−2197
18	4	−12	144	−1728
19	6	−11	121	−1331
20	10	−10	100	−1000
21	16	− 9	81	− 729
22	23	− 8	64	− 512
23	31	− 7	49	− 343
24	41	− 6	36	− 216
25	51	− 5	25	− 125
26	61	− 4	16	− 64
27	69	− 3	9	− 27
28	75	− 2	4	− 8
29	79	− 1	1	− 1
30	80	0	0	0
31	77	1	1	1
32	72	2	4	8
33	64	3	9	27
34	56	4	16	64
35	46	5	25	125
36	37	6	36	216
37	29	7	49	343
38	22	8	64	512
39	16	9	81	729
40	11	10	100	1000
41	8	11	121	1331
42	5	12	144	1728
43	3	13	169	2197
44	2	14	196	2744
45	1	15	225	3375
46	1	16	256	4096

$\Sigma f = 1000$

$\Sigma xf = -27$

$\Sigma x^2 f = 24687$

$\Sigma x^3 f = 11259$

$\mu_{2x} = 24.6863$

$\mu_{3x} = 13.2586$

$M_0 = 30$

$M_x = -.027$

$\mu'_{2x} = 24\ 687$

$\mu'_{3x} = 11.259$

$\sigma_x = 4\ 96853$

$\sigma_x \mu_{2x} = 122.655$

$\alpha_{3x} = .108097$

$M_V = 29\ 973$, $\sigma_V = 4.96853$, $\alpha_{3V} = .108097$

Explanation Since this distribution of discrete variates is an exact reproduction of the original data listed serially, we know that the moments obtained by the frequency distribution method must be identical with those which would have resulted had the serial method been employed In fact

$$(17)\qquad \begin{cases} \sum f = N, \\ \sum xf = \sum x, \\ \sum x^2 f = \sum x^2; \text{ and} \\ \sum x^3 f = \sum x^3 \end{cases}$$

Numerically, $\sum x^n f$ is absolutely equivalent to $\sum x^n$. However, $\sum x^n f$ implies more; it indicates a brief and systematic method of attaining a total in which multiplication replaces repeated additions. Thus, in the serial method the value $x = 5$ would be added 46 times during the numerical determination of $\sum x$. In the frequency distribution method one multiplication, 5 x 46, represents likewise the contribution of this variate to the total $\sum xf = \sum x$.

If a computing machine be not available, the headings of Table VII should be

v	f	x	xf	$x^2 f$	$x^3 f$

and the totals $\sum x^n f$ obtained by a detailed process With the aid of a computing machine the values of $\sum x^n f$ may be obtained readily by a continuous process, and it is necessary to record only the totals.

Since

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

it follows that

$$(18)\qquad \sum (x+1)^3 f = \sum x^3 f + 3\sum x^2 f + 3\sum xf + \sum f$$

Formula (18) is known as *Charlier's check* By associating with each value, f the value of x^3 appearing on the next lower line, the value of $\sum (x+1)^3 f$ may be obtained as readily as that of $\sum x^3 f$. Then if equation (18) be satisfied we may assume with a considerate degree of confidence that all five summations have been accurately determined

It follows that we may now write, employing (17),

$$(19)\qquad \begin{cases} \mu_{2x} = \dfrac{\Sigma x^2 f}{\Sigma f} \\ \mu_{3x} = \dfrac{\Sigma x^3 f}{\Sigma f} \end{cases}$$

and observe that here, as in the serial method,

$$\mu_{2:x} = \mu'_{2:x} - M_x^2$$

$$\mu_{3:x} = \mu'_{3:x} - 3M_x \mu'_{2x} + 2M_x^3$$

$$M_v = M_0 + M_x$$

$$\mu_{2v} = \mu_{2x}$$

etc.

21 *The Grouping of Discrete Variates* Occasionally frequency distributions of discrete variates contain so many different variates that some sort of grouping must be employed. Thus, the distribution of Table VII and the numerical calculations may be abbreviated as in Table VIII.

Explanation. The *class mark* of a class is defined as the arithmetic mean of the greatest and least variates that can occur within that class. In Table VIII we might have used the class marks as values of v, but the use of a provisional mean, as has already been demonstrated, saves a large amount of labor.

TABLE VIII (Unadjusted)

Class	Class Mark	f	$M_o = 30,\ \lambda = 3$ x
14-16	15	2	- 5
17-19	18	12	- 4
20-22	21	49	- 3
23-25	24	123	- 2
26-28	27	205	- 1
29-31	30	236	0
32-34	33	192	1
35-37	36	112	2
38-40	39	49	3
41-43	42	16	4
44-46	45	4	5

$\Sigma f = 1000$ $\qquad M_o = 30, \quad \lambda = 3$
$\Sigma xf = -9$ $\qquad M_x = -.009$
$\Sigma x^2 f = 2817$ $\qquad \mu'_{2x} = 2.817$
$\Sigma x^3 f = 405$ $\qquad \mu'_{3x} = .405$
$\mu_{2,x} = 2.81692$ $\qquad \sigma_x = 1.67837$
$\mu_{3,x} = .481058$ $\qquad \sigma_x \mu_{2x} = 4.72783$

$\alpha_{3x} = .101750$

$M_v = 29.973, \qquad \sigma_v = 5.03511, \qquad \alpha_{3v} = .101750$

The *class interval* is defined as the common difference between two consecutive class marks. In the example of Table VIII, the class interval has been chosen as the *unit* of x, consequently M_x and σ_x are expressed in class units. If λ denotes the class interval for a distribution, then

(20) $\qquad M_v = M_o - \lambda M_x$, and

(21) $\qquad \sigma_v = \lambda \sigma_x$

Thus in Table VIII we had

$$M_v = 30 + 3(-.009) = 29.973$$
$$\sigma_v = 3(1.67837) = 5.03511$$

Since the skewness is an abstract number, completely independent of the unit employed

$$(22) \qquad \alpha_{3,y} = \alpha_{3,x}$$

22 Table IX shows in the second, third, and fourth columns the values of M_y, σ_y and $\alpha_{3,y}$ which are obtained by various groupings of the data of Table VII. The grouping employed in Table VIII is listed as $D(3\ 2)$ in Table IX, the 3 denoting the number of different variates in each group, and the 2 designating the position of the first observed variate (i. e. $\nu = 15$) in the first grouping. Thus the classes of the grouping symbolized by $D\ (6;4)$ would be

12-17
18-23
24-29
etc.

From Table IX it may be observed that, although all of the values of M_y agree to a rather remarkable extent, nevertheless the unadjusted values of σ_y reveal the fact that an increase in the class interval is as a rule accompanied by an increase in the associated standard deviation and a decrease in the corresponding skewness.

23 In computing the moments $\mu'_{1,x}$, μ'_{2x}, and μ'_{3x} for distributions of grouped frequencies, the assumption is made that each variate in a class may be treated as being numerically equal to the class mark. A mathematical investigation that lies beyond the scope of an elementary course shows that in the computation of M_x and μ_{3x} it is entirely legitimate to treat each variate after this manner, but the demonstration also reveals that grouping tends to introduce a systematic error into the value of μ_{2x}. To eliminate this systematic tendency we find that one should introduce a correction and write

$$(23) \qquad \mu_{2x} = \mu'_{2x} - M_x - \frac{1 - 1/k^2}{12}$$

where k denotes the *number of different variates* that are grouped together in each class Thus, in Table VIII we should have introduced as a correction

$$\frac{3^2 - 1}{12 \cdot 3^2} = \frac{2}{27} = .074074$$

TABLE IX

Comparison of Adjusted and Unadjusted Values of σ_v and α_{3v}

(1) Grouping	(2) M_v	(3) Unadjusted	(4)	(5) Adjusted	(6)
		σ_v	α_{3v}	σ_v	α_{3v}
$D(1:1)$	29.973	4.969	.108	4.969	.108
$D(2:1)$	29.972	4.992	.106	4.967	.108
$D(2:2)$	29.974	4.995	.107	4.970	.108
Avg. $D(2)$	29.973	4.935	.106	4.968	.108
$D(3:1)$	29.974	5.030	.109	4.963	.113
$D(3:2)$	29.973	5.035	.102	4.968	.106
$D(3:3)$	29.972	5.041	.101	4.974	.105
Avg. $D(3)$	29.973	5.035	.104	4.968	.108
$D(4:1)$	29.968	5.089	.096	4.964	.103
$D(4:2)$	29.976	5.094	.104	4.970	.112
$D(4:3)$	29.976	5.078	.104	4.970	.112
$D(4:4)$	29.972	5.096	.098	4.970	.105
Avg. $D(4)$	29.973	5.089	.100	4.968	.108
$D(5:1)$	29.975	5.160	.105	4.962	.118
$D(5:2)$	29.975	5.170	.097	4.972	.109
$D(5:3)$	29.970	5.167	.094	4.970	.105
$D(5:4)$	29.970	5.163	.085	4.966	.096
$D(5:5)$	29.975	5.170	.100	4.972	.112
Avg. $D(5)$	29.973	5.166	.096	4.968	.108
$D(6:1)$	29.974	5.247	.107	4.961	.126
$D(6:2)$	29.976	5.256	.090	4.971	.117
$D(6:3)$	29.972	5.259	.087	4.971	.102
$D(6:4)$	29.974	5.250	.085	4.965	.109
$D(6:5)$	29.970	5.251	.080	4.966	.094
$D(6:6)$	29.972	5.259	.091	4.971	.108
Avg. $D(6)$	29.973	5.254	[illegible]	4.968	.108
$D(7:1)$	29.977	5.347	[illegible]	[illegible]	.121
$D(7:2)$	29.971	5.347	[illegible]	4.958	.112
$D(7:3)$	29.972	[illegible]	.087	[illegible]	.104
$D(7:4)$	29.966	[illegible]	.070	[illegible]	[illegible]
$D(7:5)$	29.971	[illegible]	[illegible]	[illegible]	.110
$D(7:6)$	29.975	[illegible]	[illegible]	[illegible]	.108
$D(7:7)$	29.976	[illegible]	[illegible]	[illegible]	[illegible]
Avg. $D(7)$	29.973	[illegible]	[illegible]	[illegible]	[illegible]

This would have resulted in the following revision:

$$\begin{array}{ll} \mu_{2x} = 2.74285 & \sigma_x = 1.65616 \\ \mu_{3x} = .481058 & \sigma_x \mu_{1x} = 4.54260 \end{array}$$

$$\alpha_{3x} = .105899$$

$$M_v = 29.973, \qquad \sigma_v = 4.96848, \qquad \alpha_{3v} = .105899$$

Again, for $k = 7$ we would use

$$\mu_{2x} = \mu'_{2x} - M_x^2 - \frac{7^2 - 1}{12 \cdot 7^2} = \mu'_{2x} - M_x^2 - \frac{4}{49}$$

When the simple adjustment of formula (21) is made, Table IX shows that the *systematic* errors in the values of σ_v and α_{3v}, caused by grouping, are eliminated. Thus in columns 5 and 6 the averages for each group are constant, consequently the errors remaining are accidental variations, which, due to a complete lack of compensation, still remain, but such discrepancies are not serious.

It should be noted that for distributions of discrete variates in which no grouping occurs, as in Table VII, the correction vanishes, since for $k = 1$

$$\frac{1 - 1/k^2}{12} = 0 \tag{24}$$

24. *Frequency Distributions of Continuous Variates.* The following will serve as an illustration of the method of obtaining the fundamental functions for a distribution of continuous variates.

TABLE X

Weights of 1000 Female Students

(Original Measurements Made to Nearest 1/10 lb.)

Class (Pounds)	Class Mark $\lambda = 10$	f	$M_o = 114.95$ x
70- 79.9	74.95	2	–4
80- 89.9	84.95	16	–3
90- 99.9	94.95	82	–2
100-109.9	104.95	231	–1
110-119.9	114.95	248	0
120-129.9	124.95	196	1
130-139.9	134.95	122	2
140-149.9	144.95	63	3
150-159.9	154.95	23	4
160-169.9	164.95	5	5
170-179.9	174.95	7	6
180-189.9	184.95	1	7
190-199.9	194.95	2	8
200-209.9	204.95	1	9
210-219.9	214.95	1	10
Total		1000	

$\sum f = 1000$ $\qquad M_o = 114.95$

$\sum xf = 379$ $\qquad M_x = .379$ class units

$\sum x^2f = 3089$ $\qquad \mu'_{2x} = 3.089$

$\sum x^3f = 8131$ $\qquad \mu'_{3x} = 8.131$

$\mu_{2x} = 2.86203$ $\qquad \sigma_x = 1.69175$

$\mu_{3.x} = 4.72769$ $\qquad \sigma_x \mu_{2x} = 4.84184$

$\alpha_{3x} = .976424$

$M_y = 118.74$ lbs., $\sigma_y = 16.9175^+$ lbs., $\alpha_{3.y} = .976424$

Explanation: The class mark has previously been defined as the mean of the greatest and least variates that can be included in a class Since the original measurements were made to the nearest tenth of a pound, the *true limits* of the 150-159.9 class are 149.95-159.95, and their mean is 154.95, which accordingly is the class mark in this instance. If the original measurements had been made to the *nearest pound*, then the classes would be written

150.0-159.0
160.0-169.0

and the true limits of the 150.0-159.0 class would be 149.5 and 159.5 pounds respectively, and the corresponding class mark would be 144.5 lbs. It is apparent, therefore, that a table of continuous variates should specify clearly the accuracy with which the original measurements were made, for the values of the class marks and consequently that of the mean, hinges on this point.

It will be noticed that in this example the class interval has again been taken as the unit of x, and this fact must be taken into consideration in determining the value of M_v and σ_v.

Since the assumption is also made that the class mark may represent the magnitudes of all variates occurring in that class, the question of correcting the second moment, μ_{2x} again arises. Since in each class of a distribution of continuous variates an infinite number of different variates may occur, the correction is in this case

$$\frac{1 - 1/k^2}{12} = \frac{1}{12}$$

Therefore, corresponding to formula (24), we must write, in order properly to adjust the second moment of a distribution of continuous variates

(25) $$\mu_{2x} = \mu'_{2x} - M_x^2 - \frac{1}{12}$$

As before, neither the values of M_x nor μ_{3x} require adjustment.

Summary of Section III. The frequency distribution is a device for presenting an extensive series of variates in a systematic and compact form. Not only are the phenomena of aggregation more readily perceptible by this method of presenting the data, but the calculations of the fundamental functions are facilitated.

The formulae for obtaining the mean, standard deviation and skewness are, with the exception of a single adjustment that may arise, identical with those employed in the serial method One need only observe that

$$N = \sum f$$
$$\sum x = \sum xf$$
$$\sum x^2 = \sum x^2 f$$
$$\sum x^3 = \sum x^3 f$$

The adjustment referred to is that we should in general regard

$$\mu_{2x} = \mu'_{2x} - M_x^2 - \frac{1-1/k^2}{12}$$

For ungrouped distributions of discrete variates this correction vanishes, since in this instance $k = 1$. For distributions of continuous variates, since here k would equal infinity, the correction is numerically equal to 1/12.

These corrections will remove systematic errors in the standard deviation and skewness that arise from the phenomenon of grouping complete frequency distributions.

Editor's Note. This abstract of Elementary Mathematical Statistics will be continued in the May issue of the ANNALS.

BAYES' THEOREM[1]

By

JOSEPH BERKSON

As for all established sciences, the typical problems of practical statistics have become inveterately attached to their several neat and convenient formulary solutions. To recall consideration of the basic reasoning underlying every-day statistical practice that applies to an elementary question may appear in the nature of an unnecessary disturbance of prevailing peace. If the experience of the writer is typical, however, vagueness or dubiousness of the premises inherent in a rule applied by rote will emerge to plague one in the conclusions, and a periodic return to fundamentals is as salutary for mental comfort as for the integrity of science itself' In what follows, an attempt will be made to go over the ground covered by Bayes' Theorem, and to point out its import for sound statistical reasoning. No claim is laid to mathematical originality at any specific points, but in the approach and synthesis will be found, we hope, a measure of instructive novelty.

A large class of statistical problems is typified in the following. A standard machine is known, from long experience, to produce a certain fraction P of imperfect products. What is the probability that in the next issue of n products, a fraction p will be imperfect?

We now present a related but not identical question. There is no available knowledge concerning the general practice of a machine; n products are examined and a fraction p found to be imperfect. What is the probability that the machine turns out generally a fraction P of imperfect products? The distinction between the two questions may be schematized as in Figure 1.

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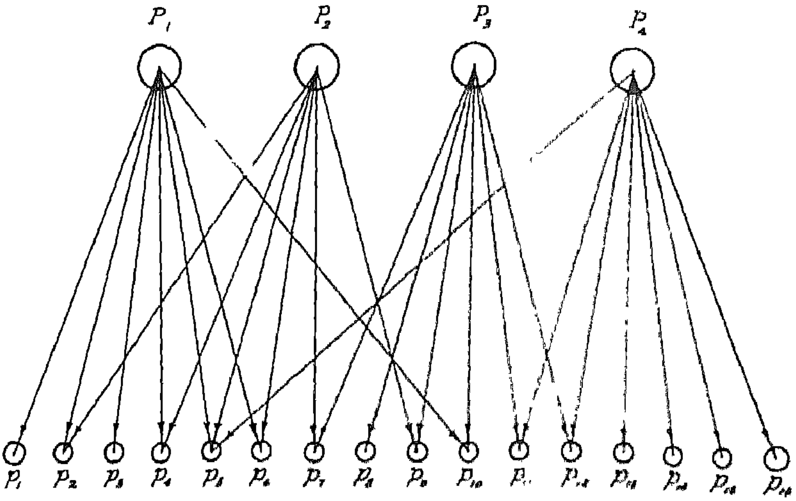
1. From the Department of Biometry and Vital Statistics of the School of Hygiene and Public Health (Paper No. 125), and the Institute for Biological Research of the Johns Hopkins University.



The values P_1, P_2, P_3, P_4 represent serially all the various fractions of imperfect products which might characterize particular machines, each one, let us say, determined by some definite combination of mechanical defects. Values p_1, p_2, p_3, etc., are the fluctuating fractions of imperfect products that might appear in the samples produced by these machines. Connected by arrows with P_1 are the randomly varying values of p that might result from P_1, with P_2 those that might result from P_2, etc., the weight of the arrows being proportional to the probability of the particular p concerned. It is to be noticed that each P may give rise to any of a number of p's and that some of the p's may result from any of a number of P's.

The first question in terms of the diagram is: "Given P_1, how probable is it that p_5 shall result?" The second is: "Given p_5, how probable is it that P_1 has been its source?" Answering the first, we calculate in the realm of the p's connected with P_1. In the second we calculate in the realm of the P's connected with p_5.

An answer to the first is given directly in terms of our every day statistical reasoning. We say that the p's which result from P_1 can be adequately described as a normal distribution with $\sigma = \sqrt{\frac{P(1-P)}{n}}$, and from this the probability of any particular p calculated. The answer to the second is more difficult, and was given in general terms first by Bayes (1) in the theorem known by his name. Bayes' theorem is not frequently used in applied statistics, yet the problems that arise in practical situations would often seem to demand just such an answer as it provides More often than not do we have a specific sample and inquire about the probable character of the universe from which it was drawn, in contra-distinction to the situation in which the universe is known, and the questions concern the possible samples.

The method of presenting the theorem here given will not follow rigidly any historical demonstration. Actually the calculation quantitatively of an "inverse probability" or the "probability of causes," was first given by Bayes But he considered a purely geometric set-up and his solution was in terms of this conception. By implication he utilized a general principle first clearly stated later by Laplace, and furthermore, Laplace generalized the solution still more by arguing from the probability of a cause given by a particular sample, to the probability of the next sample. With this realized, then, that Bayes is to be credited with the original demonstration and Laplace for an important extension, we may proceed to a demonstration which is not exactly that of either

I. *Problem* We have an urn containing three balls. Each ball is colored black or white, and each color is equally likely. We draw one ball and it is black What are the probable contents of the urn? We argue—the following are the possibilities·

I	II	III	IV
w w w	w w b	w b b	b b b

All of these possibilities, we say, are equally likely *a priori* and we have for the probabilities of the sample the following:

P_s I, the probability of a black sample from I = 0
P_s II, the probability of a black sample from II = 1/3
P_s III, the probability of a black sample from III = 2/3
P_s IV, the probability of a black sample from IV = 3/3

where P_s I is the probability of the sample s being drawn from urn I, P_s II from urn II, etc. We say now that the relative probabilities of the various urns are in proportion to the probabilities of the sample drawn, and we have

$$(a) \qquad P\,\mathrm{I}\ \ P\,\mathrm{II}:P\,\mathrm{III}:P\,\mathrm{IV} = 0\ \ 1/3:2/3:3/3$$

where P I is the probability that, having drawn the ball, urn I was its source, P II that urn II was the source, etc

Also, since the ball must have been drawn from some one of the urns, the total probability of one or another of the urns is unity and we have

(b) $P\text{ I} + P\text{ II} + P\text{ III} + P\text{ IV} = 1$

From (a) and (b) we have therefore

$$\begin{aligned} P\text{ I} &= 0 \\ P\text{ II} &= 1/6 \\ P\text{ III} &= 2/6 \\ P\text{ IV} &= 3/6 \end{aligned}$$

We now extend the problem to the case where the *a priori* probabilities of the various possible urns are not equal

Suppose we say that there are many urns of the description I, II, III, IV in a large chamber, and that these are in proportion I : II . III · IV = 1 . 2 : 3 : 4. We now pick an urn at random and draw from it a ball, which turns out to be black. What is the probability that the urn is of some particular description? Proceeding as before, we have for the probabilities of the sample being drawn from the various urns the following:

p_s I $= 1/10 \times 0 = 0$ (Probability of urn × probability of sample)

p_s II $= 2/10 \times 1/3 = 2/30$

p_s III $= 3/10 \times 2/3 = 6/30$

p_s IV $= 4/10 \times 3/3 = 12/30$

where p_s I is the probability that such a sample s be drawn from urn I, etc.

And again on the principle that the probabilities of the urns are in proportion to the probabilities of the sample drawn, we have

$$P\text{ I} : P\text{ II} : P\text{ III} : P\text{ IV} = 0 : 2/30 : 6/30 \quad 12/30$$

and as preceding

$$P\,\mathrm{I} + P\,\mathrm{II} + P\,\mathrm{III} + P\,\mathrm{IV} = 1$$

Therefore

$$\begin{aligned} P\,\mathrm{I} &= 0 \\ P\,\mathrm{II} &= 2/20 \\ P\,\mathrm{III} &= 6/20 \\ P\,\mathrm{IV} &= 12/20 \end{aligned}$$

We shall now generalize this solution.

Let π_1, π_2, π_3, etc. be the *a priori* probabilities of the various possible universes from which a sample is to be drawn. Let p_1, p_2, p_3, etc., be the probability of the sample being drawn from the respective universes. Then, a sample s having been drawn, the probability that its source is universe r is given by

$$P_r = \frac{\pi_r p_r}{\sum \pi p}$$

If all the universes are equally likely (our first case above), $\pi_1 = \pi_2 = \pi_3 = \pi_4$ · and we have

$$(1) \qquad P_r = \frac{p_r}{\sum p}$$

If the equally probable universes are infinite in number, the P's varying by infinitesimal gradations from zero to unity, and p may assume any positive value less than 1, we may extend the last formula (1) by use of the calculus as follows:

Let x = any possible P between 0 and 1. From a universe x I draw a sample containing $r + s$ individuals, designated hereafter as a sample (r, s). The probability that it will contain r successes and s failures is given by

$$P_{(r,s)} = E_{r,s}\, x^r (1-x)^s$$

where $P_{(r,s)}$ is the probability that the sample (r, s) coefficient of the $(r+1)$th term in the Bernoulli expansion $= \frac{(r+s)!}{r!\,s!}$.

The probability of the sample of (r, s) coming from a universe the P of which lies between x and $(x + dx)$ is therefore

$$_{x}^{x+dx}P_{(r,s)} = E_{r,s}\, x^{r}(1-x)^{s}dx$$

where $_{x}^{x+dx}P_{(r,s)}$ is the probability that the sample (r, s) emanates from a universe whose P lies between x and $(x + dx)$. If the universe from which the sample is drawn may have a P anywhere between a and b, the probability of the sample (r, s) is

$$(2) \qquad _{a}^{b}P_{(r,s)} = E_{r,s} \int_{a}^{b} x^{r}(1-x)^{s}dx$$

and the probability that x is between a and b is therefore as in (1)

$$(3) \qquad _{a}^{b}P = \frac{\int_{a}^{b} x^{r}(1-x)^{s}dx}{\int_{0}^{1} x^{r}(1-x)^{s}dx}$$

where $_{a}^{b}P$ is the probability that the universe from which the sample (r, s) was drawn has a P between a and b. This is Bayes' Theorem in terms of the integral calculus.

Now, we ask the further question, what is the probability of a second sample containing m successes and n failures[1] being drawn?

If x be the p of the universe from which the sample (m, n) is drawn, and if P may vary from 0 to 1 we have analogously with (2)

$$(4) \qquad _{0}^{1}P_{(m,n)} = E_{m,n} \int_{0}^{1} x^{m}(1-x)^{n}dx$$

where $_{0}^{1}P_{(m,n)}$ is the probability that a sample (m, n) be drawn from universes whose P's vary between 0 and 1, and

$$E_{m,n} = \frac{(m+n)!}{m!\,n!}$$

---

[1] Designated hereafter as the sample (m, n).

The probability of the event (m, n) occurring from any particular universe is given by the product of the probability of that universe and the probability of the event The total probability of the event (m, n), i. e, the probability that the event (m, n) occurs at all from any universe, is, therefore, given by the product of form (3) with 0 and 1 substituted for a and b and (4), as follows.

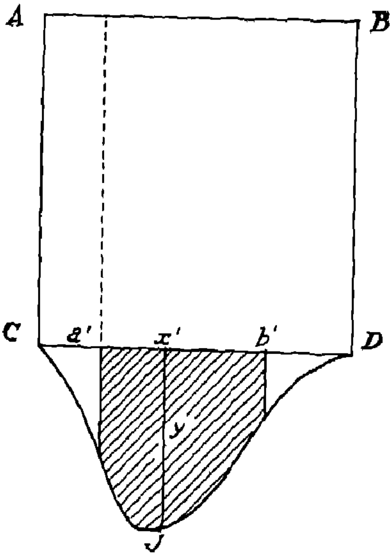
(5)
$$P_{(m,n)\,(r,s)} = \frac{(m+n)!}{n!\,m!} \, \frac{\int_0^1 x^{r+m}(1-x)^{s+n}\,dx}{\int_0^1 x^{r}(1-x)^{s}\,dx}$$

where $P_{(m,n)\,(r,s)}$ is the probability of a second sample (m, n) after a first sample (r, s) has been drawn.

This is Laplace's extension of Bayes' Theorem, somewhat modified.

Bayes' Solution.

It will be illuminating to derive this result by the method of Bayes. We shall follow his proof except to simplify his notation and to use the integral calculus where he used geometric demonstration.



ABCD is a square billiard table. A ball is thrown and comes to rest at a', through which a line is drawn parallel to AC. A second ball is thrown; if it stops to the left side of the line a', we designate a success, to the right, a failure. Before the first ball is thrown, what is the probability of the second ball succeeding r and failing s times in r plus s trials?

If the first ball comes to rest at x', the probability of a successful second throw is $\frac{Cx'}{CD} = p$ and of failure $\frac{Dx'}{CD} = q$. The probability of r successes and s failures with the first ball at x is then $\frac{(r+s)!}{r!s!} p^r q^s$.

Let us erect at each point x' along CD a distance y', so that

$$\frac{y'}{CD} = \frac{(r+s)!}{r!s!} p^r q^s \tag{6}$$

and connect the summits forming a figure as shown in Figure 2. At each point, of course, y' will be different because $p = \frac{Cx'}{CD}$, and $q = \frac{Dx'}{CD}$ will be different, but for any particular case, r and s remain constant

The probability that the first ball shall fall between a and $(a+dx)$ is $\frac{dx'}{AD}$ and that the second ball shall therefore succeed r and fail s times is $\frac{y'}{CD}$. That both shall happen is therefore

$$\frac{y'}{CD} \times \frac{dx'}{CD}$$

and if x is to be between a' and b', the total probability is

$${}_{a'}^{b'}P_{(r,s)} = \frac{1}{CD^2} \int_{a'}^{b'} y'dx'$$

where ${}_{a'}^{b}P_{(r,s)}$ is the probability that the first ball fall between a and b' and that a ball thrown subsequently $r+s$ times, succeed r and fail s times.

But CD^2 = Area of AD and $\int_{a'}^{b'} y'dx'$ = Area of the shaded portion, $a'Jb'$. Therefore

(7) $$ {}_{a'}^{b'}P_{(r,s)} = \frac{Area\ a'Jb'}{Area\ AD} $$

The probability that the first ball fall between C and D and thereafter there occur r successes and s failures is similarly $\frac{Area\ CJD}{Area\ AD}$. But the first ball must fall somewhere between C and D; therefore the total probability of the second throws having r successes and s failures is given by

(8) $$ P_{(r,s)} = \frac{Area\ CJD}{Area\ AD} $$

With this established, the analysis proceeds.

Given the result of a series of throws to be r successes and s failures, what is the probability that the first ball has fallen between a' and b'? This we may obtain by the use of the solution already derived and the principle of compound probability[1].

Let x be the desired probability that the first ball fell between a' and b' We have seen that the probability of r successes and s failures in the second series of throws is

$$ \frac{Area\ CJD}{Area\ AD} $$

from (8)

therefore the probability of the first falling between a' and b' and the experience (r, s) following is

$$ x \cdot \frac{Area\ CJD}{Area\ AD} $$

But we have shown that this combined probability is equal to

$$ \frac{Area\ a'Jb'}{Area\ AD} $$

from (7)

Therefore

(9) $$ x = \frac{Area\ a'Jb'}{Area\ CJD} $$

---

[1] This step is very elaborately proved in Bayes' original paper by a circuitous demonstration.

This is Bayes' Theorem, as its author gives it. The additional part of his work is concerned with the quantitative estimate of the ratio.

We may now show that his solution is the same as that given in (3), as follows:

(10) $$y' = CD \times E_{r,s}\left(\frac{x'}{CD}\right)^{r}\left(1-\frac{x'}{CD}\right)^{s}$$ from (6)

where

x' = distance from C to x'

$$E_{r,s} = \frac{(r+s)!}{r!\,s!}$$

Now

$$a' = a \times CD$$
$$b' = b \times CD$$

a and b having the meaning of equation (3) Assume the relationship

(11) $$x' = CD \times x$$

(12) $$dx' = CD \times dx$$

Then

$$\text{Area } aJb = \int_{x'=a'}^{x'=b'} y'\,dx'$$

$$= CD^{2} \times E_{r,s} \int_{x=a}^{x=b} x^{r}(1-x)^{s}\,dx$$

(Substituting from (11) and (12)).

Similarly

$$\text{Area } CJD = CD^{2} \times E_{r,s} \int^{x=1} x^{r}(1-x)^{s}\,dx$$

Therefore

$$\text{Area } \frac{aJb}{CJD} = \frac{\int_a^b x^r(1-x)^s\,dx}{\int_0^1 x^r(1-x)^s\,dx}$$

which is the same as formula (3) previously derived

To be directly applicable to statistical problems formula (5) must be numerically evaluated This is accomplished exactly for most practical instances only with a great amount of labor, and methods of approximation have been resorted to For a few simple special cases the solution may be easily derived as follows

An event has been tried N times with p successes and q failures. What is the probability that in the next single trial it will succeed?

Applying formula (5) to this instance, we have

$$r = p \qquad m = 1$$
$$s = q \qquad n = 0$$

and the desired probability is given by

$$P = \frac{\int_0^1 x^{p+1}(1-x)^q\,dx}{\int_0^1 x^p(1-x)^q\,dx}$$

Now

$$\int_0^1 x^a(1-x)^b\,dx = \frac{a!\,b!}{(a+b+1)!}$$

From which we have

$$P = \frac{m+1}{m+n+2} = \frac{m+1}{N+2}$$

So that if nothing is known concerning an event except that it has been tried three times and succeeded twice, the probability that it will succeed in the next trial is 3/5, not 2/3 as the more usual procedure would indicate. Again, if an event has occurred a thousand times without a failure, and we know concerning it nothing except that fact, the probability that it will fail next instance is 1/1002. If an event has never been tried at all, the probability that it will succeed on the first trial is 1/2

An event has been tried N times and succeeded each instance What is the probability that in the next d trials it will again succeed each time? Here

$$r = N \qquad m = d$$

$$s = 0 \qquad n = 0$$

and the desired probability is given by

$$P = \frac{\int_0^1 x^{N+d}dx}{\int_0^1 x^N dx}$$

$$= \frac{N+1}{N+d+1}$$

From this we conclude that if an event has succeeded 25 times and never failed, the probability that in 25 further trials it will again not fail even once is 26/51, or in general if an event has never failed in N trials, the probability that N further trials will yield no failure is about 1/2.

Discussion.

To precisely what position in the methodology of applied statistics Bayes' Theorem will eventually become adjusted, it is impossible at this point in its development to say with certainty. The literature on the subject, as soon as it leaves the realm of purely hypothetical situations, is rife with disagreement, and clarification remains a contemporary problem. In this brief presentation, no attempt can be made to adequately summarize the various views concerning the questions at issue. We may, however, consider a few points that have disciplinary value for statistical thinking rather than any immediate practical utility.

It is basic to the aims of statistical calculations to estimate the probability of given experiences from assumptions of pure random variation. A consideration of the logic involved in the development of Bayes' Theorem is useful in bringing out the inadequacy of the reasoning by which our most ordinary statistical procedures attempt to accomplish this. If, having observed a probability p , we estimate the standard deviation of succeeding samples of n by $\sqrt{\frac{pq}{n}}$, we imply tacitly that in the universe from which the sample was drawn, the chance of a success is the p of our observation The reasoning leading to, and formula (3) itself, indicate how unwarranted this is Our knowledge of the universe which generated the sample is never given with certainty by the sample. Indeed, formula (3) states a probability for any particular universe that may be assumed. With only a sample as the source of knowledge, and without Bayes' Theorem, we have no clue as to the nature of the generating universe. But, if we do not know the universe, how are we to calculate the character of its samples? One answer is to take refuge in formula (5), i. e. use Bayes' Theorem. As a practical solution of the difficulty this has two major objections. first, there are no existing tables for making the necessary calculations without prohibitive arithmetic labor; second, even if the evaluation could be effected there are reasons to doubt the validity of its application. For the formula in question rests on the assumption that all the probabilities from zero to unity which might characterize the universes from which we draw samples are *a priori* equally likely, the socalled assumption of the equal distribution of ignorance. Now this is an exceedingly questionable assumption, and it is partly on these grounds that Keynes rejects outright the possibility of applying probability to actual experience It must be admitted, we think, that it is difficult to see what there is to justify the assumption that every sort of general universe from which arise the events of experience is equally likely. Would it not appear the more reasonable hypothesis that these universes are themselves "events," samples of some larger universe, and why should this be extremely different in the distribution of its probabilities from the universes that we ordinarily meet? There are writers, however, who, admitting that the assumption is to be questioned, believe it may be subjected to experimental test, and have essayed to actually sample at random the probabilities that characterize the universes of our experience It would be impertinent to assert that an experimental investigation is bound to be futile, but the utility of this sort of procedure seems to us exceedingly dubious. We doubt indeed that any clear meaning can be assigned to the concept of "the universes of our experience," of which random samples are to be obtained. But granting the existence of such a distribution of *a priori* probabilities we doubt the relevancy of its estimation to any practical problem In any actual investigation, we deal with a definite slice of possible experience; an anthropologist is not concerned with the universes dealt with in the investigation of an economist or an epidemiologist. If *a priori* probabilities are of interest to him, they are those that obtain in his peculiar world of observation It appears to us quite as wide of the mark aimed at, to call in a formula which obtains its *a priori* probability from experience in general, as to obtain it from the unique experience at hand, and indeed it may be argued that, as between the two, the latter is the more reasonable.

What then does all this come to? Does it mean that the entire structure of established statistical procedure rests on quicksand, to be toppled over by anyone armed with a reading of Bayes' Theorem? We are inclined to the belief held by Keynes that, so far as *logic* is concerned, this is substantially true As regards this, however, it is at bottom in no worse plight than any current scientific procedure when its fundamental assumptions are hard pressed. But we do not rest the matter here. All this admits is that applied statistics, like all applied science, is not founded on unquestionable premises and invulnerable logic. It is perfectly consistent to add that *in general* its formulae are good *approximations*. How good? This is a question permitting no dogmatic comprehensive answer. Differently good for different situations. Some idea of the degree of approximation may be obtained for given assumed conditions by direct calculation. It may be shown, for instance, that under certain conditions results obtained by way of Bayes' Theorem or the more usual "normal" distribution render not very different results, and these conditions, indeed, approach the ones we most frequently encounter. But, in general, a more satisfactory answer is furnished in the pragmatic consideration that our formulae have in fact been widely used and experience has not violated their anticipations This is the fact that we would stress, because it throws into relief the experimental as opposed to the mathematical foundation of statistics. Comforted on the one hand that experience in general supports our procedures, the considerations we have elicited in this discussion will emphasize equally their shifting approximation. The clear minded and careful worker will keep this constantly in mind and shun literal interpretation of conclusions drawn from formulae applied to extreme cases. No scientist worth his salt will permit himself the use of formulae the premises of which he has not examined But the statistician, because of the great variability of the data with which he is likely to deal, stands in special need of this precaution. Where statistics run counter to what appears to be the general experience, it is a wise rule to re-examine the statistics rather than to indict forthwith the dependability of the experience Such an attitude would modify considerably much that is found in current statistical literature and it would modify it in the direction of greater soundness.

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A MATHEMATICAL THEORY OF SEASONALS

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The graph of any time series may be assumed to be a compound curve which is dependent upon the following factors:

Secular trend,	$f(x)$
Cycle,	$c(x)$
Seasonal	$s(x)$, and
Residual errors,	ϵ_x

If we designate the xth term of the observed time series by ${}_oy_x$, we have that

$$(1) \qquad {}_oy_x = f(x) \cdot c(x) \cdot s(x) + \epsilon_x$$

It also follows that the standard error, based on our hypothesis, is

$$(2) \qquad \sigma_\epsilon = \sqrt{\frac{\sum \epsilon_x^2}{N}}$$

In making predictions, we desire that the standard error of estimate be a minimum, and this requires that $\sum \epsilon^2$ be also a minimum.

In dealing with data covering a period of years, i. e. $12n$ months, we observe that

$$\sum \epsilon^2 = [{}_0y_1 - f(1)\cdot c(1)\cdot s(1)]^2$$

$$+ [{}_0y_2 - f(2)\cdot c(2)\cdot s(2)]^2$$

$$\text{-----------------------}$$

$$+ [{}_0y_{12} - f(12)\cdot c(12)\cdot s(12)]^2$$

$$+ [{}_0y_{13} - f(13)\cdot c(13)\cdot s(1)]^2$$

$$+ [{}_0y_{14} - f(14)\cdot c(14)\cdot s(2)]^2$$

$$\text{-----------------------}$$

$$+ [{}_0y_{12n-11} - f(12n-11)\cdot c(12n-11)\cdot s(1)]$$

$$+ [{}_0y_{12n-10} - f(12n-10)\cdot c(12n-10)\cdot s(2)]$$

$$\text{-----------------------}$$

$$+ [{}_0y_{12n} - f(12n)\cdot c(12n)\cdot s(12)]$$

Let us now find the values of $s(1)$, $s(2)$ . . . $s(12)$ that will minimize the standard error of estimate. Placing the partial derivative of $\sum \epsilon^2$ with respect to $s(1)$ equal to zero, yields

$$\frac{\partial}{\partial s1} = 2[{}_0y_1 - f(1)\cdot c(1)\cdot s(1)][-f(1)\cdot c(1)]$$

$$+ 2[{}_0y_{13} - f(13)\cdot c(13)\cdot s(1)][-f(13)\cdot c(13)]$$

$$\text{-----------------------}$$

$$+ 2[{}_0y_{12n-11} - f(12n-11)\cdot c(12n-11)\cdot s(1)][-f(12n-11)\cdot c(12n-11)] = 0$$

Solving

$$s(1) = \frac{\sum^{(1)} {}_0y_x\, f(x)\cdot c(x)}{\sum^{(1)} f^2(x)\cdot c^2(x)}$$

where we understand that $\sum^{(1)}{}_0 y_x \cdot f(x)\ c(x)$ means the sum of the products of ${}_0 y_x$, $f(x)$ and $c(x)$ taken from the *first* month of each year, and similarly for $\sum^{(1)} f^2(x)\cdot c^2(x)$

The partial derivative with respect to $s(2)$ yiel

$$s(2)=\frac{\sum^{(2)}{}_0 y_x \cdot f(x)\cdot c(x)}{\sum^{(2)} f^2(x)\cdot c^2(x)}$$

and in fact

$$(3) \qquad s(i)=\frac{\sum^{(i)}{}_0 y_x \cdot f(x)\cdot c(x)}{\sum^{(i)} f^2(x)\cdot c^2(x)}$$

Thus the seasonal for July is a function only of the various July values of the observed series, the secular trend and the cycle factors.

Since both $f(x)$ and $c(x)$ are smooth functions, it follows that their product, which we shall designate by $\psi(x)$, represents a smooth function which is merely that part of the time series which would remain if the accidental and seasonal fluctuations were eliminated. The formula for the seasonal index for the i th month may therefore be written

$$(4) \qquad s(i)=\frac{\sum^{(i)}{}_0 y_x \cdot \psi(x)}{\sum^{(i)} \psi^2(x)}$$

At this point we may recall the fact that in fitting a curve of the type $y=k\psi(x)$ to observed data by the *Method of Least Squares,*

$$k=\frac{\sum{}_0 y_x \cdot \psi(x)}{\sum \psi^2(x)}$$

whereas if the *Method of Moments* be employed

$$k=\frac{\sum{}_0 y_x}{\sum \psi(x)}$$

Experience in various statistical applications demonstrates that the two methods yield approximately the same results. Borrowing from this experience, we shall choose the simpler form and write instead of formula (4)

$$s(i) = \frac{\sum^{(i)} y}{\sum^{(i)} \psi(x)} \tag{5}$$

So far as theoretical considerations are concerned, (4) is [illegible] superior to (5), but the fact that the latter formula enables us to obtain seasonals by a method far simpler than would result by using formula (4), requires that we choose (5) in preference to (4) Ordinarily the difference in results obtained by using both formulae is less than one-half of one per cent.

Verbally, formula (5) states merely that *the seasonal index for any month is the ratio of the total of the variates for the month in question to the total that would have been experienced if neither accidental nor seasonal influences were present*

We now are forced to find a simple method of obtaining values of $\sum^{(i)} \psi(x)$.

Let T_{i-3}, T_{i-2}, T_{i-1}, T_i, T_{i+1}, T_{i+2}, and T_{i+3} denote the total production for seven consecutive years. If we assume that the effect of both seasonal influences and accidental or residual fluctuations is to shift the production from one month to another, but nevertheless *to leave the total production for each year practically unchanged*, then a smooth curve passing over the seven year period, and preserving the annual totals, may be assumed to afford a representation of $\psi(x)$. We, therefore, determine the equation of a parabola of the sixth degree in such a manner that the areas under this curve for seven equidistant unit intervals are equal respectively to $T_{i-3}, T_{i-2}, \cdots T_i, \cdots T_{i+2}, T_{i+3}$ Fitting six degree parabolae to successive seven year intervals it is possible to deal with a time series of any length

By adding together the interpolated values for all the January values of $\psi(x)$, and similarly for the other months, we can show that

$$\sum^{(i)} \psi(x) = c_{1,i} T_1 + c_{2,i} T_2 + c_{3,i} T_3 + c_{4,i} \left[T_4 + T_5 + \cdots T_{n-3} \right] + c_{5,i} T_{n-2} + c_{6,i} T_{n-1} + c_{7,i} T_n \tag{6}$$

where the values of the coefficients are as given in Table I.

In order to compare the efficiency of this method with another method of computing seasonals, it is necessary that each formula be tried out on some series for which the true values of the seasonal indices are known. We know in advance, of course, that there exist many satisfactory methods of obtaining seasonals, but we also desire to know something about the amount of time that each method requires as well as their relative accuracy.

The theoretical series, on which we shall try out two methods of computing seasonals, is built up from data taken from an article, "Statistical Analysis and Projection of Time Series," written and published by the statistical division of the American Telephone and Telegraph Company. After eliminating from the *Production of Pig Iron* series both trend and seasonal influences, the factors of Table II remained. We shall consider these, therefore, as the combination of "Cycle and residual" factors.

Although smoothing this data by a proper mathematical formula would eliminate the residual errors, nevertheless such procedure would introduce a bias in favor of the formula for computing seasonals that is proposed in this paper. The reason for this bias lies in the fact that most smoothing formulae are developed on the assumption that the smoothed ordinate lies on a parabola of a chosen degree, and since a similar assumption was made in our theory, it is evident that the proposed method will benefit most by employing a parabolic smoothing formula in obtaining the hypothetical cycle series.

For this reason the data of Table II, with additional data for one year on either side, was given to a draftsman with instructions to

(1) Plot the data of Table II

(2) draw free hand a *smooth* curve that to his mind best represented the general run of the data

(3) read off from his curve the approximate value of the smoothed statistics.

The data of Table III resulted.

In essential agreement with the American Telephone and Telegraph article, we shall assume a linear trend, the value for the first month being 1511 and the monthly increment 8. The product of trend by cycle produces the theoretical values of $\psi(x)$ presented in Table IV.

TABLE I

Constants for computing seasonal indices

i	$c_{1,i}$	$c_{2,i}$	$c_{3,i}$	$c_{4,i}$	$c_{5,i}$	$c_{6,i}$	$c_{7,i}$
1	.12530	.07897	.08392	083333	08259	.08959	.03963
2	.11822	.07914	08389	.083333	.08269	.08849	.04757
3	.11094	.07955	.08382	.083333	.08283	.08723	.05563
4	.10345	08018	.08373	.083333	.08299	.08590	.06375
5	.09577	.08104	.08361	.083333	.08315	.08456	.07187
6	.08792	.08208	08347	.083333	08331	.08327	.07995
7	.07995	.08327	.08331	.083333	08347	.08208	.08792
8	.07187	.08456	.08315	.083333	.08361	.08104	.09577
9	.06375	08590	.08299	.083333	.08373	.08018	.10345
10	.05563	08723	.08283	.083333	.08382	.07955	.11094
11	04757	.08849	.08269	.083333	.08389	.07914	.11822
12	.03963	.08959	08259	.083333	.08392	.07897	.12530

TABLE II

Cycle and Residual Series for Pig Iron Production

	1904	1905	1906	1907	1908	1909
January	–37.5	12.4	23.0	24.2	–43.3	– 7.9
February	–13.8	5.8	18.2	20.2	–36.4	– 7.8
March	–10.4	14.2	20.2	17.6	–40.8	–13.7
April	– .7	15.9	18.1	19.7	–42.0	–15.6
May	– 4.2	14.6	17.5	21.9	–43.0	–10.5
June	–15.2	10.4	15.0	23.1	–42.0	– 3.3
July	–27.4	7.0	16.8	23.9	–35.5	4.9
August	–25.3	11.1	9.8	21.6	–30.6	10.1
September	–12.5	15.4	12.7	19.0	–25.8	18.0
October	–12.1	18.6	20.2	21.4	–24.1	22.6
November	– 5.6	22.1	23.8	– 1.6	–19.0	24.2
December	1.1	20.7	24.9	–34.5	–12.1	27.0
	1910	**1911**	**1912**	**1913**	**1914**	**1915**
January	26.7	–17.7	– 8.0	19.5	–22.0	–35.9
February	21.5	–11.0	– 1.0	15.7	–16.7	–27.8
March	19.5	– 5.5	– .1	10.6	–10.5	–25.0
April	15.8	– 8.2	1.4	13.0	–10.8	–20.1
May	9.4	–17.9	5.4	13.9	–19.7	–16.3
June	8.2	–17.9	7.0	10.6	–21.9	– 6.9
July	2.5	–17.8	5.5	7.7	–20.3	.0
August	– 1.3	–13.6	8.1	5.1	–20.6	6.5
September	– 2.5	–10.1	7.1	4.7	–23.8	10.5
October	– 6.5	–10.3	11.0	.7	–33.6	15.1
November	–10.7	–10.5	12.8	– 7.8	–39.3	16.1
December	–18.1	– 9.8	17.8	–19.3	–40.6	21.0

TABLE III

Per Cent Cycle Series for Theoretical Distribution

	1904	1905	1906	1907	1908	1909
January	–38.2	6.6	15.5	16.3	–37.5	–12.2
February	–36.7	7.5	15.8	16.2	–38.1	– 7.1
March	–32.4	9.0	16.0	16.0	–39.1	– 2.4
April	–22.4	10.0	16.2	14.6	–39.4	2.1
May	–18.7	11.0	16.4	14.1	–39.4	5.0
June	–13.5	12.0	16.6	12.7	–38.5	10.0
July	– 9.8	13.1	16.6	11.6	–38.0	12.7
August	– 6.4	13.6	16.6	8.0	–36.7	15.6
September	– 3.7	14.1	16.7	5.0	–33.4	17.2
October	.0	14.5	16.6	.0	–29.1	18.4
November	2.3	14.8	16.5	–17.4	–23.8	19.6
December	4.4	15.0	16.5	–35.0	–17.0	20.0
	1910	**1911**	**1912**	**1913**	**1914**	**1915**
January	20.6	–12.3	– 7.2	10.8	–14.0	–32.3
February	20.6	–13.4	– 5.0	10.7	–17.0	–28.1
March	20.3	–13.6	– 3.0	10.6	–20.0	–23.6
April	19.5	–13.7	.0	10.0	–23.0	–18.1
May	17.9	–13.7	2.7	9.3	–24.7	–13.0
June	16.4	–13.6	4.8	8.5	–27.5	– 7.4
July	12.4	–13.4	6.9	7.1	–29.8	– 2.8
August	7.1	–12.8	7.9	6.0	–32.0	3.5
September	0	–12.0	8.5	3.6	–33.5	9.0
October	– 6.6	–11.2	9.5	.0	–35.0	13.8
November	– 8.0	– 9.6	10.0	– 5.0	–36.0	15.7
December	–10.9	– 7.9	10.6	–10.0	–35.0	17.5

TABLE IV

Theoretical Trend and Cycle Series, $\psi(x)$

1904	1905	1906	1907	1908	1909
934	1713	1967	2092	1184	1748
962	1736	1981	2100	1178	1857
1032	1769	1994	2105	1164	1959
1191	1794	2007	2089	1163	2057
1254	1819	2020	2089	1168	2124
1342	1845	2032	2073	1190	2234
1406	1872	2042	2061	1205	2298
1467	1889	2051	2003	1235	2366
1517	1907	2062	1956	1305	2408
1583	1922	2070	1871	1395	2443
1628	1937	2077	1552	1505	2477
1669	1949	2087	1227	1646	2495
1910	**1911**	**1912**	**1913**	**1914**	**1915**
2517	1914	2115	2632	2125	1738
2527	1897	2173	2638	2058	1851
2530	1900	2226	2644	1990	1973
2523	1905	2303	2639	1921	2122
2498	1912	2373	2631	1885	2261
2476	1921	2430	2620	1820	2414
2400	1932	2488	2595	1768	2542
2295	1952	2519	2577	1718	2715
2151	1977	2542	2527	1686	2868
2017	2002	2574	2447	1653	3003
1994	2046	2595	2332	1633	3063
1938	2092	2618	2217	1663	3120

TABLE V

Theoretical Seasonal Factors

January	.99	May	1.04	September	.98
February	.93	June	.98	October	1.04
March	1.05	July	.98	November	.99
April	1.02	August	1.00	December	1.00

By multiplying the data of Table IV by the seasonals of Table V, a theoretical series would be obtained which would comprise the elements of trend, cycle and seasonal—lacking only chance or residual errors.

In order to obtain a series of chance factors that might serve as residual error factors, sixty cards were marked with integers totaling 1200 The cards were distributed, after shuffling, into twelve piles of five cards each, and the totals of each pile noted. These were taken as the residual factors for the first year, and the process was repeated for the following years. The chance factors of Table VI resulted.

Making allowance for residual errors as well as the seasonal factors, we obtain finally the theoretical series which we shall attempt to analyze, Table VII.

If the various methods of analyzing time series are sound, they should be able to break up this series into its elementary components—trend, cycle, seasonal and residual errors. A comparison of the results by different methods should indicate to some extent their respective merits. In attacking the ordinary observed time series by different methods and comparing results the difficulty is to tell, when all has been done, which of the methods is best. Unfortunately, if they disagree, we do not know which one is nearest the truth. Our theoretical series, however, enables us to compare results obtained by different methods, since we know the answers in advance, and also will serve students as a detailed example of time series synthesis.

TABLE VI

Residual Factors

	1904	1905	1906	1907	1908	1909
January	98	98	98	98	106	99
February	91	98	101	95	96	106
March	105	103	98	94	105	89
April	100	108	99	98	102	99
May	103	100	101	94	99	97
June	94	94	84	103	100	94
July	91	114	102	102	105	103
August	116	93	90	108	108	102
September	98	102	104	118	100	106
October	95	95	107	100	94	104
November	99	84	112	96	90	97
December	110	111	104	94	95	104
	1910	1911	1912	1913	1914	1915
January	96	102	98	98	97	98
February	107	99	102	95	93	89
March	91	95	97	92	105	92
April	110	96	106	104	104	83
May	104	109	98	104	110	108
June	102	92	108	105	97	113
July	94	92	109	91	91	103
August	98	103	98	98	96	98
September	100	102	104	96	100	105
October	105	103	94	112	102	107
November	95	107	93	108	98	102
December	98	100	93	97	107	102

TABLE VII

Theoretical Series

	1904	1905	1906	1907	1908	1909
January	906	1662	1908	2030	1242	1714
February	814	1582	1860	1855	1052	1831
March	1138	1913	2052	2077	1283	1831
April	1215	1976	2027	2088	1210	2077
May	1343	1892	2122	2043	1203	2143
June	1236	1700	1672	2093	1166	2058
July	1254	2092	2041	2060	1240	2320
August	1702	1757	1846	2163	1334	2413
September	1457	1906	2102	2262	1279	2502
October	1564	1899	2304	1946	1364	2643
November	1596	1611	2303	1475	1341	2378
December	1836	2163	2170	1153	1564	2595
Total	16061	22153	24407	23245	15278	26505
	1910	1911	1912	1913	1914	1915
January	2392	1933	2052	2554	2041	1687
February	2514	1746	2061	2330	1780	1532
March	2417	1895	2267	2554	2194	1906
April	2830	1865	2490	2800	2037	1796
May	2702	2167	2419	2845	2156	2539
June	2475	1732	2571	2696	1730	2674
July	2211	1742	2657	2314	1577	2566
August	2249	2011	2469	2525	1649	2661
September	2108	1976	2591	2377	1652	2952
October	2203	2144	2516	2850	1753	3342
November	1875	2168	2389	2494	1585	3093
December	1899	2092	2435	2150	1779	3182
Total	27875	23471	28917	30489	21933	29930

To obtain the values of the seasonal factors by means of formula (5) and Table I we need only observe that for the theoretical series

$$
\begin{aligned}
T_1 &= 16061\\
T_2 &= 22153\\
T_3 &= 24407\\
T_4+T_5+\ldots T_9 &= 145291\\
T_{10} &= 30489\\
T_{11} &= 21933\\
T_{12} &= 29930
\end{aligned}
$$

Consequently we have

TABLE VIII

Seasonals by Interpolation Method

Month	$\Sigma_a y_x$	$\Sigma \psi(x)$	s
January	22121	23587	.938
February	20957	23693	.885–
March	23527	23801	.988
April	24411	23911	1.021
May	25574	24023	1.065–
June	23803	24135	.986
July	24074	24246	.993
August	24779	24358	1.017
September	25164	24468	1.028
October	26528	24576	1.079
November	24308	24682	.985
December	25018	24785	1.009
Total	290264	290265	11.994

It is interesting to compare the seasonals of Table VIII with the corresponding set obtained by the method of "link relatives." The following table presents the series of link relatives for the theoretical series of Table VII.

TABLE IX

Link Relatives for the Series of Table VII

	1904	1905	1906	1907	1908	1909
January	.898	.952	975	.914	847	1.068
February	1.398	1.209	1.103	1 120	1.220	1.000
March	1 068	1 033	.988	1.005	.943	1.134
April	1 105	957	1 047	.978	.994	1.032
May	920	.899	.788	1 024	.969	.960
June	1 015	1.231	1.221	.984	1.063	1.127
July	1 357	.840	.904	1.050	1.076	1.040
August	.856	1 085	1 139	1 046	.959	1.037
September	1 073	.996	1 096	860	1.066	1.056
October	1.020	.848	1.000	.758	.983	.900
November	1.150	1.343	942	.782	1.166	1.091
December	.905	882	.935	1.077	1.096	.922

	1910	1911	1912	1913	1914	1915
January	1 051	.903	1.004	.912	.872	.908
February	.961	1.085	1.100	1.096	1.233	1.244
March	1.171	.984	1.098	1.096	.928	.942
April	.955	1 162	.971	1.016	1.058	1.414
May	.916	.799	1.063	.948	.802	1.053
June	.893	1.006	1.033	.858	912	.960
July	1.017	1 154	.929	1.091	1.046	1.037
August	937	.983	1.049	.941	1.002	1.109
September	1 045	1.085	971	1.199	1.061	1.132
October	851	1 011	.950	.875	.904	.925
November	1.013	.965	1.019	.862	1.122	1.029
December	1 018	.981	1.049	.949	.948	

From the above we obtain the following.

TABLE X

Link Relative Seasonal Indices

Months	(1) Medians	(2) Chain Relatives	(3) (2) Adjusted	(4) Seasonal Indices
January	.913	100.0	100.0	97.5
February	1.112	91.3	91.3	89.0
March	1.019	101.5	101.4	98.8
April	1.024	103.5	103.3	100.7
May	.934	105.9	105.7	103.0
June	1.010	98.9	98.7	96.2
July	1 043	99.9	99.7	97.2
August	1.020	104.2	103.9	101.3
September	1.064	106.3	106.0	103.3
October	.914	113.1	112.7	109.9
November	1.024	103.4	103.0	100.4
December	.949	105.9	105.4	102.7
January		100.5	100.0	

The following exhibit of the results obtained by the two methods is interesting.

TABLE XI

Comparison of Interpolation and Link Relative Methods

Months	Actual Values	Interpolation Method		Link Relative Method	
		Seasonal	Error	Seasonal	Error
January	.990	.938	-.052	.975	-.015
February	930	.885	-.045	.890	-.040
March	1.050	.988	-.062	.988	-062
April	1 020	1.021	001	1.007	-.013
May	1 040	1 065	.025	1.030	-.010
June	.980	.986	.006	.962	-.018
July	980	.993	.013	.972	-.008
August	1.000	1.017	.017	1.013	.013
September	.980	1.028	.048	1 033	.053
October	1 040	1.079	.039	1.099	.059
November	990	.985	-.005	1.004	.014
December	1 000	1.009	.009	1.027	.027

The mean deviations and the standard deviations of the two methods show that both methods are about equally effective. This advantage of the interpolation method is scarcely worth mentioning. Nevertheless, the fact that the results are obtained with but a trivial amount of labor is important.

	Mean Deviation of Errors	Standard Deviation of Errors
Interpolation Method	.0269	.0337
Link Relative Method	0277	.0338

STIELTJES INTEGRALS IN MATHEMATICAL STATISTICS

By

J. Shohat

(Jacques Chokhate)

Introduction. Stieltjes integrals, introduced into analysis in 1894-5[1], play an increasingly important role not only in pure mathematics, but also in theoretical physics and in the theory of probability. In mathematical statistics, however, their use, it seems, still remains very limited. And yet, one of the most remarkable features of Stieltjes integrals is that they represent, as the case may be, an integral proper or a sum of an finite or an infinite number of *discrete* aggregates. Thus *the statistician is enabled to treat in a single formula a continuous, as well as a discontinuous distribution.* This means far more than a mere simplification of writing. In fact, since Stieltjes integrals have many properties in common with Riemann and Lebesgue definite integrals, we can use all known resources of the theory of definite integrals (mean-value theorem, various inequalities), and therefore readily obtain general results which, otherwise, require special (often complicated) proofs. The advantage of such a treatment is particularly evident in the theory of interpolation, approximation, and mechanical quadratures.

Hence, the object of this paper is to present a general exposition of the properties and applications of Stieltjes integrals. Many of the results stated below are well known[2], and the proofs may be omitted. Some results are believed to be new (for example, extension of Tchebycheff and Hölder inequalities) and may prove useful in mathematical statistics. We close, as an illustration, with the theory of interpolation, for here, even in recently published books, the continuous and discontinuous cases are treated *separately* while the underlying ideas are *identical.*

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[1] Stieltjes: (a) Recherches sur les fractions continues, Oeuvres, v. II, p. 402-559; (b) Correspondence d'Hermite et de Stieltjes, v. II, p. 272, where these integrals are first mentioned in a letter (No. 351) to Hermite under date of October 25, 1892.

[2] (a) Hobson, The Theory of Functions of a Real Variable, 2d. ed. (1921), v. I, p. 506-16, 605-09; (b) O. Perron, Die Lehre von den Kettenbrüchen (1913), p. 362-69.

I. *Definition and general properties.* Let $f(x)$ be continuous and $\psi(x)$ be bounded monotonic non-decreasing on the finite interval (a, b) $(a < b)$ Then, as is well known, the following limits exist:

$$\left.\begin{aligned} \psi(x+0) &= \lim_{\epsilon \to 0} \left[\psi(x+\epsilon) - \psi(x) \right] \\ \psi(x-0) &- \lim_{\epsilon \to 0} \left[\psi(x-\epsilon) - \psi(x) \right] \end{aligned}\right\} \quad (a \leq x \leq b)$$

If x is a point of discontinuity of $\psi(x)$, $\psi(x+0) - \psi(x-0)$ (>0) is called "saltus" of $\psi(x)$ at this point. The number of such points is at most denumerably infinite; the points of continuity of $\psi(x)$ are, therefore, everywhere dense in (a, b) $\psi(x)$ is R- integrable, and so is $\psi(x)x^k$ $(k = 0, 1 \cdots\cdots)$. The Riemann-Stieltjes integral (of $f(x)$ with respect to $\psi(x)$ $\int_a^b f(x)d\psi(x)$) is defined as follows:

(S)

$$\int_a^b f(x)d\psi(x) = \lim_{\substack{n \to \infty \\ \max(x_{i+1} - x_i) \to 0}} \sum_{i=0}^{n-1} f(\xi_i)\left[\psi(x_{i+1}) - \psi(x_i)\right]$$

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

$$x_i \leq \xi_i \leq x_{i+1} \quad (i = 0, 1, \cdots n-1)$$

The existence of the right-hand limit can be easily established. The continuity of $f(x)$ is here sufficient, but not necessary[1].

In many phases of mathematical statistics the case of a continuous $f(x)$ is evidently the more important, although many problems arising in the theory of probability require applications of the discontinuous case.

From the very definition (S) one may obtain many properties of Stieltjes integrals in common with the ordinary definite integrals. Thus:

$$(1) \qquad \int_a^b d\psi(x) = \psi(b) - \psi(a)$$

---

[1] (a) Hobson, l-c; (b) T Hildebrandt, On Integrals Related to and Extension of the Lebesque Integrals, Bulletin of the American Mathematical Society (2), V 24 (1918), p. 177-202; (c) Lebesque, Leçon sur l'intégration, 2d ed. (1928), p. 252-313

(2) $$\int_a^c f\,d\psi + \int_c^b f\,d\psi = \int_a^b f\,d\psi \qquad (a < c < b)$$

(3) $$\int_a^b (f_1 \pm f_2)\,d\psi = \int_a^b f_1\,d\psi \pm \int_a^b f_2\,d\psi$$

(4) $$\int_a^b A f\,d\psi = A\int_a^b f\,d\psi \qquad (A = \text{Const.})$$

(5) $$\left|\int_a^b f\,d\psi\right| \leq \int_a^b |f|\,d\psi$$

(6) $$\int_a^b f\,d\psi = f(\xi)\int_a^b d\psi \qquad (a \leq \xi \leq b \quad : \text{mean-value theorem})$$

(7) $$\int_a^b f_1\,d\psi \leq \int_a^b f_2\,d\psi\,, \quad \text{if } f_1(x) \leq f_2(x) \text{ for } a \leq x \leq b$$

(8) $$\int_a^b \sum_{i=1}^{\infty} f_i\,d\psi = \sum_{i=1}^{\infty}\int_a^b f_i\,d\psi$$

if $\sum_{i=1}^{\infty} f_i(x)$ converges uniformly in (a, b)

(9) $$\int_a^b f\,d\psi = f\psi\Big|_a^b - \int_a^b \psi\,df \qquad (\text{integration by parts})$$

(10) $$\int_a^b f\,d\psi = \int_a^b f(x)p(x)\,dx\,, \quad \text{if } \psi(x) = \int \phi(x)\,dx + c$$

with $p(x) \geqq 0$ in (a, b).

(10-bis) $$\int_a^b f\,d\psi = \int_a^b f(x)\psi'(x)\,dx \qquad , \text{ if } \psi'(x) \text{ exists}$$

and is R- integrable in (a, b).

Let $\psi(x)$ have only a *finite* number of points of increase in (a, b).

$$(a = x_0 <)\,x_1 < x_2 < \cdots\cdots < x_n\,(< x_{n+1} = b)$$

with the saltus σ_i at $x = x_i$ $(i = 1, 2 \cdots\cdots n)$, so that $\psi(x)$ remains constant $= \sum_{s=1}^{i}\sigma_s$ for $x_i \leq x < x_{i+1}$ and $\psi(b)\sum_{s=1}^{n}\sigma_s$. Such functions, called *stepwise* functions ("fonction en escalier"), prove very useful. Here

(11) $\int_a^b f d\psi = \sum_{i=1}^{n} \sigma_i f(x_i) \qquad \{\sigma_i = f(x_i+0) - f(x_i-0)\}$

If the number of points of increase is infinite

$$(a<)\, x_1 < x_2 < \cdots < x_n < \cdots, \quad \lim_{n\to\infty} x_n = b$$

(12) $\int_a^b f d\psi = \sum_{i=1}^{\infty} \sigma_i f(x_i)$.

Conversely, any sum $\sum_{i=1}^{n} u_i v_i$ can be represented as a Stieltjes integral in infinitely many ways. Let us introduce n positive numbers $\sigma_1, \sigma_2, \cdots \sigma_n$ a certain interval (a, b), n points $(a<)\, x_1 < \cdots < x_n (<b)$ (the choice of x_i σ_i depends upon the nature of the problem involved), and a stepwise function $\psi(x)$ having at $x = x_i$ a saltus σ_i $(i = 1, 2, 3 \cdots n)$. Then, writing $u_i = \sigma_i w_i$, we may consider v_i, w_i as values taken respectively by some functions $f(x), \phi(x)$ at $x = x_i$ $(i = 1, 2, \cdots\cdots n)$. Hence,

(13) $\sum_{i=1}^{n} u_i v_i = \int_a^b f(x)\phi(x)\, d\psi(x)$

Formulae (11-13) show clearly the use of Stieltjes integrals for the representation of sums of discrete aggregates.

(14) $\int_a^b f d\psi \geq 0$, if $f(x) \geq 0$ in (a, b)

Here "=" takes place if and only if $\psi(x)$ has a finite or denumerably infinite number of points of increase in (a, b) (not everywhere dense) and $f(x)$ vanishes at all these points, for we exclude, of course, functions $f(x)$ which vanish at all points of continuity of $\psi(x)$ and therefore vanish identically in (a, b). *If $\psi(x)$ has infinitely many points of increase, while $f(x)$ vanishes in (a, b) only a finite number of times, without changing sign, then $\int_a^b f(x)\, d\psi(x) \neq 0$ and has the sign of $f(x)$.*

(15) $\int_a^b f(x)\, x^k d\psi(x) = 0 \quad (k = 0, 1, \cdots n-1)$ implies: $f(x)$

has at least n distinct roots inside (a, b) assuming that $\psi(x)$ has at least n points of increase[1].

---

[1] This is a form of a theorem due to Perron (l-c, p. 368-69). If the number of such points is $m < n$, (15) shows only that $f(x)$ vanishes at all such points

(16) $\int_a^b x^n d\psi(x)$ [illegible] implies:

$\psi(x)$ constant [illegible]

Since in the definition [illegible] the [illegible] $\psi(x_i)$ enter, it follows that a Stieltjes integral [illegible] value if we replace $\psi(x)$ by $\psi(x)+c$. More [illegible]

(17) $\int_a^b f d\psi_1 = \int_a^b f d\psi_2$

if the two monotonic non-decreasing [illegible] differ by an additive constant only at all points [illegible] Applying the mean value theorem to $\int_x^{x+h} f(x) d\psi(x)$ [illegible]

(18) $F(x)=\int_a^x f(t)d\psi(t)$ is continuous [illegible] continuity of

$\psi(x)$ and therefore, almost everywhere [illegible]

(19) $\lim_{h \to 0} \frac{F(x+h) - F(x)}{\psi(x+h) - \psi(x)} = f(x)$ [illegible]

(20) $F'(x)=f(x)\,\psi'(x)$ at any point x [illegible] $\psi'(x)$ exists [illegible]

One recognizes in (18-20) [illegible] properties of the ordinary definite integral which [illegible] $\psi(x)=x$.

(21) $\phi(t)=\int_a^b f(x,t)d\psi(x)$ is continuous [illegible]

if $f(x,t)$ is continuous in x [illegible] with respect to $t(t_0 \le t \le t_1)$ for all values of [illegible]

(22) $\frac{d\phi(t)}{dt}=\int_a^b \frac{\partial f(x,t)}{\partial t}\, d\psi(x)$

if $\frac{\partial f(x,t)}{\partial t}$ exists and is continuous [illegible] in t $(a \le x \le b;\ t_0 \le t \le t_1)$

---

[1] If $\psi(x)$ has a finite number n of points [illegible] imply the same conclusion

Notes. (i) The above results hold, with proper limitations and modifications, if $\psi(x)$ be of *bounded variation* in (a, b), for such a function can be represented as a difference of two monotonic non-decreasing functions $\psi_{1,2}(x)$ and we define in accordance with (S),

$$\int_a^b f\,d\psi = \int_a^b f\,d\psi_1 - \int_a^b f\,d\psi_2 \ .$$

(ii) In applications to probability and mathematical statistics $\psi(x)$ stands for the "cumulative law of distribution," so that

(23) $\psi(x)$ is monotonic non-decreasing from $\psi(a)=0$ to $\psi(b)=1$.

(24) For $(a \le c < d \le b)$ the integral $\int_c^d d\,\psi(x)$ = probability $P:[c \le x \le d]$; $\int_a^b d\,\psi(x) = 1$.

(25) $\int_a^b f(x)d\,\psi(x) = E(f)$, i. e., the expected value or mathematical expectation of $f(x)$.

Let $w(x)\,f(x)$ be continuous in (a, b), and $\alpha(x)$ be of bounded variation. Then,

(26) $\psi(x) = \int_a^x w(x)d\alpha(x)$ is of bounded variation,[1]

$$\int_a^b f(x)d\,\psi(x) = \int_a^b f(x)w(x)d\alpha(x)$$

Given an infinite sequence of functions $\psi_n(x)$ $(n=1,\ 2,\ \cdots)$ of bounded variation in (a, b). If the total variation in (a, b) of all $\psi_n(x)$ does not exceed a fixed quantity M independent of n, and if, in addition, $\lim_{n\to\infty} \psi_n(x) = \psi(x)$ exists for $a \le x \le b$, then[2]

(27) $\lim_{n\to\infty} \int_a^b f(x)d\,\psi_n(x) = \int_a^b f(x)d\,\psi(x)$, for any continuous $f(x)$.

Notes. (i) (27) holds true if we know that $\lim_{n\to\infty} \psi_n(x) = \psi(x)$ exists at all points of continuity of the sequence $\psi_n(x)$ and at $x=a, b$.

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[1] T Carleman Leçons sur les équations intégrales singulières noyau réel et symétrique (Uppsala) (1923), p. 11-12.

[2] Page 9 of preceding reference.

(ii) In applications to probability and statistics (27) is of great importance In fact, consider $\psi_n(x)$ *as a sequence of variable laws of distribution approaching, as a limit, a certain fixed law of distribution* $\psi(x)$. Then(by (23), the total variation of any $\psi_n(x)$ in (a, b) is 1; (27) thus becomes applicable and shows that *under the said conditions the expected value of any continuous function in the variable law of distribution approached, as* $n\to\infty$ *its expected value in the limiting law of distribution.*

II. *Stieltjes Integrals Over an Infinite Interval* We define

(28) $$\int_a^\infty f\,d\psi = \lim_{x\to\infty}\int_a^x f\,d\psi\ ;\qquad \int_{-\infty}^\infty f\,d\psi = \lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b f\,d\psi$$

(similarly $\int_{-\infty}^b$), provided the right-hand limits exist as finite numbers. It is assumed that $\int_a^x f\,d\psi$, $\int_a^b f\,d\psi$ exist respectively for any finite $x>a$, and for any finite interval (a, b). For the existence of (28) it is necessary and sufficient that

(29) $$\left|\int_{x_1}^{x_2} f\,d\psi\right| < \epsilon \quad \text{for } x \geq \text{ a certain number } x(\epsilon),$$

$\epsilon > 0$ – arbitrarily small.

One sees readily that

(30) $\int_{-\infty}^{\infty} f\,d\psi$ exists, if $\int_{-\infty}^{\infty} d\psi$ does, and if $f(x)$ is bounded for all real values of x. The first of these conditions is satisfied if $\psi(x)$ is a law of distribution. We notice that any $\int_a^b f\,d\psi$ can be written as $\int_{-\infty}^{\infty} f\,d\psi$, if we agree to take $\psi(x) = \psi(a)$, $\psi(b)$ respectively for $x \leq a$, $\geq b$.

The formulae given above hold, in general, for infinite limits as well, *with the exception of those which require a double limiting process*, like 8, 21, 27, etc., where ordinarily additional precautions must be taken in the form of certain assumptions specifying the behaviour of $\psi(x)$ and of other functions involved at infinity. Thus, (8) is not valid in general for $(a, b)=(-\infty, \infty)$, and requires a more detailed discussion. Furmulae 21, 22 hold true if we assume, for example, the uniform boundedness and continuity with respect to t of the functions involved for all x in $(-\infty, \infty)$, and also the existence of $\int_{-\infty}^{\infty} d\psi(x)$, i. e. definite values for $\psi(\pm\infty)$.

Formula 17 deserves *special attention: in general, it is not true* *for an infinite interval*, as was shown by Stieltjes[1].

III. *Approximate Evaluation of Stieltjes Integrals.* In practice, as in statistical computations, we evaluate $\int_a^b f d\psi$ approximately, replacing it by the right-hand member of (S), for a certain chosen n. The question arises regarding the error r_n of such an approximation. Let $\omega(\delta)$ represent the modulus of continuity of $f(x)$, i e.

$$(31) \quad |f(x)-f(y)| \leq \omega(\delta) \text{ for } |x-y| \leq \delta \; (a \leq x, \; y \leq b)$$

Then, if $x_{i+1} - x_i < h$ in (S) for $i = 0, 1, \cdots\cdots n-1$, we have

$$r_n = \int_a^b f d\psi - \sum_{i=0}^{n-1} f(\xi_i)\left[\psi(x_{i+1}) - \psi(x_i)\right]$$

$$= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left[f(x) - f(\xi_i)\right] d\psi(x)$$

$$(32) \quad |r_n| \leq \omega(h) \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d\psi = \omega(h)\left[\psi(b) - \psi(a)\right]$$

$$\{h = max.\,(x_{i+1} - x_i)\;;\quad i = 0, 1, \cdots\cdots\cdots n-1\}$$

(32) *answers the above question for any continuous* $f(x)$.

Special Case: Lipschitz condition[2]:

$$(33) \quad |f(x)-f(y)| \leq \lambda\,|x-y| \qquad (a \leq x, \; y \leq b\;;$$

$$|r_n| \leq \lambda h \left[\psi(b) - \psi(a)\right] \qquad \lambda = const.)$$

In (32, 33) *we replace* h *by* $h/2$, *if* (ξ_i) *in* (S) *is, as usual, the mid-point of the interval* (x_i, x_{i+1}) $(i = 0, 1, \cdots\cdots n-1)$.

It must be noticed, however, that the above considerations are

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[1] (l. c. p. 73, p 505-06. (17) is closely related to the so-called "Moments-problem": find a monotonic non-decreasing function $\psi(x)$ in (a, b) with infinitely many points of increase, if all its moments $\gamma_k = \int_a^b x^k d\psi(x)$ $[k = 0, 1 \cdots]$ are given *This problem, for* (a, b) *infinite, may be "indetermined," i. e it may admit infinitely many solutions, while it is always "determined," for finite* (a, b). Stieltjes gives the following example:

$$\int_0^\infty x^k \left[1 + \lambda \sin (x^{1/4})\right] e^{-x^{1/4}} dx = \int_0^\infty x^k e^{-x^{1/4}} dx$$

$[\lambda$ = constant, $k = 0, 1, \cdots\cdots]$, and $\psi(x) = 0 = \int_0^x \left[1 + \lambda \sin x^{1/4}\right] e^{x^{1/4}}$ is monotonic non-decreasing in $0, \infty$, if $|\lambda| \leq 1$.

[2] If $f'(x)$ exists for $a \leq x \leq b$, then λ can be taken equal to max. $|f'(x)|$ *in* (a, b) If $f(x)$ is given graphically, λ can be found roughly as the maximum of the absolute val [illegible] ne slope in (a, b)

not workable in general on an infinite interval, for here, in place of (31), we ordinarily have the more complicated relation

$$|f(x)-f(y)| \leq \omega(x,y,\delta) \qquad (|x-y| \leq \delta)$$

where $\omega(x,y,\delta) \to \infty$ with x, y (ex. $f(x)=x^2$). Thus here, in order to obtain an inequality for the error, we must add to the right member of (32), where a, b are finite numbers properly chosen, two more terms—the upper limits of $\left|\int_{-\infty}^{a} f\,d\psi\right|$ and $\left|\int_{b}^{\infty} f\,d\psi\right|$, which we obtain by means of a suitable hypothesis concerning the behavior of $f(x)$, $\psi(x)$ at infinity.

IV *Tchebycheff and Holder Inequalities for Stieltjes Integrals*[1]. Hereafter $\psi(x)$ stands for a monotonic non-decreasing function defined on a certain interval (a, b), finite or infinite. Let $f_i(x)$, $\phi_i(x)\,[i=1,2,\cdots n]$ be continuous on (a,b)[2]. Then we have the following *fundamental transformation*:

$$(34)\qquad \begin{vmatrix} \int_a^b f_1\phi_1\,d\psi & \int_a^b f_1\phi_2\,d\psi & \cdot & \int_a^b f_1\phi_n\,d\psi \\ \int_a^b f_2\phi_1\,d\psi & & & \int_a^b f_2\phi_n\,d\psi \\ \cdot\ \cdot\ \cdot & & & \cdot\ \cdot \\ \int_a^b f_n\phi_1\,d\psi & \cdots & & \int_a^b f_n\phi_n\,d\psi \end{vmatrix}$$

$$= \frac{1}{n!}\underbrace{\int_a^b\cdots\int_a^b}_{(n\ \text{times})} \begin{vmatrix} f_1(x_1) & f_1(x_n) \\ \vdots & \vdots \\ f_n(x_1) & f_n(x_n) \end{vmatrix} \cdot \begin{vmatrix} \phi_1(x_1) & \cdot\phi_1(x_n) \\ \vdots & \vdots \\ \phi_n(x_1) & \cdot\phi_n(x_n) \end{vmatrix} \prod_{i=1}^{n} d\psi(x_i)\,.$$

The proof is very simple for $n=2$, for we can write

$$\int_a^b u(x)\,d\psi(x)\cdot\int_a^b v(x)\,d\psi(x)$$

$$\text{as}\ \int_a^b\int_a^b u(x_1)\,v(x_2)\,d\psi(x_1)\,d\psi(x_2)$$

and it may readily be extended to any n. Formula (34) yields many interesting results by a proper choice of n, f_i, ϕ_i.

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[1] Cf. my Note: Jacques Chokhate, Sur les intégrales de Stieltjes, Comptes Rendus, v. 189 (1929), p. 618-20

[2] In case $\psi(x)$ has a finite number of points of increase in (a, b), we require only definite values of all $f_i(x)$, $\phi_i(x)$ at these points

Examples: (i) $n=2$; $f_1 \equiv \phi_1$, $f_2 \equiv \phi_2$

$$\int_a^b f_1^2\, d\psi \int_a^b f_2^2\, d\psi - \left(\int_a^b f_1 f_2\, d\psi\right)^2 = \tag{35}$$

$$\frac{1}{2}\int_a^b\!\!\int_a^b \begin{vmatrix} f_1(x_1) & f_1(x_2) \\ f_2(x_1) & f_2(x_2) \end{vmatrix} d\psi(x_1)\, d\psi(x_2) \geq 0 .$$

Schwartz inequality—("=" only if f_1 and f_2 are linearly dependent.

(ii) $n=2$; $f_2 \equiv \phi_2 \equiv 1$. Write f, ϕ in place of f_1, ϕ_1:

$$\int_a^b f\phi\, d\psi \cdot \int_a^b d\psi - \int_a^b f\, d\psi \cdot \int_a^b \phi\, d\psi \tag{36}$$

$$= \frac{1}{2}\int_a^b\!\!\int_a^b [f(x)-f(y)][\phi(x)-\phi(y)]\, d\psi(x)\, d\psi(y)$$

$$\int_a^b d\psi \cdot \int_a^b f\phi\, d\psi \gtrless \int_a^b f\, d\psi \cdot \int_a^b \phi\, d\psi$$

Tchebycheff inequality (derived by him for the special case $d\psi = dx$), *where* f, ϕ *are any two functions both varying monotonically in* (a, b), *either in the same sense* (*sign* $>$ *in* (36) *or in the opposite sense* (*sign* $<$). *In* (34-37) *we may replace* $d\psi(x)$ *by* $p(x)dx\,[p(x) \geq 0 \text{ in } (a, b)]$.[1]

(iii) $f_i(x) = x^{i-1}$, $\phi_i(x) = F(x)\,x^{i-1}\,[i = 1, 2, \cdots\cdots n]$:

$$\Delta_n = \begin{vmatrix} \int_a^b F\, d\psi & \int_a^b Fx\, d\psi & \cdots\cdots & \int_a^b Fx^{n-1}\, d\psi \\ \int_a^b Fx\, d\psi & \cdots & \cdots & \int_a^b Fx^{n}\, d\psi \\ \vdots & & & \vdots \\ \int_a^b Fx^{n-1}\, d\psi & \cdots & \cdots & \int_a^b Fx^{2n-2}\, d\psi \end{vmatrix}$$

$$= \frac{1}{n!}\underbrace{\int_a^b \cdots\cdots \int_a^b}_{(n\ \text{times})} \prod_{i=1}^{n} F(x_i)\, d\psi(x_i) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (x_i - x_j)^2 .$$

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[1] Cf. E. Fischer, Ueber den Hadamardschen Determinantensatz, Archiv für Mathematik and Physik (3), v. 13 (1908), p. 32-49, where (34) is derived for the particular case $d\psi(x) \equiv dx$

The determinant Δ_n plays an important role in the theory of orthogonal Tchebycheff polynomials (see below). Formula (37) gives an upper limit for Δ_n:

$$(38)\qquad |\Delta_n| < \frac{1}{n!}(b-a)^{n(n-1)} M^n \left[\int_a^b d\psi(x)\right]^n$$

$$\left[M = \max.\ |F(x)|\ in\ (a,\ b)\right].$$

Applying (13) to the above formulae, we get:

$$(39)\qquad \left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \sum_{i=1}^{n} u_i^2 \sum_{i=1}^{n} v_i^2$$

—*Cauchy inequality* (from (34))

$$(40)\qquad n\sum_{i=1}^{n} a_i b_i \gtrless \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$$

$$(41)\qquad \frac{\sum_{i=1}^{n} u_i v_i}{\sum_{i=1}^{n} v_i} \gtrless \frac{\sum_{i=1}^{n} u_i w_i}{\sum_{i=1}^{n} w_i} \quad (v_i\,,\ w_i\ > 0)^1$$

Formulae 40, 41 follow from (36) by means of (13), with $\sigma_i = 1$ (in (40)), $\sigma_i = w_i$ (in (41)) $[\, i = 1, 2, \cdots n\,]$. The sequences $\{(a_i),\ (b_i)\}$, $\{(u_i)\,,\ (\frac{v_i}{w_i})\}$ are assumed to be either increasing or decreasing, the sign $\gtrless$ being chosen as in (36). Thus *all these (and many similar) inequalities have the same origin-formula* (34). Applying (13) to Hölder-Minkowski inequalities[2].

$$(42)\qquad \sum_{i=1}^{n} |a_i\ b_i| \leq \left\{\sum_{i=1}^{n} |a_i|^s\right\}^{1/s} \cdot \left\{\sum_{i=1}^{n} |b_i|^{s/s-1}\right\}^{s-1/s} \qquad (s > 1)$$

$$\left\{\sum_{i=1}^{n} |a_i + b_i|^s\right\}^{1/s} \leq \left\{\sum_{i=1}^{n} |a_i|^s\right\}^{1/s} + \left\{\sum_{i=1}^{n} |b_i|^s\right\}^{1/s} \qquad (s > 1)$$

we get;

$$(43)\qquad \int_a^b |f\phi|\, d\psi \leq \left\{\int_a^b |f|^s d\psi\right\}^{1/s} \cdot \left\{\int_a^b |\phi|^{\frac{s}{s-1}}\ d\psi\right\}^{s-1/s} \ (s > 1)$$

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1. Cf. l. c. p. 73, I-b, pp. 142, 143, 146, 194.

2. F. Rietz, Ueber Systeme integrierbarer Funktionen, Mathematische Annalen, v. 69 (1911), pp. 449-497; p. 456.

$$(44) \quad \left\{\int_a^b |f+\phi|^s d\psi\right\}^{1/s} \le \left\{\int_a^b |f|^s d\psi\right\}^{1/s} + \left\{\int_a^b |\phi|^s d\psi\right\}^{1/s} \quad (s \ge 1) .$$

Formula (43), with $\phi \equiv 1$, $s = s_2/s_1 > 1$ and f replaced by $|f|^{s_1}$, yields:

$$(45) \quad \left\{\int_a^b |f|^{s_1} d\psi\right\}^{1/s_1} \le \left\{\int_a^b |f|^{s_2} d\psi\right\}^{1/s_2} \cdot \left\{\int_a^b d\psi\right\}^{\frac{s_2 - s_1}{s_1 s_2}}$$

$$(s_2 > s_1 > 0) .$$

The applications of the above inequalities to the theory of probability and mathematical statistics are many. A few illustrations follow:

(i) Consider even moments $u_{2s} = \int_{-a}^{a} x^{2s} f(x) dx = 2\int_0^a x^{2s} f(x) dx$ of a continuous unimodal symmetric distribution over a finite interval $(-a, a)$. Here (36) gives (with s_1, $<$, $d\psi(x) = f(x)dx$, $2\int_0^a f(x)dx = 1$).

$$(46) \quad u_{2s} = \int_{-a}^{a} x^{2s} f(x) dx < \frac{a^{2s}}{2s+1}$$

$$\left[s = 1, 2, \cdots\cdots; \ f(x) \equiv f(-x)\right] .$$

(ii) If ξ denotes an arbitrary constant, take in (42) $f(x) = x - \xi$, $\psi(x) =$ *law of distribution of* x *over* (a, b), so that $\int_a^b d\psi(x) = 1$. We get:

$$(47) \quad v_{s_1}^{1/s_1} \le v_{s_2}^{1/s_2} \quad \text{for } s_1 < s_2$$

$$\left(v_s^{1/s} \equiv \left[\int_a^b |x - \xi|^s d\psi\right]^{1/s}\right)$$

Hence, in any distribution over any interval the quantity $v_s = \left[\int_a^b |x - \xi|^s d\psi(x)\right]^{1/s}$ *increases with* s *for any constant* ξ and, in particular,

$$u_{2s}^{1/2s} = \left[\int_a^b x^{2s} d\psi\right]^{1/2s} \qquad \text{also if } a \ge 0,$$

$$u_s'^{1/s} = \left[\int_a^b f d\psi\right]^{1/s} .^{1}$$

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[1] Paul Levy, Calcul des probabilités (Paris 1925), p. 157-58.

(iii) Apply (36) to the functions $f(x)$, $\phi(x)$ *both monotonic in* (a, b), $\psi(x)$ the same as in (ii).

(48) $E(f\phi) \gtrless E(f)E(\phi)$ (for the choice of $\gtrless$ see (36))[1].

The same formula (36) gives for *any function* $f(x)$[2]

(49) $E(f^n) > \{E(f)\}^n \quad (n = 2, 3, \cdots)$[3]

Formula (45) gives with the same $\psi(x)$:

(50) $\{E(|f|^{s_1})\}^{1/s_1} \leq \{E(|f|^{s_2})\}^{1/s_2} \quad (s_1 < s_2)$.

V. *Application of Stieltjes Integrals to Some Minimum-Problems. Given a number* $m \geq 1$, *M finite points* $x_1 \leqq x_2 \leqq \cdots x_M$, *M positive quantities* $\sigma_1, \sigma_2, \cdots \sigma_M$, *and a function* $f(x)$ *with well determined values* $f(x_i)(i = 1, 2, \cdots M)$. *Find a polynomial* $P_n(x)$, *of degree not exceeding* $n(\leq M-2)$, *minimizing the expression* $\sum_{i=1}^{M} \sigma_i |f(x_i) - P_n(x_i)|^m$. *Discuss the behavior of* $P_n(x)$ *for* $m \to \infty$. We introduce a finite interval (a, b), containing in its interior all points x_i and a monotonic non-decreasing step-wise function $\psi(x)$ with the above properties (saltus σ_i at $x = x_i$, etc.; see p. 75). Then our problem can be formulated as follows *Find a polynomial* $P_n(x)$ *of degree not exceeding* n, *minimizing the integral*

$$\int_a^b |f(x) - P_n(x)|^m \, d\psi(x) \quad [m \geq 1].$$

Here the advantage of Stieltjes integrals is clearly evident, for the latter problem has been discussed by G. Polya[3], D. Jackson[3] and the writer[3]. We know that a solution always exists and is unique for $m > 1$. The behavior of $P_n(x)$, when either or both m and n in

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1. G Bohlman Formulierung und Begründung Zweier Hulfssatze der Mathematischen Statistik, Mathematische Annalen, v 74 (1913), pp 341-442, p 374-75

2. In fact, (36) holds, with sign $>$, if $f(x) - f(y)$ and $\phi(x) - \phi(y)$ have the same sign for any x, y in a, b, which, of course, is true for $\phi(x) \equiv f(x)$.

3. (a) G. Polya, Sur un algorithme toujours convergent . . ., Comptes Rendus, v. 157 (1913) p 840-43. (b) D. Jackson, On the Convergence of certain polynomial and trigonometric approximations. Transactions of the American Mathematical Society, v 22 (1921), p. 158-66 (c) Idem, Note on the Convergence of Weighted Trigonometric Series, Bulletin of the American Mathematical Society, v. 29 (1923), p. 259-63. (d) J Shohat, On the Polynomial and Trigonometric Approximation, Mathematische Annalen, v. 103 (1929), p. 157-75.

crease indefinitely, has also been discussed by the above writers. It was found that, *if* $f(x)$ *be continuous in* (a, b), *then for* n *fixed and* $m \to \infty$, $P_n(x)$ *approaches uniformly in* (a, b) *the polynomial* $\Pi_n(x)$, *of degree* $\leq n$, *of the best approximation (in Tchebycheff sense*[1]) to $f(x)$,*provided*, $\psi(x)$ *has infinitely many points of increase everywhere dense in* (a, b). *Furthermore*, $\left[\int_a^b |f(x)-P_n(x)|^m d\psi(x)\right]^{1/m} \xrightarrow[m\to\infty]{}$ *the best approximation* $E_n(f)=\max.|f(x)-\Pi_n(x)|$ *for* $a \leq x \leq b$.

This result has been supplemented by the writer (in a paper which will appear elsewhere), who showed that *the above result holds if* $\psi(x)$ *has a finite number* $M(\geq n+2)$ *points of increase*. $\Pi_n(x)$ *representing here the polynomial (of degree* $\leq n$) *giving the best approximation to* $f(x)$ *on the aggregate of the said points of increase of* $\psi(x)$. The following cases are of special interest.

(a) $n=0$, *i. e find a constant* X_m *minimizing the sum* $\sum_{i=1}^{M} \sigma_i |f(x_i)-X_m|^m$.

Very simple considerations show that the best approximation to $\{f(x_i)\}$ $(i=1, 2, \cdots)$ by means of a constant is $E_0(f)=\frac{1}{2}|f(x_1)-f(x_k)|$, $f(x_1), f(x_k)$ *being respectively the largest and the smallest of the* $f(x_i)$, *so that* $|f(x_1)-f(x_k)|$ *is the largest possible, and the "constant of the best approximation" is* $N_0=\frac{1}{2}[f(x_k)+f(x_1)]$. Thus here

$$\lim_{m\to\infty} X_m = \frac{f(x_k)+f(x_1)}{2}$$

(51)
$$\lim_{m\to\infty} \left\{\sum_{i=1}^{m} \sigma_i [f(x_i)-X_m]^m\right\}^{1/m}$$

$$= \left|\frac{f(x_k)-f(x_1)}{2}\right| = \max. \left|\frac{f(x_i)-f(x_j)}{2}\right| (i, j = 1, 2, \cdots n)$$

(52)
$$f(x_1) < f(x_2) < \cdots < f(x_M) \text{ implies :}$$

$$\lim_{m\to\infty} X_m = \frac{f(x_M)+f(x_1)}{2}$$

$$\lim_{m\to\infty} \left\{\sum_{i=1}^{m} \sigma_i |f(x_i)-X_m|^m\right\}^{1/m} = \frac{f(x_M)-f(x_1)}{2}$$

---

[1] That is: $E_n(f) \equiv \max. |f(x)-\Pi_n(x)| \leq \max. |f(x)-G_n(x)|$ $(a \leq x \leq b)$ where $G_n(x)$ is an arbitrary polynomial of degree $\leq n$, equality implying necessarily: $G_n \equiv \Pi_n$.

and the limiting results do not depend on $x_2, x_3, \cdots x_{n-1}$ As an illustration $f(x)=x^{2k+1}$ may serve, or, more generally, $f(x)=\sum_i A_i x^{2k_i+1}$ (all $A_i > 0$; all k_i and K are positive integers or zero)[1].

(b) $M=n+2$, *n arbitrary.* Here the writer showed (the paper will appear elsewhere):

$$(53)\ \lim_{m\to\infty} P_n(x)=\Pi_n(x)$$

$$=\frac{f_{n+1}+f_{n+2}}{2}-\frac{1}{2K}\sum_{i,j=1}^{n}(-1)^{i+j}K_{i,j}\,\frac{f_i-f_{i+2}}{x_i-x_{i+2}}\,(x_{n+1}^j+x_{n+2}^j-2x^j)$$

$$(54)\ \lim_{m\to\infty}\left[\sum_{i=1}^{n}\sigma_i\,|f_i-P_n(x_i)|^m\right]^{1/m}=E_n(f)$$

$$=\frac{1}{2}\left|f_k-f_{k+1}-\frac{1}{K}\sum_{i,j=1}^{n}(-1)^{i+j}\,\frac{f_i-f_{i+2}}{x_i-x_{i+2}}\,(x_k^j-x_{k+1}^j)\right|$$

$$k=1,\ 2,\cdots\ ,M$$

$$f_i\equiv f(x_i)\ \ (i=1,\ 2,\cdots M)$$

where K, $K_{i,j}$ stand respectively for the following determinant and its minors:

$$(55)\qquad K=\begin{vmatrix} \dfrac{x_1^2-x_3^2}{x_1-x_3} & \dfrac{x_1^3-x_3^3}{x_1-x_3} & \cdots & \cdots & \dfrac{x_1^n-x_3^n}{x_1-x_3} \\ \dfrac{x_2^2-x_4^2}{x_2-x_4} & \cdots & \cdots & \cdots & \dfrac{x_2^n-x_4^n}{x_2-x_4} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{x_n^2-x_{n+2}^2}{x_n-x_{n+2}} & \cdots & \cdots & \cdots & \dfrac{x_n^n-x_{n+2}^n}{x_n-x_{n+2}} \end{vmatrix}$$

We proceed now to show the application of Stieltjes integrals to interpolation. This must be preceded by a discussion of

VI. *Orthogonal Tchebycheff Polynomials. Theorem. Any*

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[1] Cf. D. Jackson, Note on the Median of a Set of Numbers, Bulletin of the American Mathematical Society, v. 22 (1920), p. 160-64, where the above results have been obtained for the particular case $f(x)=x$.

function $\psi(x)$, *monotonic non-decreasing on* (a, b) *—finite or infinite, and having all moments* $\gamma_k=\int_a^b x^k d\psi(x)$ $(k=0, 1, \cdots)$ *with* $\gamma_0>0$ *generates a sequence of polynomials* $\{\phi_n(x)\}$ *of degree* $n=0, 1, \cdots$ *uniquely determined by the relations*[1]: $\int_a^b \phi_m \phi_n d\psi=0$ $(m\neq n\ ;\ m, n=0, 1, \cdots)$ *equivalent to* $\int_a^b x^k \phi(x) d\psi(x)=0$ $(k=0, 1, \cdots n-1)$ $(n=1, 2, \cdots\cdots)$.

Proof. Take $\phi_n(x)=x^n+f_{n-1}x^{n-1}+\cdots f_1x+f_0$. The above relations lead to the following set of equations.

$$
(56)\quad
\begin{array}{llll}
f_0\gamma_0+f_1\gamma_1+ & \cdot & \cdot & f_{n-1}\gamma_{n-1}+\gamma_n=0\\
f_0\gamma_1+f_1\gamma_2+ & \cdot\ \cdot & \cdot & f_{n-1}\gamma_n+\gamma_{n+1}=0\\
\cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\\
f_0\gamma_{n-1}+f_1\gamma_n+ & \cdot & \cdot & f_{n-1}\gamma_{2n-2}+\gamma_{2n-1}=0
\end{array}
$$

The determinant Δ_n of the coefficients γ_i is (see (37)):

$$
(57)\quad \Delta_n=\frac{1}{n!}\underbrace{\int_a^b\cdots\int_a^b}_{(n\ \text{times})}\prod_{i=1}^{n}d\psi(x_i)\prod_{i,j=1,\ i\neq j}^{n}(x_i-x_j)^2>0
$$

which proves our statement. Add to (56) the identical relation $f_0+f_1x+\cdots+f_{n-1}x^{n-1}+(x^n-\phi_n)=0$, and for $\phi_n(x)$ we obtain the following expression:

$$
(58)\quad \phi_n(x)=\begin{vmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_n\\
\gamma_1 & \gamma_2 & \cdots & \gamma_{n+1}\\
\cdot & \cdot & \cdot & \cdot\\
\gamma_{n-1} & \gamma_n & \cdots & \gamma_{2n-1}\\
1 & x & \cdots & x^n
\end{vmatrix}\cdot
\begin{vmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_{n-1}\\
\gamma_1 & \gamma_2 & \cdots & \gamma_n\\
\cdot & \cdot & \cdot & \cdot\\
\gamma_{n-1} & \gamma_n & \cdots & \gamma_{2n-2}
\end{vmatrix}
$$

Note. If $\psi(x)$ has in (a, b) only a finite number M of points of increase, then $\Delta_n\equiv 0$ *for* $n>M$, and $\phi_n(x)$ exists only for $n=0, 1, \cdots M$ (See below (65), which in this case is a rational fraction).

The following table gives the most known and important Tchebycheff polynomials.

---

[1] We disregard constant factors.

[1]

(a, b)	$d\psi(x) =$	$\phi(x)$ — polynomial of [Constant factors disregarded]
finite	dx	Legendre: $\frac{d^n}{dx^n}\left[(x-a)^n(b-x)^n\right]$
finite	$(x-a)^{\alpha-1}(b-x)^{\beta-1}dx$ $(\alpha, \beta > 0)$	Jacobi. $(x-a)^{1-\alpha}(b-x)^{1-\beta}\frac{d^n}{dx^n}\left[(x-a)^{n+\alpha-1}(b-x)^{n+\beta-1}\right]$
$(0, \infty)$	$x^{\alpha-1}e^{-kx}\,dx$ $(\alpha, K > 0)$	Laguerre: $x^{1-\alpha}e^{kx}\frac{d^n}{dx^n}\left[x^{n+\alpha-1}e^{-kx}\right]$
$(-\infty, \infty)$	$e^{-kx^2}dx\,(K>0)$	Laplace-Hermite. $e^{kx^2}\frac{d^n}{dx^n}(e^{-kx^2})$

The polynomials $\phi_n(x)$ can be *normalized*, by multiplying by constant factors $a_n = 1 : \int_a^b \phi_n^2(x)d\psi$, so as to obtain *an orthogonal and normal* system of Tchebycheff polynomials $\{\phi_n(x) = a_n x^n + \cdots\}$ $(n = 0, 1, \cdots ; a_n > 0)$

$$(59) \qquad \int_a^b \phi_m(x)\phi_n(x)d\psi = 0\,(m \neq n), = 1\,(m = n)$$
$$(m, n = 0, 1, 2 \cdots\cdots)$$

The following are some of the most important properties of $\phi_n(x)$.

(a) The roots of $\phi_n(x)$ are real, distinct and lie between a, b.

(b) If all integrals $\int_a^b x^n f(x)d\psi(x)$ exist $(n = 0, 1, \cdots)$, then, by (59,) we have the formal development:

$$(60) \cdot \quad f(x) \backsim \sum_{i=0}^{\infty} A_i\,\phi_i(x), \qquad \left[A_i = \int_a^b f(x)\,\phi_i(x)\,d\psi(x)\right]^2$$

which, regardless of its convergence or divergence, has the following remarkable property, *any "section" ("Abschnitt") of (60), i. e. the polynomial* $P_n(x) = \sum_{i=0}^{n} A_i\,\phi_i(x)$, *obtained by taking its first* $n+1$ *terms* $(n = 0, 1, \cdots)$ *gives the best approximation to* $f(x)$ *in* (a, b), *in the sense of least squares, i. e it minimizes the integral* $\int_a^b [f(x) - P_n(x)]^2\, d\psi(x)$. *Moreover*

---

[1] Cf. W. Romanowsky, Sur quelques classes nouvelles des polynomes orthogonaux, Comptes Rendus, v. 188 (1929), p. 1023-25, where new polynomials are discussed arising from Pearson's frequency curves of type IV, V, VI.

[2] In the development $f(x) \backsim \sum_{i=0}^{\infty} A_i\,\phi_i(x)$, where the $\phi_i(x)$ are not normalized, $A_i = \int_a^b f\,\phi_i\,d\psi : \int_a^b \phi_i^2\,d\psi$.

$$(61)\qquad \int_a^b [f-P_n]^2 d\psi = \min. \int_a^b [f(x)-G_n(x)]^2 d\psi(x) = \int_a^b f^2 d\psi - \sum_{i=0}^{n} A_i^2 ,$$

$G_n(x) = \sum_{i=0}^{n} g_i x^i$ denoting hereafter an *arbitrary* polynomial of degree $\leq n$. The proof is very simple. Write $G_n(x)$ *as* $\sum_{i=0}^{n} H_i \phi_i(x)$ with constant coefficient H_i, substitute this expression into $I = \int_a^b [f-G_n]^2 d\psi$, and write down the conditions of minima: $\frac{1}{2}\frac{\partial I}{\partial H_i} = 0$, which, by (59), lead to

$$H_i = \int_a^b f\phi_i \, d\psi = A_i \qquad (i = 0, 1, 2, \cdots n).$$

These coefficients A_i can be written down as linear combinations of the moments

$$(62)\qquad m_k = \int_a^b f(x) x^k d\psi(x) \quad (k = 0, 1, \cdots\cdots).$$

Introduce the symbol

$$(63)\qquad \omega(G_n) = \sum_{i=0}^{n} m_i g_i$$

$$\left(G_n(x) = \sum_{i=0}^{n} g_i x^n;\ n = 0, 1, \cdots;\ g_i \text{ arbitrary}\right)$$

Then evidently,

$$(64)\qquad \begin{aligned} A_n &= \int_a^b f\phi_n \, d\psi = \omega(\phi_n) \\ f(x) &\infty \sum_{n=0}^{\infty} \omega(\phi_n)\, \phi_n(x) ; \end{aligned}$$

in other words, we have the following simple rule: *In the expression of* $\phi_n(x)$ *replace each power* x^k *by the corresponding moment* m_k *given in* (62)$(K = 0, 1, \cdots n)$, *and we obtain the coefficient* A_n *in* (60) $(n = 0, 1, \cdots\cdot)$.

(c) $\phi_n(x)$ are denominators of the successive convergents to the continued fraction

$$(65) \qquad \int_a^b \frac{d\psi(y)}{x-y} = \frac{\lambda_1 |}{| x-c_1} - \frac{\lambda_2 |}{| x-c_2} - \cdots$$

$$(\lambda_i (> 0), \quad c_i - const.).$$

Historically, it was the aforesaid minimum property which has lead Tchebycheff to the discovery and investigation of the *general class or orthogonal polynomials corresponding to any monotonic non-decreasing function,* while before, only isolated special cases of such polynomials have been known (polynomials of Legendre, Jacobi, Laguerre, Laplace, Hermite). Tchebycheff found these polynomials in connection with

VII. *Least-squares Interpolation.* The problem can be formulated with Tchebycheff[1] as follows *Given the values of a certain function* $y=F(x)$ *at* $n+1$ *real, distinct points* $x_1, x_2, \cdots x_{n+1}$, *with the corresponding weights* σ_i [2]. *Find its value at* $x=X$, *assuming for* y *the representation* $a+bx+cx^2+\cdots+hx^m$, $(m \leq n)$ *so that the errors of* $F(x_i)$ $[i=1, 2, \cdots n+1]$ *shall have the least possible influence on the required value* $F(x)$.

Using Stieltjes integrals (which greatly simplifies Tchebycheff's analysis), we are lead to the following solution:

$$(66) \qquad \begin{gathered} F(X) = P_m(X) = \sum_{k=0}^{m} A_k \phi_k(X) \\ \left[A_k = \int_a^b F(x)\,\phi_k(x)\,d\psi(x) = \sum_{i=1}^{n+1} \sigma_i F(x_i)\,\phi_k(x_i)\right], \end{gathered}$$

where $\psi(x)$ *is the stepwise function having at* $x=x_i$ *a saltus* $\sigma_i (i=1, 2, \cdots n+1)$, (a, b) *contains in its interior all points* x_i, $\{\phi_n(x)\}$ *are orthogonal and normal polynomials determined by* (59), *or, which is the same, denominators of the successive convergents to the continued fraction* (65) (we disregard constant factors), *which here reduces to*

$$(67) \qquad \sum_{i=1}^{n+1} \frac{\sigma_i}{x-x_i} = \frac{\lambda_1 |}{| x-c_1} - \frac{\lambda_2 |}{| x-c_2} - \cdots\cdots$$

---

1. Tchebycheff, (a) Sur les fractions continues, Journal des Mathematiques, (2), v. III (1858), p. 289-323; (b) On the least-squares interpolation, Collected Papers, v I, p. 473-98; (c) On interpolation with equidistant ordinates, ibid., v. II, p. 219-42 (b, c, in Russian).

2. σ_i is inversely proportional to the mean-square error of $F(x_i)$.

We see that (66) *is nothing but the first $m+1$ terms of the development* (60). Hence, *Tchebecheff's solution* (66) *yields the minimum of* $\int_a^b[F(x)-P_n(x)]^2\,d\psi(x)=\sum_{i=1}^n\sigma_i[F(x_i)-P_n(x)]^2$. Moreover, for the mean-square error of (66), we get, by (59):

$$R^2=\int_a^b F^2 d\psi-\sum_{k=0}^m A_k^2 \quad ,$$

$$(68) \qquad =\sum_{i+1}^{n+1}\sigma_i F^2(x_i)-\sum_{k=0}^m\left\{\sum_{i=1}^{n+1}\sigma_i\, F(x_i)\,\phi_k(x_i)\right\}^2 .$$

The name "least-squares interpolation" is thus fully justified, and we see the *complete identity between the two problems· least-squares interpolation and approximate representation of functions by series of Tchebycheff polynomials. Whether the data are discrete and in a finite number, or the form a continuous set, the underlying principles and the resulting formulae are identical, provided we use Stieltjes integrals.* There is no need to treat the two cases separately (as one finds even in recent books on this subject) and to introduce special symbols in the first case. Another very important feature of the above solution has been indicated by Tchebycheff If we add one more term to the expression $a+bx+\cdot+hx^m$ assumed for $y=F(x)$, we need only add one more term to $P_m(x)$ above, *without changing the preceding ones* (compare with Lagrange interpolation formula!) Formula (68) enables one to find the number of terms necessary to attain a prescribed accuracy.

Consider two special cases.

(a) *The ordinates are equidistant* : $x_{i+1}-x_i=h$ $(i=1,2,\cdots n)$ *and all weights* σ_i *are equal* $(=1)$ Here Tchebycheff (1-c. 1-b. p 91) gives very simple expressions for the polynomials $\phi_k(x)$, as well as for the coefficients A_k of (66) ·

$$\phi(z)=\Delta^k\left[\left(z+\frac{n-1}{2}\right)\left(z+\frac{n-3}{2}\right)\cdot\quad\cdot\left(z+\frac{n-2k+1}{2}\right)\left(z-\frac{n+1}{2}\right)\left(z-\frac{n+3}{2}\right)\right.$$

$$(69) \qquad \left.\cdots\left(z-\frac{n+2k-1}{2}\right)\right]\quad k=0,1,2,\cdots;\ z=\frac{x-\frac{1}{2}(x_1+x_n)}{x_2-x_1};$$

$$\Delta^k - k^{th}\ \text{difference}.$$

$$(70) \qquad u(z)\equiv F(x)=\frac{1}{n}\sum_{i=1}^n u_i\phi_0(z)+\frac{3}{n(n^2-1^2)}\sum_{i=1}^n\frac{i}{1}\cdot\frac{n-i}{1}\Delta u_i\,\phi_1(z)$$

$$+\frac{5}{n(n^2-1^2)(n^2-2^2)}\sum_{i=1}^n\frac{i(i+1)}{1\cdot2}\cdot\frac{(n-i)(n-i-1)}{1\cdot2}\Delta^2 u_i\,\phi_2(z)+\cdots$$

$$[(u_i\equiv F(x_i)]$$

(We have replaced $n+1$ in our above formulae by n). All $\phi_n(x)$ can be easily computed by means of the relations:

$$\phi_0(x) = \Delta^0 1 = 1 \quad ; \quad \phi_1(x) = 2x ,$$

(71) $$\phi_k(x) = 2(2k-1)\phi_{k-1}(x) - (k-1)^2\left[n^2-(k-1)^2\right]\phi_{k-2}(x)$$

$$(k \geq 2)$$

(b) $m=1$, x_i *arbitrary* $(i=1, 2, \cdots n)$. We take in (67)

(72) $$\lambda_1 = \int_a^b d\psi(x) = \sum_{i=1}^{n} \sigma_i .$$

We get now (by successive division, for ex.)

(73) $$c_1 = \frac{\int_a^b x\,d\psi}{\int_a^b d\psi} = \frac{\gamma_1}{\gamma_0}$$

$$\left(\gamma_k = \int_a^b x^k d\psi(x) = \sum_{i=1}^{n} \sigma_i x_i^k\right)$$

$$\phi_0(x) = \frac{1}{\sqrt{\gamma_0}}$$

(74) $$\phi_1(x) = \frac{x_1 - c_1}{\sqrt{\int_a^b (x-c_1)^2\,d\psi}} = \frac{\gamma_0 x - \gamma_1}{\sqrt{\gamma_0^2\gamma_2 - \gamma_1^2}}$$

(75) $$P_1(x) = A_0\phi_0(x) + A_1\phi_1(x)$$

$$= \sum_{i=1}^{n} \frac{\sigma_i y_i}{\gamma_0} + \sum_{i=1}^{n} \frac{\sigma_i y_i(\gamma_0 x_i - \gamma_1)}{\gamma_0^2\gamma_2 - \gamma_1^2}(\gamma_0 x - \gamma_1)$$

$$\left[y_i \equiv F(x_i)\right]$$

R^2 (mean-square error) $= \sum_{i=1}^{n} \sigma_i \left[F(x_i) - P_n(x_i)\right]^2$

(76) $$= \left(\sum_{i=1}^{n} \frac{\sigma_i y_i}{\sqrt{\gamma_0}}\right)^2 + \left(\sum_{i=1}^{n} \frac{\sigma_i' y_i(\gamma_0 x_i - \gamma_1)}{\sqrt{\gamma_0^2\gamma_2 - \gamma_1^2}}\right)^2$$

(See 68)

Let $\psi(x)$ *represent a law of distribution.* Then, $\gamma_0 = 1$, $\sqrt{\gamma_0^2\gamma_2 - \gamma_1^2} =$ standard deviation σ, and the above formulae become:

$$\phi_0(x)=1\,, \qquad \phi_1(x)=\frac{x}{\sigma} \tag{77}$$

$$P_1(x)=\frac{x\int_a^b F\,x\,d\psi}{\int_a^b x^2 d\psi} \tag{78}$$

$$=\frac{x\sum_{i=1}^{n}\sigma_i\,x_i\,y_i}{\sum_{i=1}^{n}\sigma_i\,x_i^2} \qquad \left[(y_i)=F(x_i)\right]$$

$$R^2=\int_a^b F^2 d\psi-\left(\int_a^b F\phi_1\,d\psi\right)^2 \tag{79}$$

$$=\sum_{i=1}^{n}\sigma_i\,y_i^2-\frac{\left(\sum_{i=1}^{n}\sigma_i\,x_i\,y_i\right)^2}{\sigma^2}$$

One recognizes in (78, 79) formulae quite similar to those for the line of regression of y on x and for the standard error of estimate of y. Introduce

$$\sigma_x^2=\int_a^b x^2 d\psi\,; \quad \sigma_y^2=\int_a^b y^2 d\psi\,, \tag{80}$$

$$r=\frac{\int_a^b x\,y\,d\psi}{\sqrt{\int_a^b x\,d\psi\cdot\int_a^b y\,d\psi}}$$

and our formulae become the classical ones:

$$P_1(x)=r\,\frac{\sigma_y}{\sigma_x}\,x\;;\quad R=\sigma_y\,(1-r^2)^{1/2}. \tag{81}$$

We thus obtained, using Stieltjes integrals, elegant, simple and easily memorizable formulae for σ_x, σ_y and for the coefficient of correlation r. Moreover, we see by inspection (Schwartz inequality) that $-1\leq r\leq 1$, equality attainable if and only if x and y are linearly dependent. We see also that *the theory of linear regression is but a very special case of the general theory — due to Tchebecheff — of least-squares interpolation.*

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1. Cf. D. Jackson The Elementary Geometry of Function Space, American Mathematical Monthly, v. 31 (1924), p. 461-71.

Paris (France).

SIMULTANEOUS TREATMENT OF DISCRETE AND CONTINUOUS PROBABILITY BY USE OF STIELTJES INTEGRALS

By

WILLIAM DOWELL BATEN

The object of this paper is to present several theorems pertaining to the probability that certain functions lie within certain intervals. The first theorem is a generalization of Markoff's Lemma, which is proven for the discrete and continuous cases by use of the accumulative frequency function and Stieltjes integrals Tchebycheff's Theorem is obtained as a corollary to a very general theorem, the proof of which is based upon the first theorem. Other corollaries are given.

Three theorems, due to Guldberg, which follow are concerned with the probability that a non-negative chance variable be less than certain functions of the expected value of the variable. These are proved for the discrete and continuous cases by employing accumulative frequency functions and Stieltjes integrals. This is the first time, as far as the writer knows, the discrete and continuous cases for these theorems have been included in a single proof.

Theorem 1 If A denotes the expected value of the non-negative variable x and t is any number greater than 1, then the probability that $x \leq At^2$ is greater than $1-1/t^2$.

Proof: If x is a discrete variable with values at x_i, ($i = 1, 2, \cdots, n$) with corresponding probabilities p_i, then it is understood that the probability that x takes other values is zero. If x is a continuous variable having a probability function defined over the interval (a, b), then it is understood that the probability that x lies outside of (a, b) is zero in case (a, b) is different from $(-\infty, +\infty)$. In both cases x is a continuous variable in the interval $(-\infty, +\infty)$. Let the probability that x lies in the interval $(-\infty, x)$ be $F(x)$, with $F(-\infty) = 0$ and $F(+\infty) = 1$. Then the probability that x lies in the interval (x_1, x_2) is

$$F(x_2)-F(x_1)+\frac{1}{2}\left\{F(x_2+0)-F(x_2-0)\right\}+\frac{1}{2}\left\{F(x_1+0)-F(x_1-0)\right\}$$

where the last two limits are different from zero when there is probability different from zero at x_1 and x_2. This exists since $F(x)$ is a non-decreasing function over the interval $(-\infty, +\infty)$. In the special case when $F(x) = \int_{-\infty}^{x} f(x)\,dx$ where $f(x)$ is summable, $f(x)\,dx$ represents the probability that x lies in the interval $(x, \quad x+dx)$.

In either case, by definition

$$A = \int_{-\infty}^{\infty} x \; dF(x)$$

$x > At^2$ in the interval $(At^2+\epsilon, \infty)$, where ϵ approaches 0, hence

$$A > \int_{At^2+0}^{\infty} x \cdot dF(x)$$

But

$$\int_{At^2+0}^{\infty} x\,dF(x) = \lim_{\epsilon \to 0} \int_{At^2+\epsilon}^{\infty} x\,dF(x) \doteq \lim_{\epsilon \to 0} z_\epsilon \int_{At^2+\epsilon}^{\infty} dF(x) = \lim_{\epsilon \to 0} Z_\epsilon \cdot \lim \int_{At^2+\epsilon}^{\infty} dP(x)$$

by the first theorem of the mean, which holds for Stieltjes integrals in this case Here $z > At^2 + \epsilon$, hence $\lim_{\epsilon \to 0} z_\epsilon \geq At^2$, therefore $A > At^2 \int_{At^2\,0}^{\infty} dF(x)$ But $\int_{At^2\,0}^{\infty} dF(x)$ is the probability P that x is greater that At^2, hence

$$A > At^2 P, \quad Q > 1 - 1/t^2$$

where Q is the probability that $x \leq At^2$

This theorem is a generalization of Markoff's Lemma[1], which he proved for the discrete case. The above proof takes care of the discrete case, the continuous case and the case which is a combination of the discrete and continuous.

Theorem 2. If $f(x_1, x_2, \cdots \cdot x_n)$ is a function of n independent variables, then the probability that

$$|f-k| \leq t\sqrt{E(f^2) - 2kE(f) + k^2}$$

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[1] "Wahrscheinlichkeitsrechnung," by Markoff 1912. Page 54.

is greater than $1-1/t^2$; where E represents the expected value, k is a constant and $t>1$.

Proof: Let

$$y=\{f(x_1,x_2,\cdots x_n)-k\}^2 \quad \text{then}$$
$$E(y)=E(f^2)-2kE(f)+k^2$$

By theorem 1 the probability that

$$|f-k|\leq t\sqrt{E(f^2)-2kE(f)+k^2}$$

is greater than $1-1/t^2$.

Corollary: If $f(x_1,x_2,\cdots x_n)=x_1+x_2+\cdots x_n$ and $k=\sum_{i=1}^{n}E(x_i)$, then theorem 2 becomes the famous Tchebycheff theorem[1]. This theorem is: If $x_1, x_2, \cdots x_n$ be n independent variables, then the probability that

$$\left|\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}E(x_i)\right|\leq t\sqrt{\sum_{i=1}^{n}E(x_i^2)-2\sum_{\substack{i,j=1\\ i\neq j}}^{n}E(x_ix_j)+\left\{\sum_{i=1}^{n}E(x_i)\right\}^2},$$

is greater than $1-1/t^2$.

This proof is by far simpler than that given by Tchebycheff, while it is similar to that given by Markoff.

In the corollary if $k=\sum_{i=1}^{n}E(x_i)$, $E(x_i)=a$ $E(x_i^2)=A$, then the probability that

$$\left|\frac{\sum x_i}{n}-a\right|\leq\frac{t}{\sqrt{n}}\sqrt{A-a}$$

is greater than $1-1/t^2$.

If $f(x_1,x_2,\cdots x_n)=\sum_{i=1}^{n}x_i^s$ then the probability that

$$\left|\sum_{i=1}^{n}x_i^s-k\right|\leq t\sqrt{\sum_{i=1}^{n}A_{i,2s}+2\sum_{i=1}^{n}a_{i,s}\,a_{j,s}-2k\sum_{i=1}^{n}a_{i,s}+k^2}$$

is greater than $1-1/t^2$, where the variables are independent, $E(x_i^{2s})=A_{i,2s}$; $E(x_i^s)=a_{i,s}$. If s is negative it is under-

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1. "Des Valeurs Moyennes," by Tchebecheff, Journal de Math 1867 (2). Vol. 12.

stood that x can not take on the value zero, if $s = a/b$, where a is odd and b is even, it is understood that x is non-negative.

Other interesting results may be obtained from this theorem if $f(x_1, x_2, \cdot\cdot\cdot\cdot\cdot\cdot\cdot x_n)$ represents various functions of the n independent variables and k be given different values. If f is the sum of the variables, theorem 2 is a more general theorem than Tchebycheff's theorem because of the constant k which may have values other than $\sum_{i=1}^{n} E(x_i)$.

Let x_i be the result of an individual throw of a coin, $x_i = 1$ if a head is thrown and $x_i = 0$ if a tail is thrown, then $E(x_i^2) = p \cdot 1 + q \cdot 0$, where p is the probability of a head and q is the probability of a tail. Let m represent the number of heads thrown in n throws and let $k = np \pm \sqrt{n - npq}$, then the probability that

$$\left| m - (np \pm \sqrt{n - npq}) \right| \leq t\sqrt{n}, \text{ or that}$$

$$\left| \frac{m}{n} - \left(p \pm \sqrt{\frac{1}{n} - \frac{pq}{n}} \right) \right| \leq \frac{t}{\sqrt{n}}$$

, is greater than $1 - 1/t^2$.

Let $t = \sqrt[4]{n}$, then the probability that

$$-\frac{1}{\sqrt[4]{n}} \pm \sqrt{\frac{1}{n} - \frac{pq}{n}} \leq \frac{m}{n} - p \leq \frac{1}{\sqrt[4]{n}} \pm \sqrt{\frac{1}{n} - \frac{pq}{n}}$$

is greater than $1 - 1/t^2$ or $1 - 1/\sqrt{n}$, which approaches unity as n increases. It is near unity for large values of n. This shows that the empirical probability approaches the true probability p as the number of throws increases, and the advantage of k.

Theorem 3. Let $u'_{n,x}$ be the expected value of the non-negative variable x raised to the power n and t any number greater than 1, then the probability that $x \leq t\sqrt[n]{u'_{n,x}}$ is greater than $1 - 1/t^n$.

Proof. Let $c > \sqrt[n]{u'_{n,x}}$, and let $F(x)$ be the probability that x lies in the interval $(-\infty, x)$, then by definition

$$u'_{n,x} = \int_{-\infty}^{\infty} x^n dF(x); \quad and \quad \frac{u'_{n,x}}{c} = \int_{-\infty}^{\infty} x^n dF(x)/c^n$$

Now

$$\frac{u'_{n\cdot x}}{c^n} > \int_{-\infty}^{\infty} x^n dF(x)/c^n = \lim_{\epsilon\to 0} \int_{c+\epsilon}^{\infty} x^n dF(x)/c^n$$

$$= \lim_{\epsilon\to 0} \frac{(z_\epsilon)^n}{c^n} \cdot \lim_{\epsilon\to 0} \int_{c+\epsilon}^{\infty} dF(x) > i \cdot \int_{c+0}^{\infty} dF(x),$$

by the first theorem of the mean, and since $\lim\limits_{\epsilon\to 0} \frac{(z_\epsilon)^n}{c^n} \geq 1.$

Since $\int_{c+0}^{\infty} dF(x)$ is the probability P that x is greater than c,

$$\frac{u'_{n:x}}{c^n} > p \quad or \quad Q > 1 - \frac{u_{n:x}}{c^n}$$

where Q is the probability that $x \leq c$.

Let $t = \frac{c}{\sqrt[n]{u'_{n\,x}}}$, then $\frac{1}{t^n} = \frac{u'_{n.x}}{c^n}$, hence

$$Q > 1 - 1/t^n .$$

But Q becomes the probability that $x \leq \sqrt[n]{u'_{n\,x}}$, since c was any number greater than $\sqrt[n]{u'_{n\,x}}$.

Let $y = |x-k|$, then theorem 3 becomes: If $u'_{n\,y}$ is the expected value of $|x-k|^n$ and t is greater than 1, then the probability that $|x-k|$ does not surpass the multiple $t\sqrt[n]{u'_{n.y}}$, is greater than $1-1/t^n$, where k is a constant

If $k = \int_{-\infty}^{\infty} x \cdot dF(x)$, then $u'_{n\,y}$ becomes $u'_{n\,\bar{x}}$ and theorem 3 states that the difference $|x-k|$ does not surpass the multiple $t\sqrt[n]{u'_{n\,\bar{x}}}$, is greater than $1-1/t^n$. In this special case theorem 3 becomes Guldberg's theorem[1], but this is more general than his theorem, for it includes the continuous case, the discrete case and the case which is a combination of the discrete and continuous

If $y = |f(x)-k|$ is used for the variable, a more general theorem is obtained Here $f(x)$ is a function of x. Of course, the probability law for $f(x)$ must be secure from that of x if the continuous case is under consideration Certain restrictions must be placed upon $f(x)$ concerning continuity, summability and concerning the inverse

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[1] "Sur un théorème de M Markoff," by Alf. Guldberg. Compte Rendue, Vol 175 (1922) page 679.

Theorem 4. The probability that the difference $|x-m|$ is not greater than the multiple $t\,u_{r,x}$, $t > 1$, is greater than $1-\left(\sqrt[n]{u'_{1,x}}/u'_{r,x}\right)^n$ $(1/t^n)$, where $u_{r,x}$ is the expected value of $|x-m|^n$, and m is the expected value of x.

Theorem 5. The probability that the positive quantity x does not surpass the multiple tm, $(t > 1)$, is greater than $1-\left(\frac{\sqrt[n]{u'_{1,x}}}{m}\right)^n \cdot \frac{1}{(t-1)^n}$, where $u_{n,x}$ is the expected value of $|x-m|^n$ and $m = E(x)$.

These last two theorems are due to Guldberg[1] for the discrete case. By the method used in theorem 3 these can be proven for the continuous case, the discrete case, and the case which is a combination of the discrete and continuous.

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[1] "Sur quelques inéqualités des le calcul de probabilités," by Guldberg Comp Rend Vol 175 (1922), p. 1382.
"Sur le théorème de, Tchebecheff," by Guldberg. Comptes Rendue, Vol 175 (1922), p. 418.

EDITORIAL

FUNDAMENTALS OF THE THEORY OF SAMPLING

I Sampling from a Limited Supply

We shall consider first a population of s individuals, in which each individual possesses a common attribute that can be measured quantitatively. The sum of the associated variates may be expressed as follows.

$$x_1 + x_2 + x_3 + \qquad x_s = \sum^{s} x = sM_x$$

From this so-called *parent population* it is possible to select $\binom{s}{r}$ different *samples*, each consisting of r individuals, $(r \leq s)$. These samples may be ordered after any fashion, and the algebraic sum of the variates for the respective samples may be designated

$$\begin{array}{lllllll} z_1 = x_1 & + x_2 & + x_3 & \cdot \quad \cdot \quad \cdot & + x_r & = \sum^{r,1} x \\ z_2 = x_2 & + x_3 & + x_4 & \cdot \qquad \cdot & + x_{r+1} & = \sum^{r,2} x \\ \quad \cdot & \cdot & \cdot & \cdot \quad \cdot & & \\ \cdot \quad \cdot & \cdot & \cdot & \cdot & & \\ \quad \cdot & & & \cdot \quad \cdot \quad \cdot \quad \cdot & & \\ z_{\binom{s}{r}} = x_{s-r+1} & + x_{s-r+2} & & \cdot \quad \cdot & + x_s & = \sum^{r,\binom{s}{r}} x \end{array}$$

Thus, while $\sum^{s} x$ represents the sum of all the s variates in the parent population, $\sum^{r,i} x$ designates the sum of the r variates occurring in the i th sample.

We face now the problem of describing adequately, from a statistical point of view, the distribution of these $\binom{s}{r}$ values of z, that is to say, we must express the moments $\mu_{n:z}$ in terms of the moments of the parent population, $\mu_{n:x}$.

By definition $\qquad M_z = \dfrac{\sum z}{\binom{s}{r}}$

Since each value of z will contribute r terms to the value of $\sum z$, this latter expression will consist of $r\cdot\binom{s}{r}$ terms involving each of the s variates of the parent population alike. Therefore, each variate, x_i ($i = 1, 2, 3, \ldots s$), will occur in the expression for $\sum z$ exactly $\frac{r}{s}\cdot\binom{s}{r}$ times. Consequently

$$(1) \quad M_z = \frac{\sum z}{\binom{s}{r}} = \frac{1}{\binom{s}{r}} \frac{r}{s} \binom{s}{r} \{x_1 + x_2 + \cdots x_s\} = \frac{r}{s} \sum^{s} x = r M_x$$

We shall now investigate the values of

$$\mu_{n\,z} = \frac{\sum \bar{z}^{\,n}}{\binom{s}{r}}$$

where we choose to represent a deviation from the mean as

$$\bar{z}_i = z_i - M_z$$

Observing that

$$\bar{z}_1 = z_1 - M_z = x_1 + x_2 + \cdots x_r - rM_x = \bar{x}_1 + \bar{x}_2 + \cdots \bar{x}_r$$

we note that

$$\bar{z}_1^{\,2} = \sum^{r,1} \bar{x}^{\,2} + 2\sum^{r,1} \bar{x}_i \bar{x}_j$$

$$\bar{z}_2^{\,2} = \sum^{r,2} \bar{x}^{\,2} + 2\sum^{r,2} \bar{x}_i \bar{x}_j$$

$$\cdots$$

$$\bar{z}_{\binom{s}{r}}^{\,2} = \sum^{r,\binom{s}{r}} \bar{x}^{\,2} + 2\sum^{r\binom{s}{r}} \bar{x}_i \bar{x}_j$$

Therefore

$$\mu_{2\,z} = \frac{\sum \bar{z}^{\,2}}{\binom{s}{r}} = \frac{1}{\binom{s}{r}} \left\{ \frac{r\cdot\binom{s}{r}}{s} \sum^{s} \bar{x}^{\,2} + 2\,\frac{\binom{r}{2}\binom{s}{r}}{\binom{s}{2}} \sum^{s} \bar{x}_i \bar{x}_j \right\},$$

or, writing

$$\rho_i = \frac{r^{(i)}}{s^{(i)}} = \frac{r(r-1)(r-2)\cdots\cdots(r-\overline{i-1})}{s(s-1)(s-2)\qquad\qquad(s-\overline{i-1})}$$

$$(2a) \qquad \mu_{2\,z} = 2!\left\{ \rho_1 \frac{\sum^{s} \bar{x}^{\,2}}{2!} + \rho_2 \frac{\sum^{s} \bar{x}_i \bar{x}_j}{(1!)^2} \right\}$$

By utilizing further the multinomial theorem, it follows easily that

$$(3a) \qquad \mu_{3x} = 3!\left\{\rho_1 \frac{\sum^{s} \bar{x}^3}{3!} + \rho_2 \frac{\sum^{s} \bar{x}_i^2 \bar{x}_j}{2!\,1!} + \rho_3 \frac{\sum^{s} \bar{x}_i \bar{x}_j \bar{x}_k}{(1!)^3}\right\}$$

$$(4a) \qquad \mu_{4x} = 4!\left\{\rho_1 \frac{\sum^{s} \bar{x}^4}{4!} + \rho_2 \frac{\sum^{s} \bar{x}_i^3 \bar{x}_j}{3!\,1!} + \rho_2 \frac{\sum^{s} \bar{x}_i^2 \bar{x}_j^2}{(2!)^2}\right.$$

$$\left. + \rho_3 \frac{\sum^{s} \bar{x}_i^2 \bar{x}_j \bar{x}_k}{2!\,(1!)^2} + \rho_4 \frac{\sum^{s} \bar{x}_i \bar{x}_j \bar{x}_k \bar{x}_l}{(1!)^4}\right\}$$

, etc.

The rule for writing down the terms is as follows. The number of terms in the expression for $\mu_{n,x}$ equals the number of partitions that can be formed from the integer n. The subscript of ρ equals the number of elements in the corresponding partition, and exponents of $\bar{x}$ and the factorials in the denominators are in fact the elements of the partitions.

Our next problem is to express the summations in terms of moments of the parent population, $\mu_{n\,x}$.

First order summation

$$\sum^{s} \bar{x} = s\,\mu_{1x} = 0$$

Second order summations

$$\sum \bar{x}^2 = s\,\mu_{2\,x}$$

$$2\sum \bar{x}_i \bar{x}_j = -s\,\mu_{2\,x}$$

since $$\Big(\sum^{s} \bar{x}\Big)^2 = 0 = \sum^{s} \bar{x}^2 + 2\sum^{s} \bar{x}_i \bar{x}_j$$

Third order summations

$$\sum^{s} \bar{x}^3 = s\,\mu_{3\,x}$$

$$\sum^{s} \bar{x}_i^2 \bar{x}_j = -s\,\mu_{3\,x}$$

$$3\sum^{s} \bar{x}_i \bar{x}_j \bar{x}_k = s\,\mu_{3\,x}\,.$$

since $\sum^{s} \bar{x}^{2} \sum^{s} \bar{x} = 0 = \sum^{s} \bar{x}^{3} + \sum^{s} \bar{x}_i^{2} \bar{x}_j$

and $\left(\sum^{s} \bar{x}\right)^{3} = 0 = \sum^{s} \bar{x}^{3} + 3\sum^{s} \bar{x}_i^{2} \bar{x}_j + 6\sum^{s} \bar{x}_i \bar{x}_j \bar{x}_k$

Fourth order summations

$$\sum^{s} \bar{x}^{4} = s\mu_{4x}$$

$$\sum^{s} \bar{x}_i^{3} \bar{x}_j = -s\mu_{4\cdot x}$$

$$2\sum^{s} \bar{x}_i^{2} \bar{x}_j^{2} = -s\mu_{4.x} + s^{2}\mu_{2.x}^{2}$$

$$2\sum^{s} \bar{x}_i^{2} \bar{x}_j \bar{x}_k = 2s\mu_{4.x} - s^{2}\mu_{2.x}^{2}$$

$$8\sum^{s} \bar{x}_i \bar{x}_j \bar{x}_k \bar{x}_l = -2\,s\mu_{4\cdot x} + s^{2}\mu_{2.x}^{2}$$

Utilizing these summations, (2a), (3a) and (4a) may be written

(2) $\mu_{2\bar{x}} = s\mu_{2x}\{\rho_1 - \rho_2\}$

(3) $\mu_{3\bar{x}} = s\mu_{3\cdot x}\{\rho_1 - 3\rho_2 + 2\rho_3\}$

(4) $\mu_{4\bar{x}} = s\mu_{4x}\{\rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4\} + 3s^{2}\mu_{2\cdot x}^{2}\{\rho_2 - 2\rho_3 + \rho_4\}$.

Continuing after this fashion, one can show after a lavish use of symmetric functions that

(5) $\mu_{5\bar{x}} = s\mu_{5x}\{\rho_1 - 15\rho_2 + 50\rho_3 - 60\rho_4 + 24\rho_5\}$
$\quad + 10s^{2}\mu_{3x}\mu_{2x}\{\rho_2 - 4\rho_3 + 5\rho_4 - 2\rho_5\}$,

(6) $\mu_{6\bar{x}} = s\mu_{6x}\{\rho_1 - 31\rho_2 + 180\rho_3 - 390\rho_4 + 360\rho_5 - 120\rho_6\}$
$\quad + 15s^{2}\mu_{4\cdot x}\mu_{2x}\{\rho_2 - 8\rho_3 + 19\rho_4 - 18\rho_5 + 6\rho_6\}$
$\quad + 10s^{2}\mu_{3\cdot x}^{2}\{\rho_2 - 6\rho_3 + 13\rho_4 - 12\rho_5 + 4\rho_6\}$
$\quad + 15s^{3}\mu_{2x}^{3}\{\rho_3 - 3\rho_4 + 3\rho_5 - \rho_6\}$,

$$
\begin{aligned}
(7)\quad \mu_{7x} = {} & s\mu_{7x}\{\rho_1 - 63\rho_2 + 602\rho_3 - 2100\rho_4 + 3360\rho_5 \\
& \qquad - 2520\rho_6 + 720\rho_7\} \\
& + 21s^2\mu_{5\cdot x}\mu_{2x}\{\rho_2 - 16\rho_3 + 65\rho_4 - 110\rho_5 \\
& \qquad + 84\rho_6 - 24\rho_7\} \\
& + 35s^2\mu_{4.x}\mu_{3,x}\{\rho_2 - 10\rho_3 + 35\rho_4 - 56\rho_5 + 42\rho_6 - 12\rho_7\} \\
& + 105s^3\mu_{3x}\mu_{2x}^2\{\rho_3 - 5\rho_4 + 9\rho_5 - 7\rho_6 + 2\rho_7\},
\end{aligned}
$$

$$
\begin{aligned}
(8)\quad \mu_{8x} = {} & s\mu_{8x}\{\rho_1 - 127\rho_2 + 1932\rho_3 - 10206\rho_4 \\
& + 25200\rho_5 - 31920\rho_6 + 20160\rho_7 - 5040\rho_8\} \\
& + 28s^2\mu_{6x}\mu_{2x}\{\rho_2 - 32\rho_3 + 211\rho_4 - 570\rho_5 \\
& \qquad + 750\rho_6 - 480\rho_7 + 120\rho_8\} \\
& + 56s^2\mu_{5.x}\mu_{3x}\{\rho_2 - 18\rho_3 + 97\rho_4 - 240\rho_5 + 304\rho_6 \\
& \qquad - 192\rho_7 + 48\rho_8\} \\
& + 35s^2\mu_{4x}^2\{\rho_2 - 14\rho_3 + 73\rho_4 - 180\rho_5 + 228\rho_6 - 144\rho_7 + 36\rho_8\} \\
& + 210s^3\mu_{4.x}\mu_{2.x}^2\{\rho_3 - 9\rho_4 + 27\rho_5 - 37\rho_6 + 24\rho_7 - 6\rho_8\} \\
& + 280s^3\mu_{3.x}^2\mu_{2x}\{\rho_3 - 7\rho_4 + 19\rho_5 - 25\rho_6 + 16\rho_7 - 4\rho_8\} \\
& + 105s^4\mu_{2x}^4\{\rho_4 - 4\rho_5 + 6\rho_6 - 4\rho_7 + \rho_8\}.
\end{aligned}
$$

It is convenient, at this point, to define the "*n th sampling polynomial*" as follows:

$$(9)\qquad P_n(\rho) = D_x^n \log(\rho e^x + 1 - \rho)\Big]_{x=0}$$

If we place $y = \log(\rho e^{x} + 1 - \rho)$, then

$$y^{(1)} = \frac{\rho e^{x}}{\rho e^{x} + 1 - \rho} \quad \text{or} \quad (\rho e^{x} + 1 - \rho)\, y^{(1)} = \rho e^{x}$$

Taking the nth derivative of both sides by utilizing Leibnitz' Theorem, we obtain

$$(\rho e^{x} + 1 - \rho)\, y^{(n+1)} + \binom{n}{1} \rho e^{x} y^{(n)} + \binom{n}{2} \rho e^{x} y^{(n-1)} + \cdot \quad \cdot = \rho e^{x}.$$

Placing $x = 0$ in this equation yields, by definition,

$$P_{n+1}(\rho) + \binom{n}{1} \rho P_n(\rho) + \binom{n}{2} \rho P_{n-1}(\rho) + \cdot \quad \cdot = \rho$$

That is, for $n = 0,\ 1,\ 2,\ \cdot\ \cdot\ \cdot$

$$P_1(\rho) = \rho$$

$$P_2(\rho) + \rho P_1(\rho) = \rho$$

$$P_3(\rho) + 2\rho P_2(\rho) + \rho P_1(\rho) = \rho$$

$$P_4(\rho) + 3\rho P_3(\rho) + 3\rho P_2(\rho) + \rho P_1(\rho) = \rho$$

etc

Thus.

$$(10) \quad \begin{cases} P_1(\rho) = \rho \\ P_2(\rho) = \rho - \rho^2 \\ P_3(\rho) = \rho - 3\rho^2 + 2\rho^3 \\ P_4(\rho) = \rho - 7\rho^2 + 12\rho^3 - 6\rho^4 \\ P_5(\rho) = \rho - 15\rho^2 + 50\rho^3 - 60\rho^4 + 24\, p^5 \\ P_6(\rho) = \rho - 3\rho^2 + 180\rho^3 - 390\rho^4 + 360\rho^5 - 120\rho^6 \\ P_7(\rho) = \rho - 63\rho^2 + 602\rho^3 - 2100\rho^4 + 3360\rho^5 - 2520\rho^6 + 720\rho^7 \\ P_8(\rho) - \rho - 127\rho^2 + 1932\rho^3 - 10206\rho^4 + 25200\rho^5 - 31920\rho^6 \end{cases}$$

$$\left[+ 20160\rho^7 - 5040\rho^8\right]$$

etc

The law of formation of the coefficients is obvious. for if $c_{i,n}$ designates the coefficient of ρ^i in the expression for $P_n(\rho)$,

$$c_{i,n} = i c_{i,n-1} - (i-1) c_{i-1,n-1}$$

Comparing the polynomials of equations (9) with formulae (2) to (8) inclusive, suggests writing the expressions for $\mu_{n,x}$ in the following symbolic form.

$$\begin{aligned}
&\mu_{2,x} = 2!\left\{P_2 \frac{s\mu_{2,x}}{2!}\right\} \\
&\mu_{3,x} = 3!\left\{P_3 \frac{s\mu_{3,x}}{3!}\right\} \\
(11)\quad &\mu_{4,x} = 4!\left\{P_4 \frac{s\mu_{4,x}}{4!} + \frac{P_2^2}{2!} \frac{s\mu_{2,x}^2}{(2!)^2}\right\} \\
&\mu_{5,x} = 5!\left\{P_5 \frac{s\mu_{5,x}}{5!} + P_3 P_2 \frac{s^2\mu_{3,x}\mu_{2,x}}{3!\,2!}\right\} \\
&\mu_{6,x} = 6!\left\{P_6 \frac{s\mu_{6,x}}{6!} + P_4 P_2 \frac{s^2\mu_{4,x}\mu_{2,x}}{4!\,2!} + \frac{P_3^2}{2!} \frac{s^2\mu_{3,x}^2}{(3!)^2}\right. \\
&\qquad\qquad \left. + \frac{P_2^3}{3!} \frac{s^3\mu_{2,x}^3}{(2!)^3}\right\} \\
&\qquad\qquad\qquad \text{etc}
\end{aligned}$$

By P_n we understand an expression derived from the sampling polynomial, $P_n(\rho)$, by writing ρ^i as ρ_i. Thus,

$$P_4(\rho) = \rho - 7\rho^2 + 12\rho^3 - 6\rho^4 \text{, whereas}$$

$$P_4 = \rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4$$

Again, since

$$P_3(\rho)\cdot P_1(\rho)\cdot P_1(\rho) = (\rho - 3\rho^2 + 2\rho^3)\rho\cdot\rho = \rho^3 - 3\rho^4 + 2\rho^5,$$

$$P_3 P_1^2 = \rho_3 - 3\rho_4 + 2\rho_5$$

The number of terms in the expression for $\mu_{n,x}$ will equal the number of partitions that can be formed from the integer n. The subscripts of the P and μ factors for any selected term correspond to the elements of the corresponding partition, and the exponent of s equals the number of elements in the partition. The factorials beneath the μ factors agree with the order of these moments, and the factorials appearing occasionally under the P factors depend upon the number of times that any P is repeated as a factor in that term All terms arising from a partition in which unity is an element have been neglected, since such terms will contain $\mu_{1,x}$ as a factor and consequently be equal to zero.

Illustration I. For the parent population we shall select the following (it will be noted that graphically the ordinates terminate on the hypotenuse of an isosceles right triangle):

TABLE I

Parent Population

x	f_x
1	24
2	23
3	22
4	21
5	20
.	. .
22	3
23	2
24	1
Total	300

The mean, standard deviation and moments about the mean for this distribution are as follows:

$M_x = 8.666$

$\mu_{2,x} = 33.222$ $\qquad \sigma_x = 5.76387$

$\mu_{3,x} = 108.526$ $\qquad \alpha_{3,x} = .566749$

$\mu_{4,x} = 2642.27$ $\qquad \alpha_{4,x} = 2.39398$

$\mu_{5,x} = 20525.2$ $\qquad \alpha_{5,x} = 5.69279$

$\mu_{6,x} = 322570$ $\qquad \alpha_{6,x} = 27.3878$

It may well be remarked at this point that the standard variate corresponding to an observed variate, x_i is

$$t_i = \frac{x_i - M_x}{\sigma_x} = \frac{\bar{x}_i}{\sigma_x} , \tag{12}$$

and is consequently an *abstract number.* The n th moment of the standard variates is also without unit, i. e.

$$\alpha_{n|x} = \frac{\Sigma t^n}{N} = \frac{1}{N\sigma_x^n} \Sigma \bar{x}_i^n = \frac{\mu_{n|x}}{\sigma_x^n} \tag{13}$$

In dealing with distributions one should always bear in mind that the mean and standard deviation determine merely the *position* of the centroid vertical and the *scale* of the distribution, but that the standard moments are influenced by the *shape* of the distribution alone. Consequently a study of the mathematical representation of frequency distributions is essentially an investigation concerning the standard moments of observed and theoretical distributions

From the above parent population it would be possible to select $\binom{300}{25}$ samples, each consisting of 25 individuals To describe the distribution of these sampes, we proceed as follows.

$\rho_1 = \quad = .08333$

$\rho_2 = \rho_1 \cdot \frac{24}{299} = .0066 \; 8896 \; 32$

$\rho_3 = \rho_2 \cdot \frac{23}{298} = .0005 \; 1626 \; 226$

$\rho_4 = \rho_3 \cdot \frac{22}{297} = .0000 \; 3824 \; 1649$

$\rho_5 = \rho_4 \cdot \frac{21}{296} = .0000 \; 0271 \; 3090 \; 0$

$\rho_6 = \rho_5 \cdot \frac{20}{295} = 0000 \; 0018 \; 3938 \; 31$

$P_2 = .0766 \; 4437 \; 0$

$P_3 = .0642 \; 9896 \; 8$

$P_4 = .0424 \; 7628 \; 8$

$P_2^2 = .0056 \; 9468 \; 03$

$P_5 = .0065 \; 8261 \; 36$

$P_3 P_2 = .0048 \; 0969 \; 62$

$P_6 = .0450 \; 5692 \; 2$

$P_4 = .0032 \; 3772 \; 45$

$P_3^2 = .0040 \; 5670 \; 98$

$P_2^3 = .0004 \; 0949 \; 264$

and found that the distribution of the totals of 1000 samples of twenty-five variates each was as follows:

TABLE IV

Class	Freq
5000–	2
7000–	54
9000–	203
11000–	310
13000–	254
15000–	130
17000–	36
19000–	9
21000–	2
Total	1000

As a matter of fact, if r is fifty or greater and s is at least ten times as large as r, the parent population has relatively little control over the shape of the distribution of samples. But before investigating the limit towards which distributions of samples approach in shape, it is well to present a second illustration of the theory so far developed.

Illustration II Pearson's Hypergeometric Series.

If from a bag containing qs black and ps white balls, r balls are withdrawn without replacements, the chances that the r balls withdrawn will contain 0, 1, 2, .., x, . . . r white balls are given by the successive terms of the hypergeometric series

$$(16)\quad \frac{1}{\binom{s}{r}}\left\{\binom{qn}{r}+\binom{qn}{r-1}\binom{pn}{1}+\cdots\binom{qn}{r-x}\binom{pn}{x}+\cdots\binom{pn}{r}\right\}$$

A distribution of this type is equivalent to the simplest case that can arise in accordance with the theory of sampling, that is, by assuming that each variate of the parent population is equal to either zero or one, and that p denotes the proportion of the s variates that have unit value. The moments of the parent population are found as follows:

TABLE V

Parent Population for Hypergeometric (and Binomial) Series

x	f_x	xf_x	$x-M_x=\bar{x}$	$(x-M_x)^n f_x = \bar{x}^n f_x$
0	$(1-p)s$	0	$-p$	$(-1)^n p^n (1-p)s$
1	ps	ps	$1-p$	$p(1-p)^n \cdot s$
Total	s	ps		$p(1-p)s\{(1-p)^n+(-1)^n p^n\}$

Therefore

$$(17)\quad \mu_{n\cdot x}=p(1-p)\{(1-p)^{n-1}+(-1)^n p^{n-1}\}=pq\{q^{n-1}+(-1)^n p^{n-1}\},$$

where $(p+q=1)$

In numerical problems this formula should be used ordinarily as it stands, although for algebraic purposes we may use frequently the forms

$$\mu_{1\cdot x}=0$$

$$\mu_{2\cdot x}=pq=p(1-p)$$

$$\mu_{3\cdot x}=pq\ (q^2-p^2)=p(1-p)(1-2p)$$

$$\mu_{4\cdot x}=pq\ (q^3+p^3)=p(1-p)(1-3p+3p^2)$$

etc.

Using formulae 2, . . ., we may write the moments for the hypergeometric series as follows:

$$\mu_{2\,x}=s\mu_{2\cdot x}\{\rho_1-\rho_2\}$$

$$\mu_{3\,x}=s\mu_{3\,x}\{\rho_1-3\rho_2+2\rho_3\}$$

etc.

or if one prefers

$$\mu_{2,s} = spq\left\{\frac{r}{s} - \frac{r^{(2)}}{s^{(2)}}\right\}$$

$$\mu_{3,s} = spq\,(q^2-p^2)\left\{\frac{r}{s} - 3\,\frac{r^{(2)}}{s^{(2)}} + 2\,\frac{r^{(3)}}{s^{(3)}}\right\}$$

$$\mu_{4,s} = spq\,(q^3+p^3)\left\{\frac{r}{s} - 7\,\frac{r^{(2)}}{s^{(2)}} + 12\,\frac{r^{(3)}}{s^{(3)}} - 6\,\frac{r^{(4)}}{s^{(4)}}\right\}$$

$$+3s^2p^2q^2\left\{\frac{r^{(2)}}{s^{(2)}} - 2\,\frac{r^{(3)}}{s^{(3)}} + \frac{r^{(4)}}{s^{(4)}}\right\}$$

etc.

These will be found equivalent to those given by Pearson[*], namely

$$\mu_2 = \frac{\alpha\beta\,(s+\alpha)(s+\beta)}{s^2(s-1)}$$

$$\mu_3 = \frac{\alpha\beta(s+\alpha)(s+\beta)(s+2\alpha)(s+2\beta)}{s^3(s-1)(s-2)}$$

$$\mu_4 = \frac{m_2\,(s^2+m_1 s+m_2)}{s(s-1)(s-2)(s-3)}\Big\{s^4+s^3\,(3m_2+6m_1+1)$$

$$+3s^2(m_1m_2+2m_1^2+2m_2)$$

$$+3s\,m_2\,(m_2+6m_1)+18\,m_2^2\Big\}$$

where

$\alpha = -r$ $\qquad\beta = -ps$

$m_1 = \alpha+\beta$ $\qquad m_2 = \alpha\beta$

II. SAMPLING FROM AN UNLIMITED SUPPLY

Referring to the formula of the first part of this paper, we observe that as s approaches infinity, r remaining finite,

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[*] Lond., Edinburgh and Dublin Phil Mag, Jan.-June, 1899, page 236.

$$(18)\begin{cases} M_z = rM_x \\ \mu_{2z} = r\mu_{2x} \\ \mu_{3z} = r\mu_{3x} \\ \mu_{4z} = r\mu_{4\cdot x} + 3r^{(2)}\mu_{2x}^2 \\ \mu_{5\cdot z} = r\mu_{5x} + 10r^{(2)}\mu_{3x}\mu_{2x} \\ \mu_{6\cdot z} = r\mu_{6\cdot x} + 15r^{(2)}\mu_{4x}\mu_{2x} + 10r^{(2)}\mu_{3x}^2 + 15r^{(3)}\mu_{2x}^3 \\ \mu_{7z} = r\mu_{7\cdot x} + 21r^{(2)}\mu_{6\cdot x}\mu_{2x} + 35r^{(2)}\mu_{4\cdot x}\mu_{3\cdot x} \\ \qquad + 105r^{(3)}\mu_{3x}\mu_{2\cdot x}^2 \\ \mu_{8z} = r\mu_{8x} + 28r^{(2)}\mu_{6x}\mu_{2x} + 56r^{(2)}\mu_{5\cdot x}\mu_{3\cdot x} + 35r^{(2)}\mu_{4\cdot x}^2 \\ \qquad + 210r^{(3)}\mu_{4\cdot x}\mu_{2x}^2 + 280r^{(3)}\mu_{3\cdot x}^2\mu_{2\cdot x} + 105r^{(4)}\mu_{2x}^4 \end{cases}$$

From these the following equations may be obtained:

$$(19)\begin{cases} \mu_{2z} = r\mu_{2\cdot x} \\ \mu_{3\cdot z} = r\mu_{3\cdot x} \\ \mu_{4\cdot z} - 3\mu_{2z}^2 = r\{\mu_{4\cdot x} - 3\mu_{2\cdot x}^2\} \\ \mu_{5z} - 10\mu_{3z}\mu_{2\cdot z} = r\{\mu_{5x} - 10\mu_{3x}\mu_{2\cdot x}\} \\ \mu_{6z} - 15\mu_{4z}\mu_{2\cdot z} - 10\mu_{3z}^2 + 30\mu_{2\cdot z}^3 = r\{\mu_{6\cdot x} - 15\mu_{4\cdot x}\mu_{2} \\ \qquad - 10\mu_{3\cdot x}^2 + 30\mu_{2x}^3\} \\ \mu_{7\cdot z} - 21\mu_{5z}\mu_{2\cdot z} - 35\mu_{4\cdot z}\mu_{3z} + 210\mu_{3\cdot z}\mu_{2\cdot z}^2 = r\{\mu_{7\cdot x} \\ \qquad - 21\mu_{5x}\mu_{2x} - 35\mu_{4x}\mu_{3x} + 210\mu_{3\cdot x}\mu_{2\cdot x}^2\} \\ \mu_{8\cdot z} - 28\mu_{6\cdot z}\mu_{2\cdot z} - 56\mu_{5z}\mu_{3\cdot z} - 35\mu_{4z}^2 + 420\mu_{4\cdot z}\mu_{2\cdot z}^2 \\ \qquad + 560\mu_{3\cdot z}^2\mu_{2z} - 630\mu_{2\cdot z}^4 = r\{\mu_{8\cdot x} - 28\mu_{6\cdot x}\mu_{2\cdot x} \\ \qquad - 56\mu_{5x}\mu_{3\cdot x} - 35\mu_{4\cdot x}^2 + 420\mu_{4x}\mu_{2x}^2 \\ \qquad + 560\mu_{3x}^2\mu_{2x} - 630\mu_{2x}^4\} \end{cases}$$

In terms of the standard moments of the distributions these equations become

$$
(20)\begin{cases}
\alpha_{3\,\bar{x}}=\frac{1}{r^{1/2}}\alpha_{3\,x} \\
\alpha_{4\,\bar{x}}-3=\frac{1}{r}\left\{\alpha_{4\,x}-3\right\} \\
\alpha_{5\,\bar{x}}-10\alpha_{3\,\bar{x}}=\frac{1}{r^{3/2}}\left\{\alpha_{5\,x}-10\alpha_{3\,x}\right\} \\
\alpha_{6\,\bar{x}}-15\alpha_{4\,\bar{x}}-10\alpha^2_{3\,\bar{x}}+30=\frac{1}{r^{2}}\left\{\alpha_{6\,x}-15\alpha_{4\,x}-10\alpha^2_{3\,x}+30\right\} \\
\alpha_{7\,\bar{x}}-21\alpha_{5\,\bar{x}}-35\alpha_{4\,\bar{x}}\alpha_{3\,\bar{x}}+210\alpha_{3,\bar{x}}=\frac{1}{r^{5/2}}\left\{\alpha_{7\,x}-21\alpha_{5\,x}\right. \\
\qquad\left.-35\alpha_{4\cdot x}\alpha_{3\,x}+210\alpha_{3\,x}\right\} \\
\alpha_{8,\bar{x}}-28\alpha_{6\cdot\bar{x}}-56\alpha_{5\,\bar{x}}\alpha_{3\,\bar{x}}-35\alpha^2_{4\cdot\bar{x}}+420\alpha_{4,\bar{x}}+56\alpha^2_{3\cdot\bar{x}}-630 \\
=\frac{1}{r^{3}}\left\{\alpha_{8\cdot x}-28\alpha_{6\,x}-56\alpha_{5\,x}\alpha_{3,x}-35\alpha_{4\,x}+420\alpha_{4\cdot x}+56\alpha^2_{3\,x}-630\right\}
\end{cases}
$$

If, without reference to subscripts, we write

$$
(21)\begin{cases}
\lambda_2=\mu_2 \\
\lambda_3=\mu_3 \\
\lambda_4=\mu_4-3\mu_2^2 \\
\lambda_5=\mu_5-10\mu_3\mu_2 \\
\lambda_6=\mu_6-15\mu_4\mu_2-10\mu_3^2+30\mu_2^3 \\
\lambda_7=\mu_7-21\mu_5\mu_2-35\mu_4\mu_3+210\mu_3\mu_2^2 \\
\lambda_8=\mu_8-28\mu_6\mu_2-56\mu_5\mu_3-35\mu_4^2+420\mu_4\mu_2^2 \\
\qquad\qquad+560\mu_3^2\mu_2-630\mu_2^4
\end{cases}
$$

the distribution of samples from an unlimited supply is defined, so far as moments through the eighth order are concerned, by the relations

$$
(22)\qquad\begin{cases}
M_{\bar{x}}=rM_x \\
\lambda_{n\cdot\bar{x}}=r\lambda_{n\,x}
\end{cases}
$$

Working along a different line of approach, Thiele was the first to realize the importance of these λ functions. He made an extensive study of their unusual properties and was thus both directly and indirectly responsible for many important contributions to the theory of mathematical statistics. These values of λ_i are the so-called "semi-Invariants of Thiele."

Again, we may write

$$(23)\quad \begin{cases} \gamma_3 = \alpha_3 \\ \gamma_4 = \alpha_4 - 3 \\ \gamma_5 = \alpha_5 - 10\,\alpha_3 \\ \gamma_6 = \alpha_6 - 15\,\alpha_4 - 10\,\alpha_3^2 \quad + 30 \\ \gamma_7 = \alpha_7 - 21\,\alpha_5 - 35\,\alpha_4\alpha_3 + 210\,\alpha_3 \\ \gamma_8 = \alpha_8 - 28\,\alpha_6 - 56\,\alpha_5\alpha_3 - 35\,\alpha_4^2 + 420\,\alpha_4 + 560\,\alpha_3^2 - 630 \end{cases}$$

and observe that the shape of the distribution of samples is determined by the relation

$$(24)\quad \gamma_{n\,x} = \frac{1}{r^{x/2-1}} \cdot \gamma_{n.x}$$

which follows from equations (20).

The values γ_i are referred to as the "standardized semi-invariants of Thiele."

If now r be permitted to approach infinity as a limit, we observe that in this limiting situation the *shape* of the distribution of samples is entirely independent of the shape of the parent population, since

$$\lim_{r\to\infty} \gamma_{n,x} = 0$$

that is

$$\alpha_{3.x} = 0$$

$$\alpha_{4.x} - 3 = 0$$

$$\alpha_{5.x} - 10\,\alpha_{3.x} = 0$$

$$\alpha_{6.x} - 15\alpha_{4.x} - 10\,\alpha_{3.x}^2 + 30 = 0$$

etc.

Thus the limiting distribution, which is called "the Normal Curve," must have the following properties:

$$(25) \quad \begin{cases} \alpha_{3,x} = 0 \\ \alpha_{4,x} = 1 \cdot 3 \\ \alpha_{5,x} = 0 \\ \alpha_{6,x} = 1 \cdot 3 \cdot 5 \\ \alpha_{7,x} = 0 \\ \alpha_{8,x} = 1 \cdot 3 \cdot 5 \cdot 7 \end{cases}$$

THE THEOREM OF BERNOULLI

If p denotes the probability that an event will happen in a single trial and $q = 1 - p$ the probability that it will not happen in that trial, then the probability that the event will happen exactly x times during r trials is, by Bernoulli's Theorem

$$(26) \qquad B_{r,x} = \binom{r}{x} q^{r-x} p^{x}$$

From our point of view we need only regard the problem as one of sampling in which we withdraw samples of r variates from an infinite parent population, in which ,as per Table V, p designates the proportion of the variates which are zero in magnitude—the remaining variates being of unit magnitude. Then since

$$\mu_{n,x} = pq\left\{ q^{n-1} + (-1)^{n} p^{n-1} \right\}$$

we see from formulae (18) that

$$(27) \quad \begin{cases} M_{x} = rp \\ \mu_{2,x} = rpq \\ \mu_{3,x} = rpq\left\{q^{2} - p^{2}\right\} \\ \mu_{4,x} = rpq\left\{q^{3} + p^{3}\right\} + 3r^{(2)}p^{2}q^{2} \\ \mu_{5,x} = rpq\left\{q^{4} - p^{4}\right\} + 10\, r^{(2)}p^{2}q^{2}\left\{q^{2} - p^{2}\right\} \end{cases}$$

etc.

POISSON'S EXPONENTIAL BINOMIAL LIMIT

If the probability that each of 1000 individuals die in one year were 5, then the expected number of deaths in such a group for one year would be 500. On the other hand, if the probability that each of 10,000 die in the year were .05 then the expected number of deaths would also be 500. Again $r = 100000$ and $p = .005$ or $r = 1000000$ and $p = .0005$ would give the same value. If we continue after this fashion to let r approach infinity and p zero, but in such a manner that the product $rp = M$ remains constant, then it can be shown quite readily that (26) becomes

$$\lim_{\substack{r \to \infty \\ p \to 0 \\ rp = M}} B_{r,x} = \frac{e^{-M} M^x}{x!} \tag{28}$$

This is known as Poisson's Exponential Binomial Limit. For a Poisson distribution it follows from (27) that

$$\left\{ \begin{aligned} \mu_{2\cdot x} &= M_x \\ \mu_{3\cdot x} &= M_x \\ \mu_{4\cdot x} &= M_x + 3M_x^2 \\ \mu_{5\cdot x} &= M_x + 10M_x^2 \\ \mu_{6\cdot x} &= M_x + 25M_x^2 + 15M_x^3 \\ \mu_{7\cdot x} &= M_x + 56M_x^2 + 105M_x^3 \\ \mu_{8\cdot x} &= M_x + 119M_x^2 + 409M_x^3 + 105M_x^4 \end{aligned} \right. \tag{29}$$

Substituting these values back in the definitions of the semi-invariants (formulae 21), we observe that for a Poisson distribution

$$\lambda_{n\cdot x} = M_x \qquad (x = 2, 3, \cdots\cdots, 8) \tag{30}$$

DISCUSSION OF RESULTS

So far as I know, no general method has been worked out which will permit one to express complex summations, such as those on pages

103, 104, in terms of moments. Moreover, I am unable at present to justify the use of the "sampling polynomials" for the moments of the samples of an order higher than the eighth Laborious computations have established the fact that the apparent law of the sampling polynomials holds for the first eight moments, and hence we have a simple method at our disposal of writing down expressions for these moments of samples withdrawn from finite parent populations. A study of these sampling polynomials should reveal an entirely different approach to the problem. This is but one of many interesting problems of mathematical statistics that require further investigation.

Although we utilized the results of sampling from a limited supply to obtain corresponding formulae for sampling from an unlimited supply, nevertheless it can be shown that for $s = \infty$ a simple method exists for expressing the moments in terms of the moments, $\mu_{n:x}$, as in formulae (18). Moreover, this law holds for any positive integer, n.

Thus

$$\mu_{20:x} = \frac{20!}{20!} r^{(1)} \mu_{20:x} + \frac{20!}{18!\,2!} r^{(2)} \mu_{18:x}\, \mu_{2:x} + \cdots\cdots$$

$$+ \cdots \frac{20!}{9!\,7!\,4!} r^{(3)} \mu_{9:x}\, \mu_{7:x}\, \mu_{4:x} + \cdots\cdots$$

$$+ \cdots \frac{20!}{(4!)^2 (3!)^4} r^{(6)} \frac{\mu_{4:x}^2\, \mu_{3:x}^4}{2!\,4!} + \cdots$$

Since formulae, such as (3a) and (4a) are based on multinomial considerations, the rule for writing down the values of $\mu_{n:x}$ is valid for any value of n, when $s = \infty$.

Proceeding after this fashion, one can show that corresponding to formulae (25) one can write for the limiting distribution, referred to as the Normal Curve,

$$(31) \quad \begin{cases} \alpha_{2n+1:x} = 0 \\ \alpha_{2n:x} = \dfrac{(2n)!}{2^n (n!)} \end{cases}$$

And since the function

$$(32) \qquad y = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^{2}}{2}}$$

satisfies the above conditions, we say that (32) is the equation of the Normal Curve. In the Theory of Least Squares this equation is usually* developed on the so-called Hagen's hypothesis, that is "An error is the algebraic sum of an indefinitely great number of small elementary errors which are all equal, and each of which is equally likely to be positive or negative."

From the results that we have obtained it appears that it is not necessary to impose the restrictions that the elementary errors are all equal and that positive and negative values are equally likely. It is necessary only that

(1) the number of elementary errors be infinite, although of an order less than that of the number of errors in the parent population.

(2) the errors be independent. This restriction is really involved in our assumption that in evaluating summations, each of the s variates of the parent population occurs exactly as many times as every other variate.

Otherwise, the limiting shape of the distribution of samples is independent of the shape of the parent distribution. The fact that tables II and IV, arising from parent distributions that are so extremely abnormal, exhibit distributions of samples that are fairly normal, seems to bear out our point in spite of the fact that we employed in each instance a small value of r, i. e. twenty-five.

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*See Merriman's Method of Least Squares. John Wiley and Sons New York City

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SUCCESSIVE INTEGRATION AS A METHOD FOR FINDING LONG PERIOD CYCLES

By

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SYNOPSIS

A method is developed for finding the period of cycles in statistical data of longer period than can be found by ordinary periodogram method. It consists of computing the progressive summations of the deviations of the observed data from normal. (As a first approximation normal is taken as the mean of the observed data) Then the progressive summations of these accumulated discrepancies are found, and so on to the third or fourth integration. This process rapidly "irons out" all chance and short period variations and leaves a smooth cycle whose approximate period is obvious. The objection that this is but an extension of the "quadrature" method, which makes cycles appear in data where none are present, is discussed and methods are presented for determining whether the cycle found is real or fictitious.

SUCCESSIVE INTEGRATION AS A METHOD FOR FINDING LONG PERIOD CYCLES

The question of the search for hidden periodicities in statistical data is important for the civil engineer interested in stream flow, the meteorologist and agriculturist interested in rainfall or temperature, the business statistician and probably many others The ordinary method of periodogram analysis as developed by Schuster[1] and discussed by Whittaker and Robinson[2] requires a period of record several times as long as the longest cycle considered. A century's record of annual rainfall at any given station or group of stations will enable us to test for the presence of cycles with periods of say from eight to twenty years, and possibly even to 33 or 50 years, but could not reveal a cycle of period of say between fifty and one hundred fifty years if one were present. For the latter purpose the following method has been developed.

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[1] Proc. Roy Soc 77 (1906) p 136.

[2] The Calculus of Observations, Chap. XIII, pp. 343-362.

Suppose the given data is of the form $y = a + b \cos \frac{k(x-c)}{p} \pm d$ where

y = any value of the dependent variable, as for example, the total inches of rainfall at a given station in any given year.

a = a constant, the normal value about which y fluctuates, as the normal rainfall in inches per year for the given station. If the data contains a straight line trend this term would be replaced by $(a + mx)$, where m measures the rate or slope of the trend.

b = a constant, the amplitude of the cycle.

k = 360, to give the angle in degrees, or 2π, to give the angle in radians.

x = the independent variable or serial number of the particular value y, as for example the date A. D. of the year whose rainfall is y.

c = a constant specifying the phase of the cycle, as for example the earliest date A. D. when the rainfall is known to have been a maximum.

p = the period of the cycle (in years, in this example).

d = a variable—the deviation of each observed value from the value given by the rest of the equation. This term takes care of all variations due to cycles of other periods, or to the form of this cycle not being that of the cosine curve, or the other variations which for want of further knowledge we must consider to be purely fortuitous.

If we subtract a, the normal value, from each of these values and take the progressive totals or first integral of the differences, we get a series of the form $\frac{bp}{2\pi} \sin \frac{k(x-c)}{p} + \Sigma d$ or $\frac{bp}{2\pi} \cos \frac{k(x-c-p/4)}{p} + \Sigma d$

The term Σd will in general be small, since the plus chance variations tend to balance the minus, as do the variations due to short period cycles which may be present. The second integration will give a series of the form $-b\left(\frac{p}{2\pi}\right)^2 \cos \frac{k(x-c)}{p}$ or $b\left(\frac{p}{2\pi}\right)^2 \cos \frac{k(x-c-p/2)}{p}$ if we neglect the sums of the Σd terms, which will, of course, tend to cancel out. The third integral will be of the form $b\left(\frac{p}{2\pi}\right)^3 \cos \frac{k(x-c-3p/4)}{p}$

and the fourth integral of the form $b\left(\frac{p}{2\pi}\right)^4 \cos \frac{k(x-c-p)}{p}$ That is, each integration gives a cycle of the same period as existed in the original data, but advanced a quarter period in phase and with the amplitude multiplied by $\frac{p}{2\pi}$.

The way in which this method "irons out" chance variations and short period fluctuations is most amazing to one who has not tried it. Table I and Plate I illustrate the method as applied to the annual rainfall at Boston, Mass., for the years 1818-1928. Although this has been carried to the third integration to illustrate the method, it was unnecessary in this case, as the second integration gives a smooth curve. The first integration changed from minus to plus in 1865 (fractions of a year being neglected) and from plus to minus in 1912, giving a half period of 47 years But by producing the first integration curve backward we see that it would pass through zero in 1814, which gives a whole period of 98 years. As a compromise, 96 years was taken as the value of p .

The Weather Bureau gives the rainfall to the hundredth of an inch, but it was found that the results were practically the same if the rainfall were taken to the nearest inch, which was done in this table. The mean rainfall for the 111 years was 43.45 and there seems to be no trend (From *a priori* grounds, such as the relative constancy of flora, lake levels, etc., we are quite sure that the rainfall of New England has changed very slightly in the last few centuries, so that for a period of only 111 years we can assume the normal rainfall as constant. The matter could also be tested statistically by fitting a straight line to the data by least squares, but this was not done.) But the normal rainfall is not necessarily equal to the mean for the period In fact, a preliminary trial indicated that the period of the most important cycle was a little less than 100 years, and that of the 111 years the portion in excess of a full cycle was below normal, therefore that the observed mean would be below normal. So 44 inches per year was taken as the first approximation to normal.

A second point in Table I which needs explanation is the initial value in each of the summation columns. A preliminary computation was made starting from zero. This was found to give a series of numbers averaging quite a little above zero But the average of these values cannot be taken as their normal any more than in the case of the observed rainfall themselves Averaging the maximum and minimum values gave + 46, therefore it was assumed that all values should be reduced 46 (accumulated inches) and a second trial was made using −46 as the value of the first integration for 1817. When the

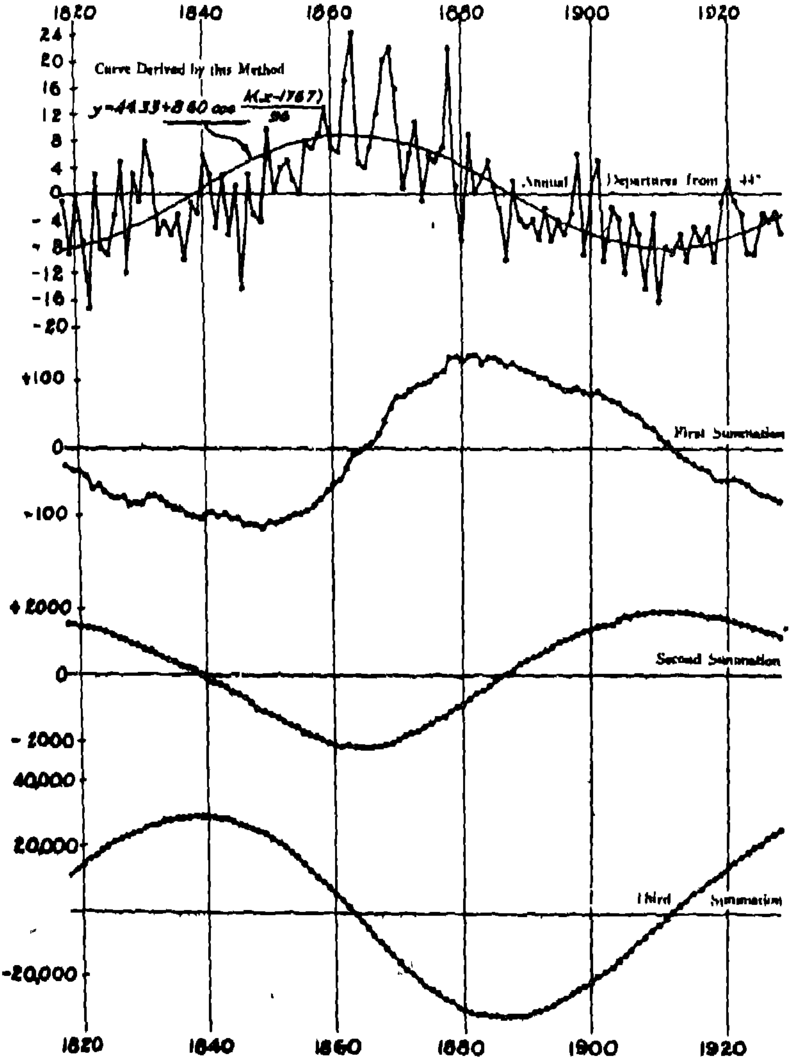
TABLE I

Search for a Long Period Cycle in the Precipitation Records of Boston, Mass.

Year	Inches of Rainfall	Excess or Deficiency From 44″	Summations		
			First	Second	Third
			- 26	1580	10400
1818	43	- 1	- 27	1553	11953
1819	35	- 9	- 36	1517	13470
1820	44	0	- 36	1481	14951
1821	37	- 7	- 43	1438	16389
1822	27	-17	- 60	1378	17767
1823	47	3	- 57	1321	19088
1824	36	- 8	- 65	1256	20344
1825	35	- 9	- 74	1182	21526
1826	41	- 3	- 77	1105	21526
1827	49	5	- 72	1033	23664
1828	32	-12	- 84	949	24613
1829	47	3	- 81	868	25481
1830	43	- 1	- 82	786	26267
1831	52	8	- 74	712	26979
1832	47	3	- 71	641	27620
1833	38	- 6	- 77	564	28184
1834	40	- 4	- 81	483	28667
1835	38	- 6	- 87	396	29063
1836	41	- 3	- 90	306	29369
1837	34	-10	- 100	206	29575
1838	43	- 1	- 101	105	29680
1839	41	- 3	- 104	1	29681
1840	49	5	- 99	- 98	29583
1841	47	3	- 96	- 194	29389
1842	39	- 5	- 101	- 295	29094
1843	47	3	- 98	- 393	28701
1844	38	- 6	- 104	- 497	28204
1845	46	2	- 102	- 599	27605
1846	30	-14	- 116	- 715	26890
1847	47	3	- 113	- 828	26062
1848	41	- 3	- 116	- 944	25118
1849	40	- 4	- 120	-1064	24054
1850	54	10	- 110	-1174	22880
1851	44	0	- 110	-1284	21596
1852	48	4	- 106	-1390	20206
1853	49	5	- 101	-1491	18715
1854	46	2	- 99	-1590	17125
1855	44	0	- 99	-1689	15436
1856	52	8	- 91	-1780	13656
1857	51	7	- 84	-1864	11792
1858	53	9	- 75	-1939	9853
1859	57	13	- 62	-2001	7852
1860	51	7	- 55	-2056	5796
1861	50	6	- 49	-2105	3691
1862	61	17	- 32	-2137	1554
1863	68	24	- 8	-2145	- 591
1864	49	5	- 3	-2148	- 2739
1865	48	4	1	-2147	- 4886
1866	51	7	8	-2139	- 7025
1867	56	12	20	-2119	- 9144
1868	64	20	40	-2079	-11223
1869	66	22	62	-2017	-13240
1870	60	16	78	-1939	-15179
1871	45	1	79	-1860	-17039

Year	Inches of Rainfall	Excess or Deficiency From 44"	Summations First	Summations Second	Summations Third
1872	50	6	85	–1775	–18814
1873	55	11	96	–1679	–20493
1874	43	– 1	95	–1584	–22077
1875	50	6	101	–1483	–23560
1876	49	5	106	–1377	–24937
1877	51	7	113	–1264	–26201
1878	66	22	135	–1129	–27330
1879	45	1	136	– 993	–28323
1880	37	– 7	129	– 864	–29187
1881	53	9	138	– 726	–29913
1882	44	0	138	– 588	–30501
1883	35	– 9	129	– 459	–30960
1884	49	5	134	– 325	–31285
1885	45	1	135	– 190	–31475
1886	42	– 2	133	– 57	–31532
1887	34	–10	123	66	–31466
1888	46	2	125	191	–31275
1889	40	– 4	121	312	–30963
1890	39	– 5	116	428	–30535
1891	40	– 4	112	540	–29995
1892	37	– 7	105	645	–29350
1893	42	– 2	103	748	–28602
1894	37	– 7	96	844	–27758
1895	40	– 4	92	936	–26822
1896	38	– 6	86	1022	–25800
1897	41	– 3	83	1105	–24695
1898	50	6	89	1194	–23501
1899	35	– 9	80	1274	–22227
1900	44	0	80	1354	–20873
1901	49	5	85	1439	–19434
1902	34	–10	75	1514	–17920
1903	42	– 2	73	1587	–16333
1904	40	– 4	69	1656	–14677
1905	32	–12	57	1713	–12964
1906	41	– 3	54	1767	–11197
1907	38	– 6	48	1815	– 9382
1908	30	–14	34	1849	– 7533
1909	41	– 3	31	1880	– 5653
1910	28	–16	15	1895	– 3758
1911	36	– 8	7	1902	– 1856
1912	35	– 9	– 2	1900	44
1913	38	– 6	– 8	1892	1936
1914	34	–10	– 18	1874	3810
1915	39	– 5	– 23	1851	5661
1916	37	– 7	– 30	1821	7482
1917	39	– 5	– 35	1786	9268
1918	34	–10	– 45	1741	11009
1919	43	– 1	– 46	1695	12704
1920	46	2	– 44	1651	14355
1921	43	– 1	– 45	1606	15961
1922	41	– 3	– 48	1558	17519
1923	35	– 9	– 57	1501	19020
1924	35	– 9	– 66	1435	20455
1925	41	– 3	– 69	1366	21821
1926	40	– 4	– 73	1293	23114
1927	41	– 3	– 76	1217	24331
1928	38	– 6	– 82	1135	25466

PLATE I



Long Period Cycle of Rainfall Variation at Boston, Mass.

second integration was made from these figures it was found that the curve had a general downward slope, and from its slope a correction was estimated for the initial term of the first integration. The initial terms of the other columns were found in a similar manner. The values given are the result of several successive approximations. It is probable that by using a different combination of values for these initial numbers an equally good cycle of a slightly different period might be developed. It is claimed only that this is a method of finding the *approximate* period of long cycles. If the exact period were important we could, by least square methods, fit cycles of periods slightly less and slightly greater than that found, and see which gave the best fit.

The minimum value of the second summation was –2148 in 1864 and the maximum was +1902 in 1911. Therefore the double amplitude was 4050 and the single amplitude was 2025. Therefore $b\left(\frac{p}{2\pi}\right)^2 = 2025$ and $b = \frac{2025 \times 4\pi^2}{96^2} = 8.67$. A similar use of the values of the third integration gave an amplitude of $\frac{29681+31532}{2} = 30606.5$ and $b = \frac{30606.5 \times 8\pi^3}{96^3} = 8.60$

As shown above the maxima of the second integration occur at about 1815 and 1911, therefore maxima of the rainfall itself should have occurred in 1767 and 1863 and c may be taken as 1767. Values of the term $8.60 \cos \frac{k(x-1767)}{96}$ were computed for each year. The first 96 terms will, of course, add to zero, but the last 15 terms add to –98.35, so that this will shift the 111-year average 0.89 inches below normal. As the average of the original data was 43.45, this indicates that the normal should be 43.45+0.89 = 44.34 and our equation becomes $y = 44.34 + 8.52 \cos \frac{k(x-1767)}{96} \pm d$.*

---

* It is not claimed that this equation as it stands can be used to forecast rainfalls at Boston. The data when examined by the periodogram method reveal several cycles of shorter periods, and even when corresponding terms are added to the formula the fortuitous variations average several inches per year. After this computation was made I found that C. F. Marvin had discussed this particular case (Monthly Weather Review, Aug., 1923, Vol. 51, pp. 383-390, "Concerning Normals, Secular Trends and Climatic Changes) By using data for Boston running back to 1750 (some actual and some "manufactured") he finds that straight line trends, rather than a cosine curve, best fit the data. He finds a normal annual rainfall of 40.06 from 1759 to 1849 and of 44.71 from 1849 to 1901, at which date the normal suddenly dropped again. The average rainfall from 1904 to 1928 inclusive was 37.80 Taking this as the normal for that period and the normals for the other two periods as given by Marvin, the average deviation from the normal for the 111 years was 5.46. The average deviation from the mean for the 111 years was 6.36. The average deviation from the formula derived above was 4.65. Adding four shorter cycles, we get the formula:

It will probably be urged against this method that the process of progressive summation often gives very misleading results, as has been pointed out by Bullock, Persons and Crum[1] and Simon Kuznets[2]. The latter gives, for example, 50 digits drawn at random, and the progressive totals of the deviations from the average, from which he deduces a pseudo cycle. Plate II shows that an even more striking cycle can be derived from this same random data by getting the second and third integrations. By following the same method as outlined above, a curve was deduced and drawn. The original data fit this curve with a mean deviation of 2 40, while they fit the average line with a mean deviation of 2.48. It may be urged that the cycle derived from the rainfall data is no more real than that derived from the chance data. But actually the cases are quite different, as we will proceed to show.

The test as to whether a time series contains cycles has been developed by Goutereau[3], Besson[4] and Woolard[5]. If the absolute values of the successive first differences of the series are averaged, this average is called the mean variability. The average of the absolue values of the differences of each value from the mean of all the values is called the mean deviation. Goutereau's Ratio, G, equals mean variation divided by mean deviation. For a random series of numbers whose distribution is Gaussian, the expected value of G will be $\sqrt{2}$. But if there is a cycle present, even if concealed by large chance varia-

---

$$y = 44.22 + 8\ 60 \cos \frac{k(x-1767)}{96} + 258 \cos \frac{k(x-1773)}{47}$$

$$+ 2\ 70 \cos \frac{k(x-1798)}{54} + 2.56 \cos \frac{k(x-1811)}{18} + 188 \cos \frac{k(x-1810)}{10}$$

which gives an average deviation of 4.28. (Some of these figures were obtained by using the annual rainfall to the nearest inch and would be slightly different if the data to hundredths of an inch were used) The reality of the four shorter cycles is very doubtful, but they produce a curve which fits the data much closer than the straight mean, or than Marvin's proposed normals, and somewhat closer than the simple 96 year cycle.

[1] "A Reply to Karl Karsten's 'The Harvard Business Indexes—a New Interpretation'," Review of Economic Statistics, April, 1927, pp 74-92

[2] "Random Events and Cyclical Oscillations," Journ of the Amer. Statistical Assn., Sept., 1919, pp 258-275.

[3] Sur la variabilite de la temperature, Annuaire de la Soc Met. de France, 54, 122-127, 1906. Summarized by Edgar W. Wooland in Minthly Weather Review, Vol. 49 (1921), pp 132-3

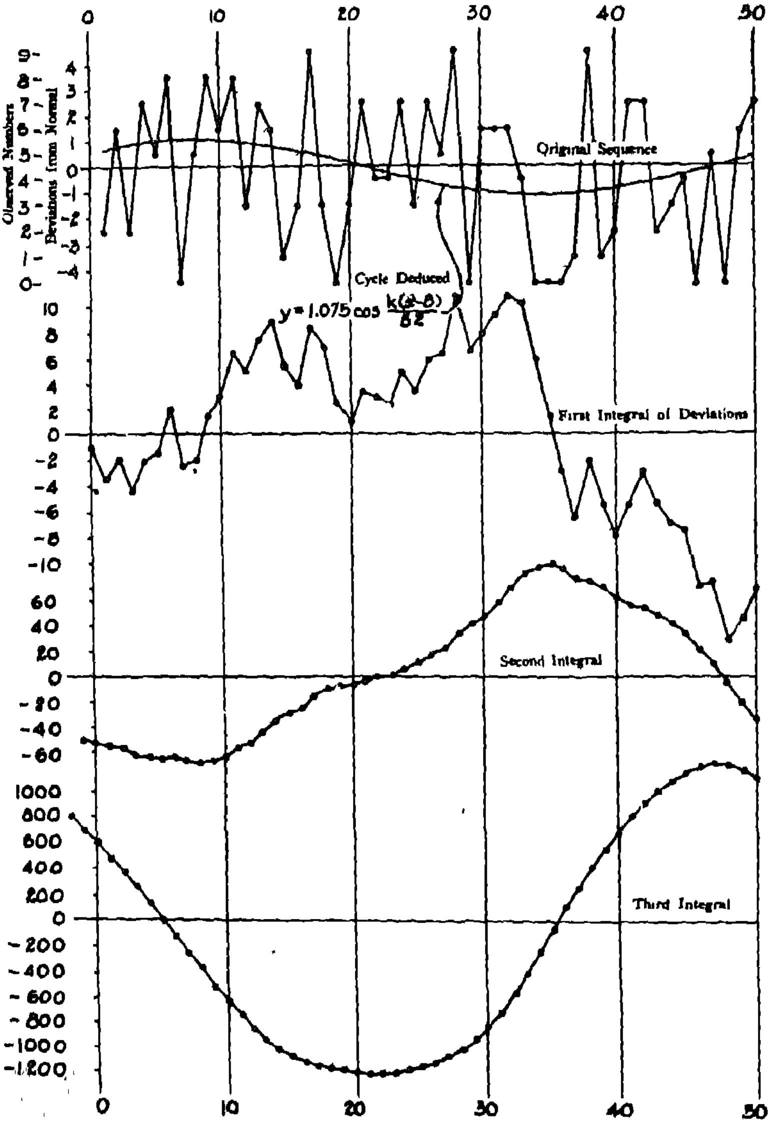
[4] "On the Comparison of Meteorological Data with Results of Chance," (Translated by E. W Woolard) Monthly Weather Review, Vol 48 (1920), pp. 89-94.

[5] Edgar W Woolard, "On the Mean Variability in Random Series," Monthly Weather Review (1925), pp. 107-111.

TABLE II

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
								- 65.0
							94.0	29.0
						-24.5	69.5	98.5
1	2	0	2		- 2.3	-26.8	42.7	141.2
2	6	1	7	5	2.7	-24.1	18.6	159.8
3	2	2	4	- 3	- 0.3	-24.4	- 5.8	154.0
4	7	3	10	6	5.7	-18.7	-24.5	129.5
5	5	3	8	- 2	3.7	-15 0	-39.5	90.0
6	8	4	12	4	7.7	- 7.3	-46.8	43.2
7	0	4	4	- 8	- 0.3	- 7.6	-54 4	- 11.2
8	5	4	9	5	4.7	- 2.9	-57.3	- 68.5
9	8	3	11	2	6.7	3.8	-53.5	-122.0
10	6	3	9	- 2	4 7	8.5	-45.0	-167.0
11	8	2	10	1	5 7	14.2	-30.8	-197.8
12	3	1	4	- 6	- 0 3	13.9	-16.9	-214.7
13	7	0	7	3	2 7	16.6	- 0.3	-215.0
14	6	- 1	5	- 2	0.7	17.3	17.0	-198.0
15	1	- 2	- 1	- 6	- 5.3	12.0	29.0	-169.0
16	3	- 3	0	1	- 4.3	7.7	36.7	-132 3
17	9	- 3	6	6	1.7	9.4	46.1	- 86.2
18	3	- 4	- 1	- 7	- 5.3	4.1	50.2	- 36.0
19	0	- 4	- 4	- 3	- 8.3	- 4.2	46.0	10.0
20	3	- 4	- 1	3	- 5 3	- 9.5	36 5	46.5
21	7	- 3	4	5	- 0.3	- 9.8	26 7	73.2
22	4	- 3	1	- 3	- 3.3	-13 1	13.6	96.8
23	4	- 2	2	1	- 2.3	-15.4	- 1.8	95.0
24	7	- 1	6	4	1.7	-13.7	-15 5	79.5
25	3	0	3	- 3	- 1 3	-15 0	-30.5	49.0
26	7	1	8	5	3 7	-11 3	-41.8	7.2
27	5	2	7	- 1	2 7	- 8 6	-50.4	- 43.2
28	9	3	12	5	7.7	- 0.9	-51.3	- 94.5
29	0	3	3	- 9	- 1 3	- 2 2	-53 5	-148 0
30	6	4	10	7	5.7	3.5	-50.0	-198 0
31	6	4	10	0	5.7	9 2	-40.8	-238 8
32	6	4	10	0	5.7	14.9	-25.9	-264.7
33	4	3	7	- 3	2 7	17.6	- 8 3	-273 0
34	0	3	3	- 4	- 1.3	16 3	8.0	-265.0
35	0	2	2	- 1	- 2 3	14 0	22.0	-243.0
36	0	1	1	- 1	- 3 3	10.7	32.7	-210.3
37	1	0	1	0	- 3.3	7.4	40.1	-170.2
38	9	- 1	8	7	3.7	11.1	51.2	-119.0
39	1	- 2	- 1	- 9	- 5.3	5 8	57.0	- 62 0
40	2	- 3	- 1	0	- 5.3	0 5	57.5	- 4.5
41	7	- 3	4	5	- 0.3	0.2	57.7	53.2
42	7	- 4	3	- 1	- 1 3	- 1.1	56.6	109.8
43	2	- 4	- 2	- 3	- 6.3	- 7.4	49.2	159 0
44	3	- 4	- 1	- 1	- 5.3	-12.7	36.5	195 5
45	4	- 3	1	2	- 3.3	-16.0	20.5	216.0
46	0	- 3	- 3	- 4	- 7.3	-23.3	- 2.8	213 2
47	5	- 2	3	6	- 1.3	-24.6	-27 4	185.8
48	0	- 1	- 1	- 4	- 5.3	-29.9	-57 3	128.5
49	6	0	6	7	1.7	-28.2	-85.5	43.0
50	7	1	8	2	3.7	-24.5	-110 0	- 67.0
Sums				92	91.4			
	214			-86	-91 4			
Sum of absolute values				178	182.8			

PLATE II



Showing How Apparent Cycles May Be Generated by Random Data.

tions superimposed on the cycle, the mean variation will be smaller than otherwise, and G will be less than 1.41. If the distribution is not Gaussian, the expected value of G will no longer be $\sqrt{2}$, but it will still be true that the presence of cycles will make G less than the expected value. Woodard, in the reference cited, gives a method for computing the expected value of G for a random order with any sort of distribution.

For the example of 50 random numbers used in Plate II, the mean deviation was 2.48 and the average variation was 3.45, making $G = 1.39$. The expected average deviation of numbers drawn from a universe containing equal numbers of each of the digits from 0 to 9 is 2.50 and the expected deviation by Woolard's method is 3.30, making the expected value of $G = 1.32$[1] In the data for Boston rainfall, given in Table I, the mean deviation was 6.356 and the mean variability 6.54, or $G = 1.03$, while Woolard's method would give an expected value of 1.38 ± 0.11 for random succession. The distribution of the "universe" of which this is a sample is not Gaussian, but it is not *much* different from Gaussian, so that the true value of G is not far from 1.41 An investigation of a much larger sample of rainfall data, which the writer hopes to publish soon, gives $G = 1.386 \pm .027$ It is therefore quite certain that the departure of the value of G from the expected value for random numbers is not accidental but indicates that we have here a real cycle, while in the case of drawn numbers we had only an apparent one.

To test the operation of the method in a case where it was known that there were both chance and cyclic elements present, Table II was prepared. The first two columns give the same random numbers from which Plate II was plotted. Column (3) is an artificial cycle which approximates a sine curve of period 24 and amplitude 4.00. Column (4) gives the algebraic sum of (2) and (3), and column (5) the first differences of column (4) Column (6) gives the deviations of the values in column (4) from the mean (4.30) Columns (7), (8) and (9) give the first, second and third integrals of these values. These figures are plotted on Plate III The period of the cycle in the third integral curve is 21 5 and the amplitude is about 200. The original cycle which would give this as a third integral would be $4.30+5.00 \cos \frac{k(x-7.5)}{21.5}$ This curve is plotted on the upper figure of Plate III as a solid line. The cyclic amounts added to the random data are plotted (above or below the line of the mean, 4.30) as crosses and connected by a dotted line. The cycle that emerges from the process is not quite the one that went in—it has been combined with the pseudo-cycle which arises from the fortuitous variations—but it is still a fair approximation of the original cycle. We can then assume that cycles derived by this process from statistical data that contain real cycles will be approximations of true cycles.

---

[1] The sample of 50 drawings gave by Woolard's method an expected mean variation for random succession of $3.27 \pm .39$ or $G = 1.32 \pm .16$. Thus the observed value fell within the range of the probable error. But the agreement is often much closer. The results of a little experiment made by the writer are as follows. A pack of cards was thoroughly shuffled and the cards turns up and their value recorded in order (Ace = 1, Jack = 11, Queen = 12, King = 13). The mean deviation is forced in this case to be 42/13 or 3.231. The observed mean variation was 1.314 making $G = 1.335$, while the expected value of G as given by Woolard's method for this case of rectangular distribution is 1.333

Goutereau's ratio gives a means of determining whether real cycles are present If there are none it is not necessary to search for them. If there are cycles present two courses are open to us. We may first construct a periodogram and find whether there are short cycles present. If not, we can assume that a long-period cycle determined by the method of this paper will be real. If there are short period cycles present, we can eliminate them and test the residue by Goutereau's ratio. If it still contains a cycle, we can assume that there is a real long-period cycle. The other procedure would be to first find the long-period cycle by the method of this paper If the amplitude of the cycle is large, it is quite certainly a real cycle. If it is very small, it may perhaps be negligible, even if real, and is probably unreal. In doubtful cases the cycle deduced may be subtracted from the given data and the residue tested again by Goutereau's ratio. If G is markedly larger than it was in the original data, we may assume that the cycle is real.

Criteria as to the reality of a given cycle have been proposed by C. F. Marvin[1], H W. Clough[2], Dinsmore Alter[3], and Sir Gilbert Walker[4], but they are adapted only to cycles obtained by means of the

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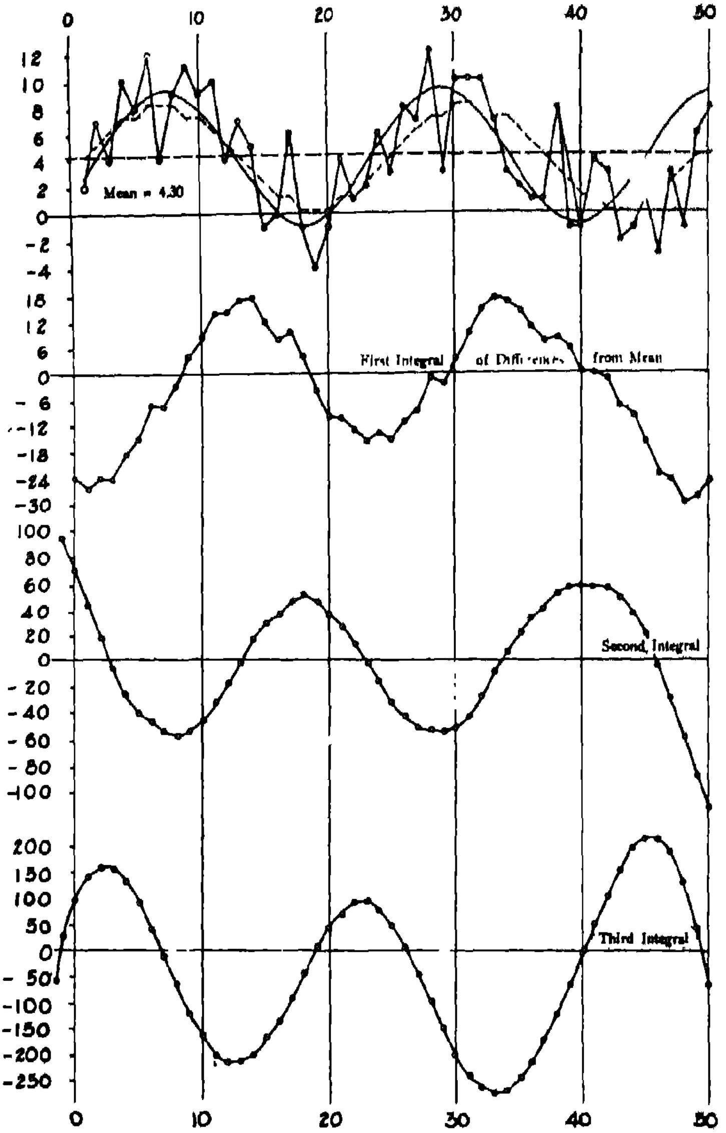
[1] Theory and Use of the Periodocite, Monthly Weather Review, Vol. 49 (1921), pp. 115-124.

[2] A Statistical Comparison of Meteorological Data with Data of Random Occurrence, Monthly Weather Review, Vol. 49 (1921), pp. 124-132.

[3] The Criteria of Reality in the Periodogram, Monthly Weather Review, Vol. 54 (1926), pp. 57-58.

[4] On Periodicity Quart. Jour. Royal Met'l Soc., 51, No. 216, pp. 337-346.

PLATE III



Random Data of Plate II Combined with a Cycle (the Dashed Curve at the Top of This Plate).

periodogram and not very well even to those[1]

The question still remains as to how much less than the expected value for random succession G may be before we are to believe that a cycle is present The writer would hazard 10 per cent as a rough guess It is to be hoped that some master of mathematical statistics will give us before long a quantitative statement of, say, the relationship between the ratio of the observed mean variation and the expected mean variation for the same numbers arranged in random succession, and the probability of a cycle of given amplitude being present.

It should be added that the germ idea of this paper is a product of the fertile mind of the writer's colleague, Professor P W. Ott.

CONCLUSIONS

(1) The method of successive integration of discrepancies will reveal the approximate period of long-period cycles if they are present.

(2) Even if no long-period cycle is present, the method will give a fictitious cycle, but there are tests by which the reality or falsity of the cycle can be investigated.

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[1] For example, Marvin's criterion depends only upon the standard deviations of the sums of the various columns of the tabulation as compared to the standard deviation of all the data, without reference to the order of the columns. Take, for example, the tabulation given on page 353 of Whitaker-Robinson's "Calculus of Observations." It is very evident that there is a real cycle But suppose that another problem had yielded exactly the same columns of data but in a random order, say the fifth column, then the twentieth, then the third, etc. Are we to suppose that a cycle of period 24 days is equally probable in this case? This objection seems to the writer to make the method of Whitaker and Robinson much inferior to that of Schuster.

Ralph W. Powell

EQUIMODAL FREQUENCY DISTRIBUTIONS

By

EDWIN D MOULTON, JR

The object of this paper is the determination of a set of frequency curves, each of which will give a better fit to the modal neighborhood of the data to which it is applied than is often found in the existing methods. Interest in this subject was aroused in the following way. First it was discovered that a great number of distributions of data derived from a study of the financial ratios of public utility companies conformed to the same general type of curve. Second, it developed that the type of curve designated by the Pearsonian criterion quite often yielded a very poor fit to the data. The mode determined by the theoretical curve was obviously unsuited to the actual data. Furthermore, in some cases, on the left extremity of the distribution, the rise of the curve to the mode was too steep for a good fit. The accompanying chart (p. 140) presents a particular instance of these conditions, together with the curve fitted by the method developed in this paper.

The curves which were used in this study of financial ratios were those developed by Pearson and Elderton from a consideration of the various cases which arose in the solution of the differential equation

$$\frac{dy}{dx} = \frac{y(x-a)}{F(x)}$$

where $F(x)$ was assumed to be expansible in ascending powers of x. The other assumptions made were that $F(x) = b_0 + b_1 x + b_2 x^2$, and that the constants a, b_0, b_1, and b_2 were determined by equating the moments of the raw data to the moments of the theoretical distribution. Here we will modify these assumptions, and under the new conditions determine the principal types of curves which arise when the polynomial in the denominator is of the third or lower degree.

The new assumption is that the value of the constant, a, the mode, is determined first from the observed data, and equated to the value of the mode in the theoretical distribution This method of procedure is particularly adapted to economic data, as it assures a good fit about the mode, notwithstanding the fact that in some raw data the mode is a rather vague concept The fit about the mode is of primary importance in much economic data.

II. We begin with the case of the cubic in the denominator, that is with the differential equation

$$\frac{dy}{dx}=\frac{(x-a)\,y}{b_0+b_1x+b_2x^2+b_3x^3}\ ,\ \text{or}$$

$$(b_0+b_1x+b_2x^2+b_3x^3)\frac{dy}{dx}=y\,(x-a)$$

where a is known. Multiplying both sides by x^n, integrating, and using the notation $\mu_n'=\int y\,x^n dx$, we have

$$nb_0\mu_{n-1}'+(n+1)\,b_1\mu_n'+(n+2)\,b_2\mu_{n+1}'+(n+3)\,b_3\mu_{n+2}'=a\mu_n'-\mu_{n+1}'$$

Putting $n = 0, 1, 2, 3$, and changing the origin to the mean, we have

$$(1)\qquad\left\{\begin{array}{rrrrl} 0b_0+ & b_1+ & 0b_2+ & 3\mu_2b_3 & =a\\ b_0+ & 0b_1+ & 3\mu_2b_2+ & 4\mu_3b_3 & =-\mu_2\\ 0b_0+ & 3\mu_2b_1+ & 4\mu_3b_2+ & 5\mu_4b_3 & =a\mu_2-\mu_3\\ 3\mu_2b_0+ & 4\mu_3b_1+ & 5\mu_4b_2+ & 6\mu_5b_3 & =a\mu_3-\mu_4 \end{array}\right.$$

Solving these equations for b_0, b_1, b_2, and b_3, we have

$$b_0=\frac{\begin{vmatrix} a & 1 & 0 & 3\mu_2\\ -\mu_2 & 0 & 3\mu_2 & 4\mu_3\\ a\mu_2-\mu_3 & 3\mu_2 & 4\mu_3 & 5\mu_4\\ a\mu_3-\mu_4 & 4\mu_3 & 5\mu_4 & 6\mu_5 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 0 & 3\mu_2\\ 1 & 0 & 3\mu_2 & 4\mu_3\\ 0 & 3\mu_2 & 4\mu_3 & 5\mu_4\\ 3\mu_2 & 4\mu_3 & 5\mu_4 & 6\mu_5 \end{vmatrix}}=\frac{A}{\Delta}$$

$$b_1 = \frac{\begin{vmatrix} 0 & a & 0 & 3\mu_2 \\ 1 & -\mu_2 & 3\mu_2 & 4\mu_3 \\ 0 & a\mu_2-\mu_3 & 4\mu_3 & 5\mu_4 \\ 3\mu_2 & a\mu_3-\mu_4 & 5\mu_4 & 6\mu_5 \end{vmatrix}}{\Delta} = \frac{B}{\Delta}$$

$$b_2 = \frac{\begin{vmatrix} 0 & 1 & a & 3\mu_2 \\ 1 & 0 & -\mu_2 & 4\mu_3 \\ 0 & 3\mu_2 & a\mu_2-\mu_3 & 5\mu_4 \\ 3\mu_2 & 4\mu_3 & a\mu_3-\mu_4 & 6\mu_5 \end{vmatrix}}{\Delta} = \frac{C}{\Delta}$$

$$b_3 = \frac{\begin{vmatrix} 0 & 1 & 0 & a \\ 1 & 0 & 3\mu_2 & -\mu_2 \\ 0 & 3\mu_2 & 4\mu_3 & a\mu_2-\mu_3 \\ 3\mu_2 & 4\mu_3 & 5\mu_4 & a\mu_3-\mu_4 \end{vmatrix}}{\Delta} = \frac{D}{\Delta}$$

The differential equation then becomes

$$\frac{dy}{y} = \frac{(x-a)\,dx}{\frac{A}{\Delta} + \frac{Bx}{\Delta} + \frac{Cx^2}{\Delta} + \frac{Dx^3}{\Delta}}$$

The solution of the differential equation depends on the nature of the zeros of the denominator of the right hand member, that is on the discriminant of the general cubic,

$$18\, b_3 b_2 b_1 b_0 - 4 b_2^3 b_0 + b_2^2 b_1^2 - 4 b_2 b_1^3 - 27 b_3^2 b_0^2$$

The cubic has three distinct real zeros, one real and two imaginary zeros, or at least two real and equal zeros, according as the discriminant is greater than zero, less than zero, or equal to zero. We will expect, therefore, three general types of curves when the integration is effected.

III. If we assume that $b_3 = 0$, we will have only three constants, b_0, b_1, and b_2 to determine, and the equations (1) become

$$
(2)\quad \left\{
\begin{array}{lcl}
b_1 & & = a \\
b_0 \quad \cdot & +3b_2\mu_2 & = -\mu_2 \\
 & 3b_1\mu_2 + 4b_2\mu_3 & = -\mu_3 + a\mu_2
\end{array}
\right.
$$

Solving these equations simultaneously, we find

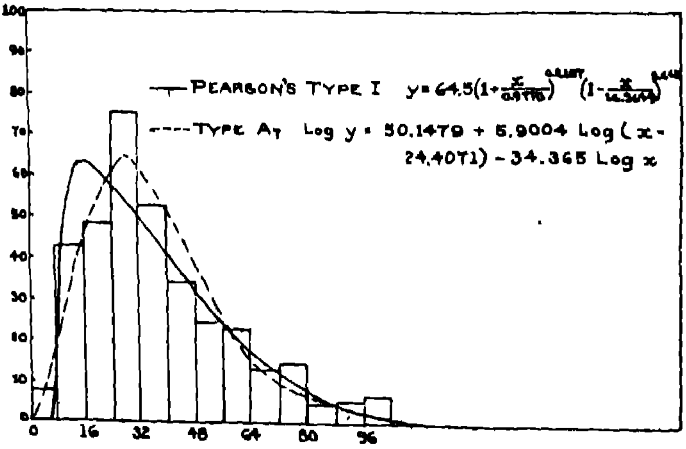
$$b_0 = \frac{-\mu_2\mu_3 + 6a\mu_2^2}{4\mu_3}$$

$$b_1 = a$$

$$b_2 = \frac{-\mu_3 - 2a\mu_2}{4\mu_3}$$

Thus, in the case of the quadratic in the denominator, we have determined the constants in terms of the mode, and the first, second, and third moments of the raw data. In other words, we are calculating the theoretical curve under the assumption that its mode, mean, standard deviation, and skewness are equal respectively to the mode,

Ratio of Revenue to Net Worth in 351 Traction Companies



mean, standard deviation, and skewness of the raw data. The differential equation then becomes

$$\frac{1}{y}\frac{dy}{dx}=\frac{x-a}{\frac{-\mu_2\mu_3+6a\mu_2^2}{4\mu_3}+ax+\frac{-\mu_3-2a\mu_2}{4\mu_3}x^2}$$

Now, the solution of this differential equation depends on the particular values of the constants in the denominator, i. e., on the quadratic discriminant $b_1^2-4b_0b_2$ Again, we will expect three general types of curves when the integration is effected.

If we assume $b_3=b_2=0$, equations (1) become

$$b_1=a$$

(3)

$$b_0=-\mu_2$$

and the differential equation is

$$\frac{1}{y}\frac{dy}{dx}=-\frac{x-a}{\mu_2-ax}$$

If we keep only b_0, we have $b_0=-\mu_2$ and $a=0$, and our equation is $\frac{1}{y}\frac{dy}{dx}=-\frac{x}{\mu_2}$

We now turn to a discussion of the various types of curves which arise from the solution of the preceding differential equations. The following classification will be made—Class A will include all curves arising from the solution of differential equations in which $F(x)$ has real and unequal zeros. Class B will include all curves arising from the solution of differential equations in which $F(x)$ has complex zeros. Class C will include all curves arising from the solution of differential equations in which $F(x)$ has at least two equal zeros.

TYPE A-1

IV. When all the zeros are positive, the differential equation may be written in the form

$$\frac{dy}{y}=\frac{(x-a)dx}{b_3(x-A_1)(x-A_2)(x-A_3)}$$

where we assume $A_1 > A_2 > A_3$. Separating into partial fractions and integrating, we have

$$\begin{aligned}\log y = {} & \frac{(A_2-A_3)(A_1-a)}{b_3(A_1-A_3)(A_2-A_3)(A_1-A_2)}\log(x-A_1)\\ & -\frac{(A_1-A_3)(A_2-a)}{b_3(A_1-A_3)(A_2-A_3)(A_1-A_2)}\log(x-A_2)\\ & +\frac{(A_1-A_2)(A_3-a)}{b_3(A_1-A_3)(A_2-A_3)(A_1-A_2)}\log(x-A_3)\\ & + \log k\end{aligned}$$

Exponentiating

$$y = \frac{k(x-A_1)^{\frac{(A_2-A_3)(A_1-a)}{b_3(A_1-A_3)(A_2-A_3)(A_1-A_2)}}(x-A_3)^{\frac{(A_1-A_2)(A_3-a)}{b_3(A_1-A_3)(A_2-A_3)(A_1-A_2)}}}{(x-A_2)^{\frac{(A_1-A_3)(A_2-a)}{b_3(A_1-A_3)(A_2-A_3)(A_1-A_2)}}}$$

Transferring the origin to the mode, i. e., putting x for $x-a$ we have

$$y = \frac{k(x+a-A_1)^{\frac{c_1a_1}{s}}(x+a-A_3)^{\frac{c_3a_3}{s}}}{(x+a-A_2)^{\frac{c_2a_2}{s}}}$$

$$= \frac{y_0(1-x/a_1)^{\frac{c_1a_1}{s}}(1-x/a_3)^{\frac{c_3a_3}{s}}}{(1-x/a_2)^{\frac{c_2a_2}{s}}}$$

where

$c_1 = A_2 - A_3$ $\qquad c_2 = A_1 - A_3$ $\qquad c_3 = A_1 - A_2$

$a_1 = A_1 - a$ $\qquad a_2 = A_2 - a$ $\qquad a_3 = A_3 - a$

$$s = b_3(A_1-A_2)(A_1-A_3)(A_2-A_3)$$

Then $c_2a_2 = c_1a_1 + c_3a_3$

Let $\frac{c_1}{s} = m_1$, $\qquad \frac{c_2}{s} = m_2$, $\qquad \frac{c_3}{s} = m_3$

Then $m_2 = m_1 - m_3$

The equation now becomes

$$y = \frac{y_0 (1 - x/a_1)^{m_1 a_1} (1 - x/a_3)^{m_2 a_3}}{(1 - \frac{x}{a_2})^{m_2 a_2}}$$

With the exception of y_0, the values of all the constants in the equation are known in terms of moments. Two methods will be given for its determination.

First—Calculate the area under the curve, using the theoretical ordinates measured in terms of y_0. Let this area equal N, the number of observations, and solve for y_0.

Thus $y = \frac{N}{B_1 + B_2 + \cdots + B_n}$, where $y_0 B_i$ represents the areas calculated from the theoretical ordinates.

Second—Calculate the value of χ^2 (Chi-square) with the theoretical ordinates measured in terms of y_0. Set the first derivative of this expression equal to zero, and determine the value of y_0 which makes χ^2 a minimum. From the goodness of fit point of view, this gives the best possible value of y_0.

Thus $\chi^2 = \sum \frac{(y_0 B_i - O_i)^2}{y_0 B_i}$, where the $y_0 B_i$ represent the theoretical areas as before, and the O_i represent the observed areas.

Setting the first derivative of this expression equal to zero, we have

$$\sum B_i - \frac{1}{y_0^2} \sum \frac{O_i^2}{B_i} = 0$$

$$y_0^2 = \frac{\sum \frac{O_i^2}{B_i}}{\sum B_i}$$

TYPE A-2

V. When there are two positive zeros and one negative zero, the equation may be written in the form

$$\frac{dy}{y}=\frac{(x-a)\,dx}{b_3\,(x-A_1)(x-A_2)\,(x+A_3)}, \quad \text{where } A_1 > A_2 > -A_3$$

Proceeding as in Type A-I, we find

$$y=\frac{y_0\left(1-x/a_1\right)^{\frac{c_1 a_1}{s}}}{\left(1-x/a_2\right)^{\frac{c_2 a_2}{s}}\left(1-x/a_3\right)^{\frac{c_3 a_3}{s}}}$$

where

$$\begin{aligned} c_1 &= A_2+A_3 & a_1 &= A_1-a & & \\ c_2 &= A_1+A_3 & a_2 &= A_2-a & s &= b_3 c_1 c_2 c_3 \\ c_3 &= A_1-A_2 & a_3 &= A_3+a & & \end{aligned}$$

y_0 is calculated as in Type A-I. The origin is at the mode.

TYPE A-3

VI. When there are two negative zeros and one positive zero, the equation may be written in the form

$$\frac{dy}{y}=\frac{(x-a)\,dx}{b_3\,(x-A_1)(x+A_2)(x+A_3)}, \quad \text{where } A_1 > -A_2 > -A_3$$

Using the same method as before, we find

$$y=\frac{y_0\left(1-x/a_1\right)^{\frac{c_1 a_1}{s}}\left(1+x/a_2\right)^{\frac{c_2 a_2}{s}}}{\left(1+x/a_3\right)^{\frac{c_3 a_3}{s}}}, \quad \text{where}$$

$$\begin{aligned} a_1 &= A_1-a & c_1 &= -A_2+A_3 & & \\ a_2 &= A_2+a & c_2 &= A_1+A_3 & s &= b_3 c_1 c_2 c_3 \\ a_3 &= A_3+a & c_3 &= A_1+A_2 & & \end{aligned}$$

y_0 is calculated as in the previous cases. The origin is at the mode.

TYPE A-4

VII. Where all three zeros are negative, the equation may be written in the form

$$\frac{dy}{y} = \frac{(x-a)\,dx}{b_3(x+A_1)(x+A_2)(x+A_3)}, \text{ where } -A_1 > -A_2 > -A_3$$

Proceeding as in the last three cases, we find

$$y = \frac{y_0\left(1+x/a_2\right)^{\frac{c_2 a_2}{s}}}{\left(1+x/a_1\right)^{\frac{c_1 a_1}{s}}\left(1+x/a_3\right)^{\frac{c_3 a_3}{s}}}, \text{ where}$$

$a_1 = A_1 + a$ $\qquad c_1 = -A_2 + A_3$

$a_2 = A_2 + a$ $\qquad c_2 = -A_1 + A_3$ $\qquad s = b_3 c_1 c_2 c_3$

$a_3 = A_3 + a$ $\qquad c_3 = -A_1 + A_2$

y_0 is determined as in the previous cases. The origin is at the mode.

TYPE A-5

VIII. In Type A-3, suppose $A_1 = A_2$. Then

$$y = \frac{y_0\left(1-x/A_1-a\right)^{\frac{(-A_1+A_3)(A_1-a)}{s}}\left(1+x/A_1+a\right)^{\frac{(A_1+A_3)(A_1+a)}{s}}}{\left(1+\frac{x}{A_3+a}\right)^{\frac{2A_1(A_3+a)}{s}}},$$

where $s = b_3\ (A_1+A_3)(-A_1+A_3)\ (2A_1)$

Suppose a, the mode, is at the mean, that is, is equal to zero. Then.

$$A_1 - a = A_1 + a = a_1$$

Then

$$y = \frac{y_0\left(1 - x/a_1\right)^{\frac{(-A_1+A_3)\,a_1}{3}}\left(1 + x/a_1\right)^{\frac{(A_1+A_3)\,a_1}{3}}}{\left(1 + x/a_3\right)^{\frac{2A_1a_3}{3}}}$$

y_0 is calculated as before. The origin is at the mode.

TYPE A-6

IX. In Type A-3, suppose one of the zeros is zero, say A_1. The equation then becomes

$$y = \frac{kx^{\frac{(-A_2+A_3)(-a)}{b_3A_2A_3(-A_2+A_3)}}\left(x+A_2\right)^{\frac{A_3(A_2+a)}{b_3A_2A_3(-A_2+A_3)}}}{\left(x+A_3\right)^{\frac{A_2(A_3+a)}{b_3A_2A_3(-A_2+A_3)}}}$$

$$= \frac{y_0x^{\frac{-a}{b_3A_2A_3}}\left(1+x/A_2\right)^{\frac{A_2+a}{b_3A_2(-A_2+A_3)}}}{\left(1+x/A_3\right)^{\frac{A_3+a}{b_3A_3(-A_2+A_3)}}}$$

The values of all the constants except y_0 are known, and it may be calculated by either of the formulas given in Type A-1.

The origin is at the mean.

TYPE A-7

X. In case $F(x)$ is quadratic, and its zeros are of like sign, we have

$$\frac{dy}{y} = \frac{(x-a)dx}{b_2(x+A_1)(x+A_2)} = \frac{A_1+a}{b_2(A_1-A_2)}\,\frac{dx}{x+A_1} - \frac{A_2+a}{b_2(A_1-A_2)}\,\frac{dx}{x+A_2}$$

Integrating, we have

$$\log y = \frac{A_1+a}{b_2(A_1-A_2)}\log(x+A_1) + \frac{-(A_2+a)}{b_2(A_1-A_2)}\log(x+A_2) + \log y'$$

Exponentiating,

$$y = y'(x+A_1)^{\frac{A_1+a}{b_2(A_1-A_2)}}(x+A_2)^{\frac{-(A_2+a)}{b_2(A_1-A_2)}}$$

Changing x to $x - A_2$, we have

$$y = y'(x+A_1-A_2)^{\frac{A_1+a}{b_2(A_1-A_2)}}x^{\frac{-(A_2+a)}{b_2(A_1-A_2)}}$$

Let

$$A_1 - A_2 = -m \;; \quad \frac{A_1+a}{-b_2 m} = p_1 \quad ; \quad \frac{A_2+a}{b_2 m} = p_2$$

Then

$$y = y_0\,(x-m)^{p_1}x^{-p_2}$$

The constants A_1 and A_2 are given by the zeros of the quadratic in the denominator, and m, p_1, and p_2 are given in terms of these above. By integration of this equation between the limits m and ∞, it has been found[1] that

$$y_0 = \frac{N\Gamma(p_2)\ m^{p_2-p_1-1}}{\Gamma(p_1+1)\,\Gamma(p_2-p_1-1)}$$

y_0 may also be determined by finite integration by either of the methods given in Type A-1.

Origin = Mean $- A_2$

Type A-8

XI. This type occurs when $F(x)$ is a quadratic and the zeros are real and opposite in sign. The equation then becomes

$$\frac{dy}{y} = \frac{1}{b_2}\,\frac{x-a}{(x+A_1)(x-A_2)}$$

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[1] Elderton, "Frequency Curves and Correlation," p. 85.

Integrating, we obtain

$$\log y = \frac{1}{b_2}\,\frac{A_1+a}{A_1+A_2}\,\log(x+A_1) + \frac{1}{b_2}\,\frac{A_2-a}{A_1+A_2}\,\log(x-A_2) - \log y'$$

Exponentiating,

$$y = y'(x+A_1)^{\frac{1}{b_2}\frac{A_1+a}{A_1+A_2}}\,(x-A_2)^{\frac{1}{b_2}\frac{A_2-a}{A_1+A_2}}$$

Now, changing the origin to the mode, i. e., putting x for $x-a$, we have

$$y = y'(x+A_1+a)^{\frac{1}{b_2}\frac{A_1+a}{A_1+A_2}}\,(x-A_2+a)^{\frac{1}{b_2}\frac{A_2-a}{A_1+A_2}}$$

$$= y_0\left(1+\frac{x}{a_1}\right)^{m_1}\left(1-\frac{x}{a_2}\right)^{m_2}$$

where

$$a_1 = A_1 + a \quad ; \quad a_2 = A_2 - a$$

$$m_1 = \frac{1}{b_2}\,\frac{a_1}{A_1+A_2} \quad ; \quad m_2 = \frac{1}{b_2}\,\frac{a_2}{A_1+A_2}$$

and

$$\frac{m_1}{a_1} = \frac{m_2}{a_2}$$

The value of y_0 has been found[1] by integration to be

$$y_0 = \frac{N m_1^{m_1} m_2^{m_2}\,\Gamma(m_1+m_2+2)}{b(m_1+m_2)^{m_1+m_2}\,\Gamma(m_1+1)\,\Gamma(m_2+1)}$$

Where $b = a_1 + a_2$, N = total frequency. y_0 may also be calculated by either of the methods given in Type A-1. The origin is at the mode.

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[1] Elderton, "Frequency Curves and Correlation," p. 59.

Type A-9

XII. If the zeros of the quadratic are A_1, and $-A_1$, the equation may be written

$$\frac{dy}{y}=\frac{1}{b_2}\,\frac{A_1-a}{2A_1}\,\frac{dx}{x-A_1}+\frac{1}{b_2}\,\frac{A_1+a}{2A_1}\,\frac{dx}{x+A_1}$$

Integrating, we obtain

$$\log y=\frac{1}{b_2}\,\frac{A_1-a}{2A_1}\log(x-A_1)+\frac{1}{b_2}\,\frac{A_1+a}{2A_1}\log(x+A_1)+\log y$$

Exponentiating,

$$y=y'(x-A_1)^{\frac{1}{b_2}\frac{A_1-a}{2A_1}}(x+A_1)^{\frac{1}{b_2}\frac{A_1+a}{2A_1}}$$

Changing the origin to the mode, i. e, putting x for $x-a$, we have

$$y=y'(x+a-A_1)^{\frac{1}{b_2}\frac{A_1-a}{2A_1}}(x+a+A_1)^{\frac{1}{b_2}\frac{A_1+a}{2A_1}}$$

$$=y_0\left(1-\frac{x}{a_1}\right)^{m_1}\left(1+\frac{x}{c_1}\right)^{m_2}$$

where

$$a_1=A_1-a \qquad a_1+2a=A_1+a=c_1$$

$$\frac{a_1}{2b_2A_1}=m_1 \qquad \frac{c_1}{2b_2A_1}=m_2$$

Then, as before,

$$y_0=\frac{N}{a_1+c_1}\cdot\frac{m_1^{m_1}m_2^{m_2}\,\Gamma(m_1+m_2+2)}{(m_1+m_2)^{m_1+m_2}\,\Gamma(m_1+1)\,\Gamma(m_2+1)}$$

Type A-10

XIII If $F(x)$ is assumed to be linear, the equation may be written

$$\frac{dy}{y} = \frac{x-a}{b_0+b_1x}\,dx = \left(\frac{1}{b_1} + \frac{-a-\frac{b_0}{b_1}}{b_1x+b_0}\right)dx$$

Integrating,

$$\log y = \frac{x}{b_1} + \frac{1}{b_1}\left(-a-\frac{b_0}{b_1}\right)\log(b_1x+b_0) + \log y'$$

Exponentiating,

$$y = y'\,e^{\frac{x}{b_1}}(b_1x+b_0)^{-\frac{1}{b_1}\left(a+\frac{b_0}{b_1}\right)}$$

Changing the origin to the mode by putting x for $x-a$, we have

$$y = y''e^{\frac{x+a}{b_1}}\left(1+\frac{x}{\frac{b_1a+b_0}{b_1}}\right)^{-\frac{1}{b_1}\left(\frac{b_1a+b_0}{b_1}\right)}$$

Now let

$$y''e^{\frac{a}{b_1}} = y_0 \quad ; \quad \frac{ab_1+b_0}{b_1} = m \quad ; \quad -\frac{1}{b_1} = \gamma$$

Then

$$y = y_0\,e^{-\gamma x}\left(1+\frac{x}{m}\right)^{\gamma m}$$

The constants may be determined as follows.

When $b_2 = b_3 = 0$, it has been found that

$$b_0 = -\mu_2 \quad ; \quad b_1 = a$$

$$\therefore\ m = \frac{ab_1+b_0}{b_1} = \frac{a^2-\mu_2}{a}$$

$$\gamma = -\frac{1}{b_1} = -\frac{1}{a}$$

The value of y_0 has been found[1] to be

$$y_0 = \frac{Nq^{q+1}}{me^{q}\Gamma(q+1)}$$

where $q = \gamma m$.

y_0 may also be found by the methods of Type A-1. The origin is at the mode.

TYPE A-11 (The normal curve)

XIV. Putting $b_1 = b_2 = b_3 = 0$, we have

$$\frac{dy}{y} = \frac{x-a}{b_0}\,dx$$

Integrating,

$$\log y = \frac{x^2}{2b_0} - \frac{ax}{b_0} + \log c = \frac{(x-a)^2}{2b_0} + \log y'$$

Exponentiating,

$$y = y' e^{\frac{(x-a)^2}{2b_0}}$$

Changing the origin to the mode, and substituting the value for b_0 when $b_1 = b_2 = b_3 = 0$, that is $b_0 = -\mu_2$, we obtain

$$y = y_0 e^{-\frac{x^2}{2\mu_2}}$$

To find the value of y_0 , integrate between the limits $-\infty$ and ∞ and find the total frequency N. It has been found[2] that

$$y_0 = \frac{N}{\sqrt{2\pi\mu_2}}$$

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[1] Elderton, "Frequency Curves and Correlation," p. 68.

[2] Elderton, "Frequency Curves and Correlation," p. 91.

y_0 may also be found by either of the other two methods. The origin is at the mode.

TYPE B-1

XV. When $F(x)$ is a cubic, and two of the zeros are complex, the differential equation may be written in the form

$$\frac{dy}{y}=\frac{(x-a)\,dx}{b_3(x-A_1)\left[x^2-(A_2+A_3)x+A_2A_3\right]},\text{ where } (A_2-A_3)^2<0$$

Separating into partial fractions and integrating,

$$\int\frac{dy}{y}=\frac{A_1-a}{b_3\left[A_1(A_1-A_2-A_3)+A_2A_3\right]}\int\frac{dx}{x-A_1}$$

$$-\frac{A_1-a}{b_3\left[A_1(A_1-A_2-A_3)+A_2A_3\right]}\int\frac{x\,dx}{x^2-(A_2+A_3)x+A_2A_3}$$

$$+\frac{a(A_1-A_2-A_3)+A_2A_3}{b_3\left[A_1(A_1-A_2-A_3)+A_2A_3\right]}\int\frac{dx}{x^2-(A_2+A_3)x+A_2A_3}$$

Now let $x'=x-\frac{A_2+A_3}{2}$; $M^2=\frac{-(A_2-A_3)^2}{2}$

$$k=a(A_1-A_2-A_3)+A_2A_3-\frac{(A_1-a)(A_2+A_3)}{2}$$

$$d=\left[A_1(A_1-A_2-A_3)+A_2A_3\right]b_3$$

Performing the integration, we have

$$\log y=\frac{A_1-a}{d}\log\left(x'+\frac{A_2+A_3}{2}-A_1\right)-\frac{A_1-a}{2d}\log\left(x'^2+M^2\right)$$
$$+\frac{k}{Md}\tan^{-1}\frac{x'}{M}+\log y_0$$

Exponentiating,

$$y=\frac{y_0\left[x'+\frac{1}{2}(A_2+A_3)-A_1\right]^{\frac{A_1-a}{d}}e^{\frac{k}{Md}\tan^{-1}\frac{x'}{M}}}{\left(x'^2+M^2\right)^{\frac{A_1-a}{2d}}}$$

$$=\frac{y_0\,(x+c)^{2m}\,e^{\frac{k}{Md}\tan^{-1}\frac{x'}{M}}}{\left(x'^2+M^2\right)^{m}}$$

where $m = \frac{A_1 - a}{2d}$, $\frac{A_2 + A_3}{2} - A_1 = c$

y_0 may be determined as in Type A-1. Origin = Mean + $\frac{A_2 + A_3}{2}$

TYPE B-2

XVI. When two of the zeros are pure imaginaries, the equation may be written:

$$\int \frac{dy}{y} = \frac{A_1 - a}{b_3(A_1^2 + A_2^2)} \int \frac{dx}{x - A_1} - \frac{A_1 - a}{2b_3(A_1^2 + A_2^2)} \int \frac{2x\,dx}{x^2 + A_2^2}$$
$$+ \frac{aA_1 + A_2^2}{b_3(A_1^2 + A_2^2)} \int \frac{dx}{x^2 + A_1^2}$$

Performing the integration, we have

$$\log y = \frac{A_1 - a}{b_3(A_1^2 + A_2^2)} \log (x - A_1) - \frac{A_1 - a}{2b_3(A_1^2 + A_2^2)} \log (x^2 + A_2^2)$$

$$+ \frac{aA_1 + A_2^2}{b_3 A_2 (A_1^2 + A_2^2)} \tan^{-1} \frac{x}{A_2} + \log y_0$$

Exponentiating, we have

$$y = \frac{y_0 (x - A_1)^{\frac{A_1 - a}{b_3(A_2^2 + A_1^2)}} e^{\frac{aA_1 + A_2^2}{b_3 A_2 (A_1^2 + A_2^2)} \tan^{-1} \frac{x}{A_2}}}{(x^2 + A_2^2)^{\frac{A_1 - a}{2b_3(A_1^2 - A_2^2)}}}$$

Let $b_3 (A_1^2 + A_2^2) = d$; $\frac{A_1 - a}{d} = 2m$; $aA_1 + A_2^2 = k$ Then

$$y = \frac{y_0 (x - A_1)^{2m} e^{\frac{k}{A_2 d} \tan^{-1} \frac{x}{A_2}}}{(x^2 + A_2^2)^m}$$

y_0 may be determined as in the previous case. The origin is at the mean.

TYPE B-3

XVII. If $F(x)$ is quadratic and the zeros are complex, the equation may be written

$$\int \frac{dy}{y} = \int \frac{(x-a)\,dx}{b_0+b_1x+b_2x^2} = \int \frac{(x-a)\,dx}{b_2\left[\left(x^2+\frac{b_1}{b_2}x+\frac{b_1^2}{4b_2^2}\right)+\left(\frac{b_0}{b_2}-\frac{b_1^2}{4b_2^2}\right)\right]}$$

Let $X = x+\frac{b_1}{2b_2}$; $A^2 = \frac{b_0}{b_2} - \frac{b_1^2}{4b^2}$

Then $x-a = X-\frac{b_1}{2b_2}-a = X+c$, where $c = -\left(\frac{b_1}{2b_2}+a\right)$

We have then

$$\log y = \int \frac{(X+c)\,dX}{b_2(X^2+A^2)} = \int \frac{X\,dX}{b_2(X^2+A^2)} + \frac{1}{b_2}\int \frac{c\,dX}{X^2+A^2}$$

$$= \frac{1}{2b_2}\log(X^2+A^2) + \frac{c}{Ab_2}\tan^{-1}\frac{X}{A} + \log y'$$

Exponentiating,

$$y = y'\,(X^2+A^2)^{\frac{1}{2b_2}}\, e^{\frac{c}{Ab_2}\tan^{-1}\frac{X}{A}}$$

$$= y_0\left(1+\frac{X^2}{A^2}\right)^{\frac{1}{2b_2}} e^{\frac{c}{Ab_2}\tan^{-1}\frac{X}{A}}$$

which may be written

$$y = y_0\left(1+\frac{X^2}{A^2}\right)^{-n} e^{-\gamma\tan^{-1}\frac{X}{A}},$$

where $n = -\frac{1}{2b_2}$; $-\gamma = \frac{c}{Ab_2}$; $A^2 = \frac{4b_0b_2-b_1^2}{4b_2^2}$

y_0 may be calculated as in Type A-1. Origin = Mean $-\ \frac{b_1}{2b_2}$

TYPE C-1

XVIII. When $F(x)$ is cubic, and two zeros of the denominator are equal, the equation may be written in the form

$$\int \frac{dy}{y} = \int \frac{(x-a)\,dx}{b_3(x-A_1)^2(x-A_2)} = \frac{A_1-a}{b_3(A_1-A_2)} \int \frac{dx}{(x-A_1)^2}$$

$$- \frac{A_2-a}{b_3(A_1-A_2)^2} \int \frac{dx}{x-A_1} + \frac{A_2-a}{b_3\,(A_1-A_2)^2} \int \frac{dx}{x-A_2}$$

Performing the integration, we have

$$\log y = \frac{-(A_1-a)}{b_3\,(A_1-A_2)(x-A_1)} - \frac{A_2-a}{b_3\,(A_1-A_2)^2} \log(x-A_1)$$

$$+ \frac{A_2-a}{b_3\,(A_1-A_2)^2} \log\,(x-A_2) + \log\,y'$$

Exponentiating,

$$y = \frac{y' e^{-\frac{A_1-a}{b_3\,(A_1-A_2)(x-A_1)}} (x-A_2)^{\frac{A_2-a}{b_3\,(A_1-A_2)^2}}}{(x-A_1)^{\frac{A_2-a}{b_3\,(A_1-A_2)^2}}} = \frac{y' e^{\frac{-m_1}{x-A_1}} (x-A_2)^{m_2}}{(x-A_1)^{m_2}}$$

where $m_1 = \dfrac{A_1-a}{b_3\,(A_1-A_2)}$; $\qquad m_2 = \dfrac{A_2-a}{b_3(A_1-A_2)^2}$

Now, changing the origin to A_1, i. e., replacing x by $x+A_1$, we have

$$y = \frac{y' e^{-\frac{m_1}{x}} (x+A_1-A_2)^{m_2}}{x^{m_2}}$$

$$= \frac{y_0\, e^{-\frac{m_1}{x}} \left(1+x/k\right)^{m_2}}{x^{m_2}}$$

where $k = A_1 - A_2$

y_0 may be determined as in the previous case.
Origin = Mean + A_1.

TYPE C-2

XIX. If all the zeros are equal, the equation may be written

$$\int \frac{dy}{y} = \int \frac{A_1 - a}{b_3 (x - A_1)^3}\, dx + \int \frac{dx}{b_3 (x - A_1)^2}$$

Integrating,

$$\log y = -\frac{A_1 - a}{2b_3 (x - A_1)^2} - \frac{1}{b_3 (x - A_1)} + \log y_0$$

Exponentiating,

$$y = y_0\, e^{\frac{-(A_1 - a)}{2b_3 (x - A_1)^2} - \frac{1}{b_3 (x - A_1)}} = y_0\, e^{\frac{A_1 - a - 2x}{2b_3 (x - A_1)^2}}$$

where y_0 may be determined as before. Origin = Mean.

TYPE C-3

XX. When $F(x)$ is quadratic, and the zeros are real and equal, the equation may be written

$$\int \frac{dy}{y} = \int \frac{1}{b_2}\, \frac{(x - a)\, dx}{\left(x + \frac{b_1}{2b_2}\right)^2}$$

$$= \int \frac{\left[\left(x + \frac{b_1}{2b_2}\right) - \left(a + \frac{b_1}{2b_2}\right)\right] dx}{b_2 \left(x + \frac{b_1}{2b_2}\right)^2}$$

$$= \int \frac{dx}{b_2 \left(x + \frac{b_1}{2b_2}\right)} - \int \frac{\left(a + \frac{b_1}{2b_2}\right)}{b_2 \left(x + \frac{b_1}{2b_2}\right)^2}\, dx$$

$$\therefore \log y = \frac{1}{b_2} \log \left(x + \frac{b_1}{2b_2}\right) + \frac{a + \frac{b_1}{2b_2}}{b_2 \left(x + \frac{b_1}{2b_2}\right)} + \log y'$$

Exponentiating,

$$y = y'\left(x+\frac{b_1}{2b_2}\right)^{\frac{1}{b_2}} e^{\frac{a+\frac{b_1}{2b_2}}{b_2\left(x+\frac{b_1}{2b_2}\right)}}$$

Now let

$$x+\frac{b_1}{2b_2} = x' \; ; \; \frac{1}{b_2} = -\rho \; ; \; \frac{a+\frac{b_1}{2b_2}}{b_2} = -\gamma$$

Then $y = y_0 x^{-\rho} e^{-\frac{\gamma}{x}}$

The constants in terms of moments are

$$-\rho = \frac{-4\mu_3}{\mu_3+2a\mu_2} \; ; \; -\gamma = \frac{4\mu_2(a\mu_3-2a^2\mu_2)}{\mu_3^2-4a\mu_2\mu_3+4a^2\mu_2}$$

It has been found[1] that $y_0 = \frac{N\gamma^{\rho-1}}{\Gamma(\rho-1)}$. Origin = Mean $-\frac{b_1}{2b_2}$

XXI. The following example, illustrated in the chart (p. 140' is given to illustrate the method. The data is fitted by Type A-7.

Ratio of Revenue to Net Worth in Traction Companies

Ratio	Observed Frequency	Theoretical Frequency
.04	7	7.3
.12	43	31.9
.20	48	55.6
.28	75	63.6
.36	53	57.6
.44	34	45.1
.52	25	32.2
.60	22	21.6
.68	12	14.1
.76	14	8.6
.84	5	5.2
.92	6	3.2
1.00	7	1.9

---

[1] Elderton, "Frequency Curves and Correlation," p. 82.

The constants calculated from the observed frequencies were

$$\begin{array}{lll}
\mu'_2 = 6.4354 & b_0 = -5.7908 & m = 24.97 \\
\mu_3 = 18.0115 & b_1 = -1.2125 & p_1 = 5.3312 \\
\text{Mean} = .376 & b_2 = -.0334 & p_2 = 35.2713 \\
\text{Mode} = .2793 & A_1 = 5.6587 & \log y_0 = 50.1479 \\
a = 1.2125 & A_2 = 30.6287 &
\end{array}$$

Origin at Mean - 30.6287 or 30.4287 to left of 53 group.

Curve starts at 30.4287 - 24.97 = 5.4587 before the center of this group.

EQUATION

$$y = y_0 (x - 24.97)^{5.3312} \; x^{-35.2713}$$

Edwin D Mouzon, Jr.

ON FITTING CURVES TO OBSERVATIONAL SERIES BY THE METHOD OF DIFFERENCES

By

HARRY S. WILL

I. PRELIMINARY STATEMENT

Curve fitting may be technically described as the representation of a series of observations by a mathematical function. Given the observations and the function to be fitted, the problem is to determine the constants of the equation in such a way as to secure a valid representation. The method to be employed in the determination of these constants must take into account the object which the fitting process is intended to serve. If the object is to interpolate for undetermined items between specified ordinates of the series, any method which will give the constants of the equation will suffice, since the representation of the given ordinates is exact. In this case, questions of method will hinge on considerations of convenience. If, however, the object is to secure the representation of *all* the items of the series by means of a single function, questions of method will hinge on the validity of the representation, which, in this case, can only be approximate.

Functions used as approximate representations of observational series fall into two general classes: first, those which have the force of a law descriptive of a necessary sequence of events; and, second, those which depict a norm as a characteristic trend in growth. These two types of representation merit separate methodological consideration; and, in what is to follow, we shall make an analysis of the problems involved and develop a method, which, it is believed, will place in the hands of the statistician a new and serviceable instrument.

II. FUNDAMENTAL TYPES OF OBSERVATIONAL SERIES

For the purpose of fixing attention on certain characteristics of observational data, let us consider two distinctly different sorts of series. Let us suppose that the first series consists of a set of observations on a comet moving through space, and that the second consists of the record of gold production in the United States.

For the sake of simplicity, let us further suppose that the movement of both series is properly represented by the function $y = f(x)$. The two sets of observations may then be represented by an equation of the form

$$(1) \qquad Y \pm d \pm \epsilon = f(x) = Y \pm v$$

In this equation, the term ϵ represents an error of observation due to factors such as faulty judgment, clerical inaccuracies, and lack of precision in the use of instruments. The term d represents the deviation of the fitted function from the true magnitude of the phenomenon undergoing examination, after the series has been corrected for the errors ϵ. Taken together, d and ϵ make up the residuals

$$v = f(x) - Y$$

Now it is quite evident that, in the case of the first series, owing to the regularity of the path of the moving body, the deviations d will be negligibly small in comparison with the errors ϵ, and that, in the case of the second series, owing to the irregularity of production, the deviations d will be large in comparison with the errors ϵ. In fitting a curve to the first series, we assume that a true value exists and that the observational errors may be defined by the fitting process; while in fitting to the second, we assume a normal value merely, and seek to define the deviations of the observations from this norm.

These considerations suggest that the procedure which is applicable to the determination of constants in the one case may not be applicable in the other. Let us therefore inquire as to the solutions best suited to each case.

III. THE CLASSICAL SOLUTION

It was in 1806 that Legendre formulated his test of the validity attaching to the functional representation of an observational series. This formulation has become known as the principle of least squares and may be stated thus: *Where the constants of a mathematical function are to be determined from a set of empirical observations, that solution is best which makes the sum of the squares of the residual errors a minimum.*

So far as its mere statement is concerned, this principle is a rule of thumb which may be adopted or discarded at the discretion of the individual. The principal has, however, been placed on a definite logical basis by Gauss and later writers, who have derived it from the normal law of error $p(x) = m\int_0^z e^{-z^2} dz$ Under the assumptions of this law, deviations from the most-probable value are fortuitous in character, the term *fortuitous* implying that individual deviations are unanalytic in the sense that the forces operating to bring them about cannot be resolved into more elemental components. All that we can clam to know *a priori* about the values of such deviations is that they are as likely to be positive as negative and that they must remain within the bounds $\pm\infty$. The function $p(x)$ gives the probability for the occurrence of a deviation of magnitude $z = x/\sigma$.

Statisticians generally have accepted the principle of least squares as providing a sufficient theoretical basis for the fitting of curves to all sorts of series. Because of this, it becomes all the more important that certain limitations of the principle and its application to the analysis of statistical series should be carefully noted.

Considering again the case where the observations are made on a body moving through space, we see that the errors of observation committed may properly be regarded as fortuitous in character, for, on the basis of our assumption of precise motion in the path $y = f(x)$, the most probable value of the residuals is clearly defined as zero, so that the errors committed are as likely to be positive as negative; no finite bound can be set as to the possible magnitude of such random errors, and the forces determining their magnitudes cannot be resolved into their components If our assumption as to the path of the moving body is valid, these errors conform to the normal law in the frequency of their occurrence, and their magnitudes may be accurately ascertained by a least squares determination of constants.

Returning now to the case where the observations consist of a record of gold production, can we claim to have the same basis for an application of least squares to the determination of our line of best fit? Two important considerations would lead us to think otherwise. The first of these is that the magnitude of deviations from trend is definitely restricted; for production is limited both by the capacity of the extractive industries and by the consumers' demand. The second is found in the highly analytic character of these deviations; for it is significant that whenever it becomes possible to resolve the forces determining the values of given deviations of a set into their elemental components, the prediction of the sign and magnitude of specified deviations becomes in some measure possible; and when this occurs, such deviations are removed from the category of the fortuitous and unpredictable and placed in that of the analytic and predictable.

The arguments are supported by the use made of weighted deviations from trend in the forecasting of economic events. A rise in price or fall in production is not explained, in comparison with the normal trend, as a circumstance which is to be expected a certain number of times in a thousand, but rather because analysis shows the rise or fall to be the necessary result of known events. Obviously, a forecast based on unanalytic and purely fortuitous deviations could have no real significance whatever.

Granting that residuals may sometimes be obtained by least squares operations which may be regarded as a random sample of an approximately normal distribution, it must be clearly borne in mind that these residuals are brought into being by the creative act of curve fitting; and the mere marking off of a deviation does not justify our regarding it as being due to the working of forces distinct and different from those effective in producing the remaining part of the ordinate. In the case of the celestial observations which we have assumed, the act of fitting defines, but does not create, the errors.

The argument may be advanced at this point that it is not necessary to regard the principle of least squares as resting on the law of error: for we may obtain the normal equations from which our least squares determination is made by treating the solution as a simple problem in maxima and minima. But if we do, we cannot claim to have determined the *most probable* values of our constants; for this claim must rest on the derivation of the normal equations from the law of error.

The justification for the arbitrary use of the least squares technique that is most likely to be made is that it minimizes extreme deviations from the fitted line. This is unquestionably true; but it appears as a weakness of the method in the present connection rather than as an element of strength; for, in a least squares deduction of normal equations, we may regard each absolute deviation as being weighted with its own magnitude, deviations less than the mean deviation receiving weights less than the mean weight, and *vice versa*; and why, the query obtrudes, should we, in our determination of constants, overweight the observations most remote from what we term the norm and underweight those which lie closest?

The argument that the least squares fit will avoid the commission of extreme errors in the projection of the curve beyond the limits of observation, or at least tend in that direction, is fallacious; for the fitting of a line to a given set of observations to secure the minimum sum of the squared residuals is unlikely to effect the same end when new observations are added. At least, we have no logical basis for the expectation of such a result unless we fall back on the position that the fitted curve describes a necessary sequence of events and that the residuals are fortuitous in character; and this is the very assumption we have found to be untenable for most economic and social series.

We may, then, say that fortuitous deviations are properly to be regarded as functions of the observations; while analytic deviations are to be regarded as functions of the hypothesis we set up with reference to the type of curve which is most appropriate to the data. In brief, our reasoning supplies a definite basis for the contention that, for data in which the errors of observation are small in comparison with the analytic deviations from trend, the least squares definitions do not lead to results which are to be regarded as necessarily best for all purposes.

IV. THE METHOD OF DIFFERENCES

The method of curve fitting which is now to be presented was originated by the writer in the spring of 1925. Since that time, it has been put to a wide variety of practical tests and has been found to yield highly satisfactory results. The designation ***method of differences*** has been given to it because of the extensive and essential use made of the calculus of finite differences.

Before undertaking the task of deriving the formulas for the determination of constants, let us state the assumptions on which the method is based, as follows:

(a) The function to be fitted is logically appropriate to the data.

(b) The data are free of constant and systematic errors.

(c) Accidental errors of observation are relatively small and unimportant.

(d) Where a set of secular values is irregular and without significant trend, the arithmetic mean is the best representation of the set.

The first of these assumptions is, in a general way, implied in any method of fitting. The effect of the second and third is to qualify the fitted deviations as analytic. The fourth is made use of constantly in the writing of formulas for the determination of parameters.

In the derivation of formulas, the essential steps are as follows: (1) equations defining each constant of the function fitted are developed by a process of differencing; (2) equations are formed from which approximations to the value of the given constant may be obtained; (3) the mean of the several approximations to the value of the given constant is taken as the most plausible value of the constant.

V. NOTATION

To avoid the possibility of misunderstanding, we shall explicitly define certain symbols made use of in this memoir.

The original observations are denoted by the symbol Y_i, $i = 0, 1, 2, \ldots n-1$; nd other capitals are used to designate empirical functions of the original observations; e. g., $U_i = Y_i - Y_0$. The symbol u_i denotes values of mathematical functions corresponding to the observations Y_i The argument is denoted by the symbol

$$x_i , \quad x_i = i\Delta_x$$

Summations within the definite bounds a and b is indicated by the symbol $\sum_a^b$: e. g., $\sum_{i=0}^{i=n-1} y_i = y_0 + y_1 + \ldots + y_{n-1}$.

Finite differences of order r and rank k are defined by the symbol Δ_k^r ; e. g, $\Delta_k^r y_i = \Delta_k^{r-1} y_{i+k} - \Delta_k^{r-1} y_i$, where the difference of zero order is taken as the quantity undifferenced. In particular, we have $\Delta_k^0 y_i = y_i$; $\Delta_k y_i = y_{i+k} - y_i$; $\Delta_k^2 y_i = y_{i+2k} - 2y_{i+k} + y_i$; $\Delta_k^3 y_i = y_{i+3k} - 3y_{i+2k} + 3y_{i+k} - y_i$.

In these relations, the values of k and r are integral. The value of y_{i+k} is precisely the value of the function $y = f(x)$ when $x = x_i + k\Delta x$. In the difference operations of the following sections, the usage $\Delta_x^2 = (\Delta_x)^2 \neq \Delta(x^2)$ is adhered to. Note that $\Delta_k \log y_i = \log y_{i+k} - \log y_i$, and also that $\log \Delta_k y_i = \log(y_{i+k} - y_i)$.

Since, on taking logarithms, ratios resolve themselves into differences between logarithms, we have, analagous to the differences $\Delta_k^r y_i$, the ratios $p_k^r y_i = p_k^{r-1} y_{i+k} : p_k^{r-1} y_i$, where the ratio of order $r = 0$ denotes the specified quantity. In particular, we have $p_k^0 y_i = y_i$; $p_k y_i = y_{i+k} : y_i$; $p_k^2 y_i = y_{i+2k} \cdot y_i : y_{i+k}^2$; $p_k^3 y_i = (y_{i+3k} \cdot y_{i+k}^3) : (y_{i+2k}^3 \cdot y_i)$.

In forming the first differences $\Delta_k y_i$, where k is the increment in the y subscript corresponding to the increment $k\Delta x$ in x_i, it will be noted that the first k values of y_i are excluded as minuend; hence we can form but $n - k$ first differences of rank k, that is, when the increment in the y subscript is k. Similarly, in forming the second differences $\Delta_k^2 y_i = \Delta_k y_{i+k} - \Delta_k y_i$, the first k values of $\Delta_k y_i$ are excluded from appearance as minuend; hence we can form but $n - k - k = n - 2k$ second differences from n values of y when the rank of differences is k. In general, when the rank of differences is k, we may form $n - rk$ differences of order r from a set of values of y. Evidently, the number of ratios which may be formed from a given number of observations follows the same rule as that which applies to differences.

VI. LINEAR SERIES

Let us write the equation of the linear series in the form

$$y_i = a + bx_i \tag{1}$$

Giving to x the increment $k\Delta x$, we get

$$y_{i+k} = a + b\,(x_i + k\Delta x) \tag{2}$$

Subtracting (1) from (2), we get

$$\Delta_k y_i = b\,k\Delta x \tag{3}$$

By making the substitution $\Delta_k Y_i$ for $\Delta_k y_i$ in equation (3), we may form $n - k$ approximations to the value of b, as follows:

$$\begin{aligned} b_0 &= \Delta_k Y_0 : k\Delta x \\ b_1 &= \Delta_k Y_1 : k\Delta x \\ &\cdots\cdots\cdots \\ b_{n-k-1} &= \Delta_k Y_{n-k-1} : k\Delta x \end{aligned} \tag{4}$$

Similarly, when b is determined, by substituting Y_i for y_i in equation (1), we are able to form n approximations to the value of a, as follows:

$$
(5) \qquad
\begin{aligned}
a_0 &= Y_0 - bx_0 \\
a_1 &= Y_1 - bx_1 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{n-1} &= Y_{n-1} - bx_{n-1}
\end{aligned}
$$

By taking mean values of the approximations specified in equations (4) and (5), we arrive at the following formulas for determining the value of the parameters b and a:

$$
(6) \qquad
\begin{aligned}
b &= \Big[\sum_{i=0}^{i=n-k-1} \Delta_k Y_i\Big] : \big[k\,(n-k)\,\Delta x\big]. \\
a &= \Big[\sum_{i=0}^{i=n-1} Y_i - b\sum_{i=0}^{i=n-1} x_i\Big] : n. \\
& \\
2k &= n \pm j \qquad j = 0 \text{ or } 1.
\end{aligned}
$$

This arbitrary determination of k will be justified in a later section

VII. Parabolic Series

The equation of the quadratic parabola is

$$(1) \qquad y_i = a + bx_i + cx_i^2 .$$

Giving to x the increment $k\Delta x$, we have

$$(2) \qquad y_{i+k} = a + b\,(x_i + k\Delta x) + c\,(x_i + k\Delta x)^2 .$$

Subtracting (1) from (2), we have

$$(3) \qquad \Delta_k y_i = bk\Delta x + ck\Delta x\,(2x_i + k\Delta x).$$

Giving to x a second increment $k\Delta x$, we have

$$\Delta_k y_{i+k} = bk\Delta x + c\,k\Delta x\ (2x_i + 3k\Delta x) \tag{4}$$

Subtracting (3) from (4), we obtain

$$\Delta_k^2 y_i = 2ck^2\Delta x^2 \tag{5}$$

From equations (5), (3), and (1), we deduce the following approximations to parameters:

$$\begin{aligned} c_i &= \Delta_k^2 Y_i : (2k^2\Delta x^2) \quad i = 0, 1, \cdots n-2k-1. \\ b_i &= [\Delta_k Y_i - ck\Delta x\ (2x_i + k\Delta x)] : [k\Delta x], \quad i = 0, 1, \cdots n-k-1. \\ a_i &= Y_i - bx_i - cx_i^2, \quad i = 0, 1, \cdots n-1 \end{aligned} \tag{6}$$

Taking mean values of the approximations indicated in equations (6), we have the following formulas for the determination of parameters.

$$\begin{aligned} c &= \left[\sum_{i=0}^{i=n-2k-1} \Delta_k^2 Y_i\right] : \left[2k^2(n-2k)\Delta x^2\right]. \\ b &= \left[\sum_{i=0}^{i=n-k-1} \Delta_k Y_i - ck\Delta x \sum_{i=0}^{i=n-k-1}(2x_i + k\Delta x)\right] : \left[k(n-k)\Delta x\right]. \\ a &= \left[\sum_{i=0}^{i=n-1} Y_i - b\sum_{i=0}^{i=n-1} x_i - c\sum_{i=0}^{i=n-1} x_i^2\right] : n. \\ 3k &= n \pm j, \quad j = 0, 1, \text{ or } 2 \end{aligned} \tag{7}$$

We shall next write the equation of the cubic parabola, which is

$$y_i = a + bx_i + cx_i^2 + dx_i^3 \tag{8}$$

Giving to x the increment $k\Delta x$, we have

$$y_{i+k} = a + b(x_i + k\Delta x) + c(x_i + k\Delta x)^2 + d(x_i + k\Delta x)^3. \tag{9}$$

Subtracting (8) from (9), we get

$$\Delta_k y_i = bk\Delta x + ck\Delta x\,(2x_i + k\Delta x) + d\,k\Delta x\,(3x_i^2 + 3x_i\,k\Delta x + k^2\Delta x^2). \tag{10}$$

TABLE I

Production of Gold in the United States

(in units of $100,000)

$y = 695.75 + 53.89x - 2.81x^2$

Year	Y	y	v	Year	Y	y	v
1900	792	696	96	1911	969	949	20
1901	787	747	40	1912	935	938	− 3
1902	800	792	8	1913	883	921	−38
1903	736	832	−96	1914	945	899	46
1904	805	866	−61	1915	1010	872	138
1905	882	895	−13	1916	926	839	87
1906	944	918	26	1917	838	800	38
1907	904	935	−31	1918	686	755	−69
1908	946	947	− 1	1919	603	705	−102
1909	997	953	44	1920	512	650	−138
1910	963	954	9				
Σ					17863	17863	0

Mean error of estimate 52.6

Again giving to x the increment $k\Delta x$, we have

$$\Delta_k y_{1+k} = bk\Delta x + ck\Delta x\,(2x_1 + 3k\Delta x) + dk\Delta x\,[3(x_1 + k\Delta x)^2 + 3k\Delta x\,(x_1 + k\Delta x) + k^2\Delta x^2]. \tag{11}$$

Subtracting (10) from (11), we obtain

$$\Delta_k^2 y_1 = 2ck^2\Delta x^2 + 6dk^2\Delta x^2(x_1 + k\Delta x). \tag{12}$$

Once more increasing x by $k\Delta x$, we have

$$\Delta_k^2 y_{1+k} = 2ck^2\Delta x^2 + 6dk^2\Delta x^2(x_1 + 2k\Delta x). \tag{13}$$

Subtracting (12) from (13), we obtain finally

$$\Delta_k^3 y_1 = 6dk^3\Delta x^3. \tag{14}$$

From equations (14), (12), (10), and (8), we deduce the following parametric approximations:

$$d_i = \Delta_k^3 Y_i : (6k^3 \Delta x^3), \quad i = 0, 1, \cdots n-3k-1.$$

$$c_i = [\Delta_k^2 Y_i - 6dk^2 \Delta x^2 (x_i + k\Delta x)] : [2k^2 \Delta x^2], \quad i = 0, 1, \cdots n-2k-1.$$

$$(15) \quad b_i = [\Delta_k Y_i - ck\Delta x(2x_i + k\Delta x) - dk\Delta x(3x_i^2 + 3x_i k\Delta x + k^2\Delta x^2)]$$

$$: [k\Delta x], \quad i = 0, 1, \cdots n-k-1.$$

$$a_i = Y_i - bx_i - cx_i^2 - dx_i^3, \quad i = 0, 1, \cdots n-1.$$

Taking mean values of the approximations indicated in equations (15), we have the following formulas for the determination of parameters:

$$d = \left[\sum_{i=0}^{i=n-3k-1} \Delta_k^3 Y_i\right] : [6k^3 (n-3k)\Delta x^3].$$

$$c = \left[\sum_{i=0}^{i=n-2k-1} \Delta_k^2 Y_i - 6dk^2\Delta x^2 \sum_{i=0}^{i=n-2k-1} (x_i - k\Delta x)\right] : [2k^2 (n-2k)\Delta x^2].$$

$$(16) \quad b = \left[\sum_{i=0}^{i=n-k-1} \Delta_k Y_i - ck\Delta x \sum_{i=0}^{i=n-k-1} (2x_i + k\Delta x)\right.$$

$$\left. - dk\Delta x \sum_{i=0}^{i=n-k-1} (3x_i^2 + 3x_i - k\Delta x + k^2\Delta x^2)\right] : [k(n-k)\Delta x].$$

$$a = \left[\sum_{i=0}^{i=n-1} Y_i - b\sum_{i=0}^{i=n-1} x_i - c\sum_{i=0}^{i=n-1} x_i^2 - d\sum_{i=0}^{i=n-1} x_i^3\right] : n.$$

$4k = n \pm j, \qquad j = 0, 1, 2$, or 3.

VIII. HYPERBOLIC SERIES

Let us write the hyperbolic series

$$(1) \quad y_i = a + bx_i + c : (x_i + 1)$$

Giving the increment $k\Delta x$ to x, we have

$$(2) \quad y_{i+k} = a + b(x_i + k\Delta x) + c : (x_i + 1 + k\Delta x)$$

By subtraction, we have

$$(3) \quad \Delta_k y_i = bk\Delta x - ck\Delta x : ((x_i + 1)(x_i + 1 + k\Delta x))$$

Giving a second increment $k\Delta x$ to x, we obtain

$$\Delta_k y_{i+k} = bk\Delta x - ck\Delta x : \left((x_i+1+k\Delta x)(x_i+1+2k\Delta x)\right). \tag{4}$$

By subtraction, we obtain

$$\Delta_k^2 y_i = 2ck^2\Delta x^2 : \left((x_i+1)(x_i+1+2k\Delta x)\right). \tag{5}$$

Making the substitutions $x_i' = (x_i+1)(x_i+1+k\Delta x)$, and $x_i'' = (x_i+1+2k\Delta x)$, we have, from equations (5), (3), and (1), the following parametric approximations:

$$c_i = (x_i'' \Delta_k^2 Y_i) : (2k^2\Delta x^2), \quad i = 0, 1, \cdots n-2k-1.$$

$$b_i = (\Delta_k Y_i + ck\Delta x : x_i') : (k\Delta x), \quad i = 0, 1, \cdots n-k-1. \tag{6}$$

$$a_i = Y_i - bx_i - c : (x_i+1), \quad i = 0, 1, \cdots n-1.$$

By taking mean values of the approximations indicated in equations (6), we have the following formulas for determining parameters:

$$c = \left[\sum_{i=0}^{i=n-2k-1} x_i'' \Delta_k^2 Y_i\right] : \left[2k^2(n-2k)\Delta x^2\right].$$

$$b = \left[\sum_{i=0}^{i=n-k-1} \Delta_k Y_i + ck\Delta x\left(\sum_{i=0}^{i=n-k-1} 1/x_i'\right)\right] : \left[k(n-k)\Delta x\right]. \tag{7}$$

$$a = \left[\sum_{i=0}^{i=n-1} Y_i - b\sum_{i=0}^{i=n-1} x_i - c\sum_{i=0}^{i=n-1} (x_i+1)^{-1}\right] : n.$$

$$3k = n \pm j, \quad j = 0, 1, \text{ or } 2.$$

If the coefficient of x in (1) is zero, we have simply

$$y_i = a + b : (x_i+1). \tag{8}$$

Formulas (7) now reduce to

$$b = -\left[\sum_{i=0}^{i=n-k-1} x_i' \Delta_k Y_i\right] : \left[k(n-k)\Delta x\right].$$

$$a = \left[\sum_{i=0}^{i=n-1} Y_i - b\sum_{i=0}^{i=n-1} (x_i - 1)^{-1}\right] : n. \tag{9}$$

$$2k = n \pm j \quad j = 0 \text{ or } 1.$$

It is evident that a term in x^2 or x^3 could be added to equation (1) and a solution be obtained by a direct extension of the general method of analysis applied to equation (1).

IX. LOGARITHMIC SERIES

Let us write the logarithmic equation

(1) $$y_i = a + b\,x_i + c \cdot \log\,(x_i + l)\,.$$

Giving x the increment $k\Delta x$, we have

(2) $$y_{i+k} = a + b\,(x_i + k\Delta x) + c \cdot \log\,(x_i + l + k\Delta x)\,.$$

Subtracting (1) from (2), we get

(3) $$\Delta_k y_i = b\,k\Delta x + c\,\Delta_k \log\,(x_i + l)\,.$$

Giving to x a second increment $k\Delta x$, we get

(4) $$\Delta_k y_{i+k} = b\,k\Delta x + c\,\Delta_k \log\,(x_i + l + k\Delta x)\,.$$

TABLE II

Deaths from Typhoid Fever in Greater City of New York

(Number of deaths per 1,000,000 inhabitants)

$$y = 143.899 + 8.695\,x - 206.652\,\Gamma x$$

Year	*Y*	*y*	*v*	*Year*	*Y*	*y*	*v*
1911	111.7	152.6	-40.9	1919	21.8	25.6	- 3.8
1912	100.5	99.3	1.2	1920	24.2	24.8	- 0.6
1913	72.0	71.7	0.3	1921	21.3	25.1	- 3.8
1914	65.0	54.7	10.3	1922	22.1	26.0	- 3.9
1915	63.5	43.4	20.1	1923	23.6	27.4	- 3.8
1916	40.6	35.8	4.8	1924	30.0	29.6	0.4
1917	42.4	30.7	11.7	1925	31.9	32.1	- 0.2
1918	35.6	27.4	8.2				
Σ					706.2	706.2	0.0

Mean error of estimate 7.6

Subtracting (3) from (4), we obtain

$$\Delta_k^2 y_i = c\Delta_k^2 \log(x_i+1)\,. \tag{5}$$

From equations (5), (3), and (1), we have the following approximations to parameters.

$$\begin{aligned}
c_i &= \Delta_k^2 Y_i : \Delta_k^2 \log(x_i+1), \quad i=0,1,\cdots n-2k-1. \\
b_i &= \Delta_k Y_i - c\Delta_k \log(x_i+1), \quad i=0,1,\cdots n-k-1. \\
a_i &= Y_i - b x_i - c\cdot\log(x_i+1), \quad i=0,1,\cdots n-1.
\end{aligned} \tag{6}$$

Taking mean values of the approximations indicated in equations (6), we have the following formulas for the determination of parameters:

$$\begin{aligned}
c &= \left[\sum_{i=0}^{i=n-2k-1}\left(\Delta_k^2 Y_i : \Delta_k^2 \log(x_i+1)\right)\right] : \left[n-2k\right]. \\
b &= \left[\sum_{i=0}^{i=n-k-1}\Delta_k Y_i - c\sum_{i=0}^{i=n-k-1}\Delta_k \log(x_i+1)\right] : \left[k(n-k)\Delta x\right]. \\
a &= \left[\sum_{i=0}^{i=n-1} Y_i - b\sum_{i=0}^{i=n-1} x_i - c\sum_{i=0}^{i=n-1}\log(x_i+1)\right] : n\,, \\
3k &= n\pm j\,, \quad j=0,1, \text{ or } 2
\end{aligned} \tag{7}$$

If the coefficient of x in equation (1) is zero, we have

$$y_i = a + b\cdot\log(x_i+1)\,. \tag{8}$$

Formulas (7) then reduce to

$$\begin{aligned}
b &= \left[\sum_{i=0}^{i=n-k-1}\left(\Delta_k Y_i : \Delta_k \log(x_i+1)\right)\right] : \left[n-k\right]. \\
a &= \left[\sum_{i=0}^{i=n-1} Y_i - b\sum_{i=0}^{i=n-1}\log(x_i+1)\right] : n\,. \\
2k &= n\pm j\,, \quad j=0 \text{ or } 1
\end{aligned} \tag{9}$$

X. GENERAL POLYNOMIAL SERIES

The solutions of polynomials presented in the preceding sections, while best for the series considered, are too specialized in mode of analysis for application to polynomials generally. We shall now develop a solution which is applicable to any polynomial in $z = f(x)$, $f(x)$ being a function of x whose value is known, as for example, x^{-1}, $\log x$, $\tan x$, etc.

We write

(1) $y_i = d + cz_i + bz_i^2 + az_i^3$.

Giving to z_i the increment $\Delta_k z_i$, we have

(2) $y_{i+k} = d + cz_{i+k} + bz_{i+k}^2 + az_{i+k}^3$.

Subtracting (1) from (2), we get

(3) $\Delta_k y_i = c\Delta_k z_i + b\Delta_k z_i^2 + a\Delta_k z_i^3$.

This is the i^{th} equation $\Delta_k y$. We write the $i+k^{th}$ equation as

(4) $\Delta_k y_{i+k} = c\Delta_k z_{i+k} + b\Delta_k z_{i+k}^2 + a\Delta_k z_{i+k}^3$.

Multiplying (3) by $\Delta_k z_{i+k}$ and (4) by $\Delta_k z_i$ and then subtracting (4) from (3), we obtain

(5) $\Delta_k' y_i = b\Delta_k' z_i^2 + a\Delta_k' z_i^3$,

where $\Delta_k' y_i = \Delta_k y_i \cdot \Delta_k z_{i+k} - \Delta_k y_{i+k} \cdot \Delta_k z_i$; $\Delta_k' z_i^2 = \Delta_k z_i^2 \cdot \Delta_k z_{i+k}$
$- \Delta_k z_{i+k}^2 \cdot \Delta_k z_i$;

and $\Delta_k' z_i^3 = \Delta_k z_i^3 \cdot \Delta_k z_{i+k} - \Delta_k z_{i+k}^3 \cdot \Delta_k z_i$.

In (5), we have the $i\,th$ equation $\Delta_k' y$. We write the $i+k^{th}$ equation as

(6) $\Delta_k' y_{i+k} = b\Delta_k' z_{i+k}^2 + a\Delta_k' z_{i+k}^3$.

Multiplying (5) by $\Delta_k' z_{i+k}^2$ and (6) by $\Delta_k' z_i^2$ and subtracting the latter result from the former, we obtain

$$(7)\qquad \Delta''_k y_i = a\,\Delta''_k z_i^3 ,$$

where $\Delta''_k y_i = \Delta'_k y_i \cdot \Delta'_k z^2_{i+k} - \Delta'_k y_{i+k} \cdot \Delta'_k z^2_i$;

and $\Delta''_k z^3_i = \Delta'_k z^3_i \cdot \Delta'_k z^2_{i+k} - \Delta'_k z^3_{i+k} \cdot \Delta'_k z^2_i$.

From equations (7), (5), (3), and (1), we are now able to write the following parametric approximations:

$$(8)\qquad \begin{aligned} a_i &= \left[\Delta''_k Y_i\right] : \left[\Delta''_k z^3_i\right], \quad i = 0,\ 1,\ \cdots\ n-3k-1 .\\ b_i &= \left[\Delta'_k Y_i - a\Delta'_k z^3_i\right] : \left[\Delta'_k z^2_i\right], \quad i = 0,\ 1,\ \cdots\ n-2k-1,\\ c_i &= \left[\Delta_k Y_i - b\Delta_k z^2_i - c\Delta_k z^3_i\right] : \left[\Delta_k z_i\right], \quad i = 0,\ 1, \cdots n-k-1 .\\ d_i &= Y_i - cz_i - bz^2_i - az^3_i , \quad i = 0,\ 1,\ \cdots\ n-1 \end{aligned}$$

When $z_0 = \pm\infty$, i takes the values 1, 2, . . . $n-rk-1$, r being the number of reductions essential to the approximation.

The mean values of equations (8) give the following determinations:

$$(9)\qquad \begin{aligned} a &= \left[a_0 + a_1 + \cdots + a_{n-3k-1}\right] : \left[n-3k\right].\\ b &= \left[b_0 + b_1 + \cdots + b_{n-2k-1}\right] : \left[n-2k\right].\\ c &= \left[c_0 + c_1 + \cdots + c_{n-k-1}\right] : \left[n-k\right].\\ d &= \left[\sum_{i=0}^{i=n-1} Y_i - c\sum_{i=0}^{i=n-1} z_i - b\sum_{i=0}^{i=n-1} z_i^2 - a\sum_{i=0}^{i=n-1} z_i^3\right] : n . \end{aligned}$$

If equation (1) is simplified to

$$(10)\qquad y_i = c + bz_i + az_i^2 ,$$

the parametric approximations become the following:

$$a_i = [\Delta'_k Y_i] : [\Delta'_k z_i^2], \quad i = 0, 1, \cdots n-2k-1.$$

$$(11) \qquad b_i = [\Delta_k Y_i - a\Delta_k z_i^2] : [\Delta_k z_i], \quad i = 0, 1, \cdots n-k-1.$$

$$c_i = Y_i - bz_i - az_i^2, \quad i = 0, 1, \cdots n-1.$$

The mean values of these approximations give the parameters sought.

XI. EXPONENTIAL SERIES

We shall now write the equation of the exponential series

$$(1) \qquad y_i = d + cx_i + be^{ax_i}.$$

Giving to x the increment $k\Delta x$, we have

$$(2) \qquad y_{i+k} = d + c(x_i + k\Delta x) + be^{a(x_i + k\Delta x)}$$

Subtracting (1) from (2), we get

$$(3) \qquad \Delta_k y_i = ck\Delta x + bhe^{ax_i},$$

where $h = e^{ak\Delta x} - 1$.

Giving to x a second increment $k\Delta x$, we obtain

$$(4) \qquad \Delta_k y_{i+k} = ck\Delta x + bhe^{a(x_i + k\Delta x)}.$$

Subtracting (3) from (4), we obtain

$$(5) \qquad \Delta_k^2 y_i = bh^2 e^{ax_i}$$

Taking logarithms, we have

$$(6) \qquad \log \Delta_k^2 y_i = \log(bh^2) + ax_i$$

Again giving x the increment $k\Delta x$, we have

$$(7) \qquad \log \Delta_k^2 y_{i+k} = \log(bh^2) + a(x_i + k\Delta x).$$

Subtracting (6) from (7), we obtain

$$\Delta_k \log \Delta_k^2 y_i = a k \Delta x \tag{8}$$

From equations (8), (5), (3), and (1), we form the following parametric approximations:

$$\begin{aligned} a_i &= (\Delta_k \log \Delta_k^2 Y_i) : (k\Delta x), \quad i = 0,\ 1, \cdots n-3k-1. \\ b_i &= (\Delta_k^2 Y_i) : (h^2 e^{ax_i}), \quad i = 0,\ 1, \cdots n-2k-1. \\ c_i &= (\Delta_k Y_i - bhe^{ax_i}) : (k\Delta x), \quad i = 0,\ 1, \cdots n-k-1. \\ d_i &= Y_i - cx_i - be^{ax_i}, \quad i = 0,\ 1, \cdots n-1. \end{aligned} \tag{9}$$

Taking mean values of the approximations indicated in equations (9), we have the following formulas for determining parameters:

$$\begin{aligned} a &= \Big[\sum_{i=0}^{i=n-3k-1} \Delta_k \log \Delta_k^2 Y_i\Big] : \big[k(n-3k)\Delta x\big]. \\ b &= \Big[\sum_{i=0}^{i=n-2k-1} (\Delta_k^2 Y_i : e^{ax_i})\Big] : \big[h^2(n-2k)\big]. \\ c &= \Big[\sum_{i=0}^{i=n-k-1} \Delta_k Y_i - bh\sum_{i=0}^{i=n-k-1} e^{ax_i}\Big] : \big[k(n-k)\Delta x\big]. \\ d &= \Big[\sum_{i=0}^{i=n-1} Y_i - c\sum_{i=0}^{i=n-1} x_i - b\sum_{i=0}^{i=n-1} e^{ax_i}\Big] : n \end{aligned} \tag{10}$$

If, in equation (1), the coefficient of x is zero, we have

$$y_i = c + be^{ax_i}, \tag{11}$$

and the formulas for determining parameters become

$$\begin{aligned} a &= \Big[\sum_{i=0}^{i=n-2k-1} \Delta_k \log \Delta_k Y_i\Big] : \big[k(n-2k)\Delta x\big]. \\ b &= \Big[\sum_{i=0}^{i=n-k-1} (\Delta_k Y_i : e^{ax_i})\Big] : \big[h(n-k)\big]. \\ c &= \Big[\sum_{i=0}^{i=n-1} Y_i - b\sum_{i=0}^{i=n-1} e^{ax_i}\Big] : n. \end{aligned} \tag{12}$$

XII. LOGISTIC SERIES

Let us write the equation of the logistic series

$$(1)\qquad y_i = \left[d + cx_i\right] \cdot \left[1 + be^{ax_i}\right].$$

Multiplying by the denominator on the right and transposing, we get

$$(2)\qquad y_i = d + cx_i - by_i e^{ax_i}.$$

Giving to x the increment $k\Delta x$, we have

$$(3)\qquad y_{i+k} = d + cx_{i+k} - by_{i+k} e^{ax_{i+k}}$$

Subtracting (2) from (3), we get

$$(4)\qquad \Delta_k y_i = ck\Delta x - by_{i+k} e^{ax_{i+k}} + by_i e^{ax_i}.$$

Again giving to x the increment $k\Delta x$, we obtain

$$(5)\qquad \Delta_k y_{i+k} = ck\Delta x - by_{i+2k} e^{ax_{i+2k}} + by_{i+k} e^{ax_{i+k}}$$

Subtracting (4) from (5), we have

$$(6)\qquad \begin{aligned} \Delta_k^2 y_i &= -by_{i+2k} e^{ax_{i+2k}} + 2by_{i+k} e^{ax_{i+k}} - by_i e^{ax_i} \\ &= -be^{ax_i}\left(y_{i+2k} e^{a2k\Delta x} - 2y_{i+k} e^{ak\Delta x} + y_i\right). \end{aligned}$$

If, in (6), we give to x the increment $k\Delta x$, we get

$$(7)\qquad \begin{aligned} \Delta_k^2 y_{i+k} &= -by_{i+3k} e^{ax_{i+3k}} + 2by_{i+2k} e^{ax_{i+2k}} - by_{i+k} e^{ax_{i+k}} \\ &= -be^{ax_i}\left(y_{i+3k} e^{a3k\Delta x} - 2y_{i+2k} e^{a2k\Delta x} + y_{i+k} e^{ak\Delta x}\right) \end{aligned}$$

On dividing (7) by (6) and multiplying the quotient on the right by the parenthetical expression of (3), we have

$$(8)\qquad \begin{aligned} &p_k \Delta_k^2 y_i \left(y_{i+2k} e^{a2k\Delta x} - 2y_{i+k} e^{ak\Delta x}\ y_i\right) \\ &= y_{i+3k} e^{a3k\Delta x} - 2y_{i+2k} e^{a2k\Delta x} + y_{i+k} e^{ak\Delta x}. \end{aligned}$$

TABLE III

Population of Ohio

U. S Census Count Interpolated to January 1; Unit, 1 000 persons

$$y = 91.8\left(1+e^{6.0266077+29.10267 \log \sin x}\right)$$

Year	Y	y	v	Year	Y	y	v
1800	41.2	91.8	+ 50.6	1870	2651.8	2751.3	+ 99.5
1810	219.7	319.8	+100.1	1880	3175.9	3237.2	+ 61.3
1820	560.3	639 3	+ 79.0	1890	3652.7	3738.1	+ 85.4
1830	922.9	1005.6	+ 82.7	1900	4137.4	4252.5	+115.1
1840	1495 3	1405 6	− 89.7	1910	4749.3	4778.9	+ 29.6
1850	1961 3	1832.8	−128.5	1920	5759.4	5316.0	−443.4
1860	2324.5	2282.3	− 42.2				
Σ					31651.7	31651.2	− 0.5

Mean error of estimate 108 2

Predicted Population

Year	1930	1940	1950	1960	1970	1980
Population	6306	6862	7425	7996	8573	9156

Simplifying (8), we have

$$y_{1+3k}e^{a3k\Delta x}-y_{1+2k}\left(2+p_k\Delta_k^2 y_1\right)e^{a2k\Delta x}+y_{1+k}\left(1+2p_k\Delta_k^2 y_1\right)e^{ak\Delta x} -y_1 p_k\Delta_k^2 y_1 = 0 \tag{9}$$

Equation (9) is evidently cubic in $e^{ak\Delta x}$, and its roots are to be found by conventional methods, care being taken to select the root which will give the parametric approximation most consistent with the hypotheses under which the function is being fitted.

From equations (9), (6), (4), and (1), we are able to form the following approximations to parameters:

$$(10)\quad \begin{aligned} &a_i = [\log e^{ak\Delta x}] : [k\Delta x], \qquad i = 0,\ 1,\ \cdots\ n-3k-1. \\ &b_i = [\Delta_k^2 Y_i] : [-Y_{i+2k} e^{ax_{i+2k}} + 2Y_{i+k} e^{ax_{i+k}} - Y_i e^{ax_i}], \quad i=0,1,\cdots n-2k-1. \\ &c_i = [\Delta_k Y_i + bY_{i+k} e^{ax_{i+k}} - bY_i e^{ax_i}] : [k\Delta x], \quad i=0,\ 1,\ \cdots n-k-1. \\ &d_i = Y_i + bY_i e^{ax_i} - cx_i, \qquad i = 0,\ 1,\ \cdots\ n-1. \end{aligned}$$

The mean values of the indicated approximations give the best values of the parameters sought.

If, in equation (1), the coefficient of x is zero, we have the Verhulst logistic,

$$(11)\quad y_i = c : (1 + be^{ax_i}).$$

The solution of this equation by the method of analysis applied to equation (1) leads eventually to the following:

$$(12)\quad y_{i+2k} e^{a2k\Delta x} - y_{i+k}(1 + \rho_k \Delta_k y_i) e^{ak\Delta x} + y_i \rho_k \Delta_k y_i = 0.$$

which is evidently quadratic in $e^{ak\Delta x}$.

The parametric approximations take the form

$$(13)\quad \begin{aligned} &a_i = [\log e^{ak\Delta x}] : [k\Delta x], \qquad i=0,\ 1,\ \cdots\ n-2k-1. \\ &b_i = -[\Delta_k Y_i] : [Y_{i+k} e^{ax_{i+k}} - Y_i e^{ax_i}], \quad i=0,\ 1,\ \cdots\ n-k-1. \\ &c_i = Y_i + bY_i e^{ax_i}, \quad i=0,\ 1,\ \cdots\ n-1. \end{aligned}$$

The mean values of these approximations give the values of parameters.

The Verhulst logistic may also be solved by applying formulas (12), section XI, to the ordinates $1/Y_i$, the solution being for $1/c$,

b/c, and a. Similarly, a solution for the serial equation

$$(14') \qquad y_i = d : \left[1 + c x_i + b e^{a x_i} \right]$$

may be had by applying formulas (10), section XI, to the ordinates $1/Y_i$, the solution giving the values of $1/d$, c/d, b/d, and a.

To solve for certain other series which are of interest, we write

$$(15) \qquad y_i = m e^{b e^{a x_i}}$$

The solution is obtained by applying formulas (12), section XI, to the ordinates log Y_i.

We have also

$$(16) \qquad y_i = y_0 \left(1 + e^{b + a z_i} \right) = y_0 + B e^{a z_i}$$

where $B = y_0 e^b$, and the argument $z = f(x)$ is chosen so that $a z_0 = -\infty$. This condition is met when $f(x)$ takes the form $1/x$, log x, cot x, log sin x, etc., the sign of a being sometimes plus and sometimes minus.

From (16), by forming the function $u_i = y_i - y_0$, we get

$$(17) \qquad \log u_i = B + a z_i .$$

On taking a first difference of rank k, this becomes

$$(18) \qquad \Delta_k \log u_i = a \Delta_k z_i .$$

From equations (18), (17), and (16), we deduce the following parametric approximations:

$$a_i = \Delta_k \log U_i : \Delta_k z_i , \qquad i = 1,\ 2,\ \cdots\ n-k-1 .$$

$$(19) \qquad \log B_i = \log U_i - a z_i , \qquad i = 1,\ 2,\ \cdots\ n-1 .$$

$$Y_{0i} = Y_i - B e^{a z_i} , \qquad i = 0,\ 1,\ \cdots\ n-1 .$$

The mean values of a_i, B_i, and y_0 give the values of the parameters sought

If a term in x is added to the exponent of equation (16), we have

$$(20) \qquad y_i = y_0 \left(1+e^{c+bx_i+ax_i^2}\right).$$

The solution for these is obtained by carrying the analysis applied to equation (16) to second order differences and applying formulas (7), section IX, to the ordinates log U_i.

If equation (20) is rewritten as

$$(21) \qquad y = y_0 \left(1+e^{c+bx_i+ax_i^2}\right),$$

The solution is obtained by applying formulas (11), section X, to the ordinates log U_i. This solution holds, it will be noted, only when the signs of b and a are such that $bz_0 = az_0^2 = -\infty$

XIII. DETERMINATION OF THE RANK OF DIFFERENCES

In the writing of formulas for the determination of parameters, the rank of differences has been fixed in a purely arbitrary manner. We shall now give a rational justification for the rank assigned.

In what follows, we shall speak of the process by which one of the parameters is eliminated from the equation of the function $y=f(x)$ as a *reduction*; and the definition shall be understood to hold whether the reduction takes place through a simple difference Δy, a logarithmic difference $\Delta \log y$, a product difference $\Delta' y$, or a ratio py, the rank of the reduction being the same as the rank of the difference or ratio involved. In this, we interpret $\Delta_k^s \log \Delta_k^r y$ and $\Delta_k^s p_k^r y$ as determining reductions of order $s+r$.

The process by which a first parameter is eliminated we shall call a first reduction; that by which a second is eliminated, a second reduction; and so on to the r th reduction. The process by which the last but one of the parameters of the original function is eliminated we shall term the *ultimate reduction*; and the parameter definied by the ultimate reduction we shall term the *ultimate parameter.*

Now, a little thought or experimentation will quickly reveal that, for any ultimate parametric approximation, the value of the approximation will vary with the rank of the reduction from which it results; for, regarding a parameter as a statistical characteristic of a series of observations, when the rank of a parametric approximation is at a minimum, or when $k = 1$, the given approximation, viewed as a single instance of a number of possible approximations, is least characteristic of the complete series; and when the rank of the given approximation is at a maximum, or when $k = n - r$, the approximation, again viewed as a single instance of a number of possible approximations, is most characteristic of the complete series

All this, of course, assumes, as we have always done in writing the equations of parametric approximations, that approximations are written in terms of the observational ordinates Y_i; for, if approximations are written in terms of the functional ordinates y_i, the value of the parameter is independent of the rank of the reduction, a fact which follows from the manner of deriving the equation defining the ultimate parameter.

Since, then, the value and representative character of an ultimate parameter varies with the rank of the reduction by which it is defined, we may, when the ultimate reduction is of the first order, express the weight of an approximation by the relation

(1) $$w_k\ (p_l) = k \cdot$$

Here $w_k\ p_l$ is used as the arbitrary symbol for the weight of a parametric approximation defined of first order and rank k.

Suppose, now, that the given ultimate parameter is arrived at by two reductions, the first of rank k and the second of rank h. Clearly, the value of the approximation will, in this case, vary with h as well as k. Under these conditions, the weight of the approximation is expressed by the relation

(2) $$w^2_{k,h}(p_l) = k\ h$$

Here, the symbol $w^2_{k,h}(p_l)$ denotes the weight of a parametric approximation involving a second reduction and the ranks k and h.

Similarly, we have

(3) $$w^3_{k,h,f}(p_l) = k \cdot h\ f\ .$$

Evidently, by a direct extension of our method of induction, we arrive at the general relation

(4) $$W^{r}_{k^1, k^2, \ldots k^r}(p_1) = k_1 \cdot k_2 \cdot \cdots \cdot k_r .$$

In the derivation of all formulas, it has been assumed that k is constant for all reductions; hence, equations (2), (3), and (4) become

(5) $$W^{2}_{k}(p_1) = k \cdot k = k^2$$

(6) $$W^{3}_{k}(p_1) = k \cdot k \cdot k = k^3$$

(7) $$W^{r}_{k}(p_1) = k^r$$

Giving verbal expression to the relation (7), we say that the weight of a parametric approximation involving a reduction of the r th order and k th rank is equal to the r th power of k.

We have already shown, section V, that the number of differences of order r and rank k which can be formed from n observations is $n - rk$; likewise, the number of parametric approximations which can be formed when reductions are of the r th order, is $n - rk$; and, since the reliability of a parameter as determined from a formula must vary with the number as well as the weight of the several approximations, we may write the following equation, conditioning the reliability of the ultimate parameter p :

(8) $$\psi(p) = k^r(n - rk).$$

Regarding k as a continuous variable, we may obtain the condition for $\psi(p) =$ a maximum by differentiating ψ with respect to k, thus,

(9) $$D_k\, \psi(p) = nrk^{r-1} - r(r+1)k^r .$$

Setting (9) equal to zero and solving, we have

(10) $$k = n : (r-1) ,$$

That $\psi(p)$ is a maximum and not a minimum when k is determined from equation (10) is shown by taking the second derivative of ψ and substituting for k, thus:

$$(11)\quad \begin{aligned} D_k^2\psi(p) &= r(r-1)nk^{r-2} - r^2(r+1)k^{r-1} = \\ & rk^{r-2}\left((r-1)n - r(r+1)n : (r+1)\right) = -rnk^{r-2}, \end{aligned}$$

which is negative, since r, n, and k are positive.

Equation (10) may give fractional values of k; but, in practice, k is always integral; hence, we write (10) in the form

$$(12)\quad k = (n \pm j) : (r+1) .$$

This is the relation from which we have determined the value of k in the writing of formulas.

We may now formulate the following rule for the determination of the rank of differences. *When the equation defining an ultimate parameter involves a reduction of order r and rank k, the value of k is to be obtained from the relation $k = (n \pm j) : (r+1)$, j being assigned the smallest integral value that will make $n \pm j$ an exact multiple of k.* In case $n + j = n - j$, that value of k is taken which gives the highest value for $\psi(p)$ when k is substituted in equation (8).

XIV Numerical Computations

In carrying through the numerical computations prescribed by formulas developed in this memoir, the following abridgments are useful in the summation of differences:

$$(1a)\quad \sum_{i=0}^{i=n-k-1} \Delta_k Y_i = \sum_{i=0}^{i=n-k-1} (Y_{i+k} - Y_i)$$

$$(1b)\quad = \sum_{i=0}^{i=n-k-1} Y_{i+k} - \sum_{i=0}^{i=n-k-1} Y_i$$

$$(1c)\quad = \sum_{i=k}^{i=n-1} Y_i - \sum_{i=0}^{i=n-k-1} Y_i$$

$$(1d)\quad = \sum_{i=n-k}^{i=n-1} Y_i - \sum_{i=0}^{i=k-1} Y_i .$$

$$(2a)\quad \sum_{i=0}^{i=n-2k-1} \Delta_k^2 Y_i = \sum_{i=0}^{i=n-2k-1} (Y_{i+2k} - 2Y_{i+k} + Y_i)$$

$$(2b)\quad = \sum_{i=0}^{i=n-2k-1} Y_{i+2k} - 2\sum_{i=0}^{i=n-2k-1} Y_{i+k} + \sum_{i=0}^{i=n-2k-1} Y_i$$

$$(2c)\quad = \sum_{i=2k}^{i=n-1} Y_i - 2\sum_{i=k}^{i=n-k-1} Y_i + \sum_{i=0}^{i=n-2k-1} Y_i$$

$$(2d)\qquad =\sum_{j=n-k}^{j=n-1} Y_j - 2\sum_{j=n-2k}^{j=2k-1} Y_j + \sum_{j=0}^{j=k-1} Y_j \,.$$

$$(3a)\qquad \sum_{j=0}^{j=n-3k-1} \Delta_k^3 Y_j = \sum_{j=0}^{j=n-3k-1} Y_j \left(Y_{j+3k} - 3Y_{j+2k} + 3_{j+k} + Y_j\right)$$

$$(3b)\qquad = \sum_{j=0}^{j=n-3k-1} Y_{j+3k} - 3\sum_{j=0}^{j=n-3k-1} Y_{j+2k} + 3\sum_{j=0}^{j=n-3k-1} Y_{j+k} - \sum_{j=0}^{j=n-3k-1} Y_j$$

$$(3c)\qquad = \sum_{j=3k}^{j=n-1} Y_j - 3\sum_{j=2k}^{j=n-k-1} Y_j + 3\sum_{j=k}^{j=n-2k-1} Y_j - \sum_{j=0}^{j=n-3k-1} Y_j$$

$$(3d)\qquad = \sum_{j=n-k}^{j=n-1} Y_j - 3\sum_{j=n-2k}^{j=3k-1} Y_j + 3\sum_{j=n-3k}^{j=2k-1} Y_j - \sum_{j=0}^{j=k-1} Y_j \,.$$

These relations evidently apply quite generally to the summation of differences. They may also be used to check the accuracy of differences formed. When j is positive in the relation $(r+1)k = n \pm j$, equations (c) are most convenient; when j is negative, equations (d) are most convenient.

A useful check on the product difference Δ' employed in section X is obtained as follows:

$$(4)\qquad \Delta_k Y_j + \Delta_k z_j + \Delta_k z_j^2 + \Delta_k z_j^3 = S_j \,.$$

$$(5)\qquad \Delta_k Y_{j+k} + \Delta_k z_{j+k} + \Delta_k z_{j+k}^2 + \Delta_k z_{j+k}^3 = S_{j+k} \,.$$

Multiplying (4) by $\Delta_k z_{j+k}$ and (5) by $\Delta_k z_j$, we have

$$(6)\qquad \Delta_k Y_j \cdot \Delta_k z_{j+k} + \Delta_k z_j \cdot \Delta_k z_{j+k} + \Delta_k z_j^2 \cdot \Delta_k z_{j+k} + \Delta_k z_j^3 \cdot \Delta_k z_{j+k} = S_j \cdot \Delta_k z_{j+k} \,.$$

$$(7)\qquad \Delta_k Y_{j+k} \cdot \Delta_k z_j + \Delta_k z_{j+k} \cdot \Delta_k z_j + \Delta_k z_{j+k}^2 \cdot \Delta_k z_j + \Delta_k z_{j+k}^3 \cdot \Delta_k z_j = S_{j+k} \cdot \Delta_k z_j \,.$$

Subtracting (6) from (7), we get

$$(8)\qquad \Delta_k' Y_j + \Delta_k' z_j^2 + \Delta_k' z_j^3 = \Delta_k' S_j \,.$$

Evidently, similar relations hold for the product differences Δ'', etc.

Another check that is constantly useful in the computations is the well known relation $\Sigma a f_j = a \Sigma f_j$.

TABLE IV

Auxiliary Functions Computed in Fitting to the
Ohio Population

U	$\log U$	x	$\log \sin x$	$a \log \sin x$	B_i	$B10^{bx}$
0.0	$-\infty$	0	$-\infty$	$-\infty$		
178.5	2.25164	1	− 1.75814	− 2.22212	29769	228.0
519.1	2.71525	2	− 1.45718	− 1.84173	36056	547.5
881.7	2.94532	3	− 1.28120	− 1.61931	36697	913.8
1454.1	3.16259	4	− 1.15642	− 1.46160	42091	1313.8
1920.1	3.28332	5	− 1.05970	− 1.33935	41944	1741.0
2283.3	3.35856	6	− 0.98077	− 1.23960	39642	2190.5
2610.6	3.41674	7	− 0.91411	− 1.15534	37332	2659.5
3134.7	3.49620	8	− 0.85644	− 1.08246	37902	3145.4
3611.5	3.55769	9	− 0.80567	− 1.01829	37669	3646.3
4096.2	3.61238	10	− 0.76033	− 0.96097	37441	4160.7
4708.1	3.67285	11	− 0.71940	− 0.90924	38202	4687.1
5718.2	3.75726	12	− 0.68212	− 0.86212	41627	5224.2
31116.1	39.22980		−12.43148	−15.71213	456372	30457.8

$\Delta_6 \log U$	$\Delta_6 \log \sin x$	a_i
1.16510	0.84403	1.3805
0.78095	0.60074	1.3000
0.61237	0.47553	1.2877
0.44979	0.39609	1.1356
0.38953	0.34030	1.1447
0.39870	0.29865	1.3350
3.79644	2.95534	7.5835

$a = \frac{1}{6}\Sigma a_i = 1.2639$; $\qquad B = \frac{1}{12}\Sigma B_i = 38031$;

$y_0 = \frac{1}{13}(\Sigma Y - \Sigma B10^{bx}) = 91.8$; $\qquad b = \log B - \log y_0 = 2.61730$.

In computing the ordinates

(9) $$y_i = a + bx_i + cx_i^2$$

the following formulas are useful:

(10) $$\Delta y_0 = (b + c\Delta x)\Delta x \,.$$

(11) $$\Delta y_{i+1} = \Delta y_i + 2c\Delta x \,.$$

(12) $$y_0 = a \,.$$

(13) $$y_{i+1} = y_i + \Delta y_i \,.$$

The formulas for Σx, Σx^2, and Σx^3 are to be found in any standard reference work on statistical computations.

As illustrations of the quantities to be obtained in actual computations, we give, in Table IV, the auxiliaries computed in fitting the curve $y = y_0(1 + e^{b + a \cdot \log \sin x})$ to the population of Ohio.

XV. CRITICAL REVIEW

We have now presented, at some length, the technique of fitting curves by the method of differences. The term, "method of differences" is doubtless sufficiently descriptive for general purposes; but the designation *method of mean difference functions* would better convey an idea of the chief features of the technique elaborated, namely, the dependence on functions of finite differences in the derivation of equations defining parameters and the determination of the best value of a given parameter by taking the mean of the several approximations.

The fundamental requirement of this method is that, under the procedure followed, the reliability of the parameters determined shall be a maximum. This requirement results in a sum of absolute residuals which is less than that to be obtained by the Gaussian method of least squares or the Pearsonian method of moments. Rigorous adherence to the Edgeworthian requirement that the sum of the absolute residuals shall be a minimum is, it will be observed, not a demand of the present method. It can be shown that Lipka's method of averages will give the same residuals for a linear series as the method of differences; but it does not, however, give the same results in general.

The following claims to merit may be advanced for the method of differences:

(1) The computations involved in the determination of parameters are simple and easily checked.

(2) The method permits of fitting to a wide variety of functions by the direct application of its fundamental principles.

(3) The general technique developed may be adapted to special solutions in particular cases; e. g., the solutions of parabolic series given in section VII are special cases of the solution for the general polynomial series given in section X.

(4) The parametric approximations or some function involved in their determination give a convenient test of fit. If these approximations are nearly constant or fluctuate irregularly about a central value, the implication is that the test function is appropriate to the data; if the approximations show a systematic change or trend in their values, the implication is that the test function is inappropriate.

(5) The method yields satisfactory results in practice.

That the residuals do register our failure to predict the values of the observations is undeniable; but it does not follow that the least squares definition of residuals leads to the equation of greatest value for predictive purposes; for we can scarcely hope to establish that a set of residuals determined from a small number of observations constitutes a system of normally distributed variates.

Let us now consider the logistic series

$$y = m e^{-be^{-ax}} \tag{2a}$$

$$y = y_0 \left(1 + e^{b+ax}\right) , \tag{3a}$$

and

$$y = m : \left[1 + be^{-ax}\right] . \tag{4a}$$

This last is the Verhulst logistic.

The origin, maximum, and point of inflection of these three functions are determined by the following relations:

$$y_0 = me^{-b},$$

$$dy = abme^{-ax}e^{-be^{-ax}}dx, \tag{2b}$$

$$d^2y = a^2bme^{-be^{-ax}}(be^{-2ax} - e^{-ax})dx^2.$$

$$y = y_0(1+e^{b-az}),$$

$$dy = aBe^{az}dz, \tag{3b}$$

$$d^2y = a^2Be^{az}dz^2 + aBe^{az}d^2z.$$

$$y_0 = m:[1+b].$$

$$dy = abm[e^{-ax}.:(1+be^{-ax})^2]dx, \tag{4b}$$

$$d^2y = 2a^2b^2me^{-2ax}(1+be^{-ax})dx^2$$

$$-2a^2bme^{-2ax}(1+be^{-ax})^2dx^2-(1+be^{-ax})^4dx^2.$$

With the origin at $f(x_0)$ and the maximum at $f(x_\infty)$, these curves show essentially the same properties and, therefore, negate the claim of Professors Pearl and Reed to have discovered in the Verhulst type logistic the unique mathematical expression for the growth of populations. This assertion, of course, makes no statement as to the type of population which is best represented by each curve.

In fitting type (2) to the population of Ohio, we have obtained, while not an ideal fit, certainly one much better than can be obtained by fitting type (3). These results, however, serve to enhance rather than diminish the general usefulness of the logistic hypothesis as an empirical generalization of the growth of populations.

As the writer conceives it, this hypothesis may be stated as follows: *When the growth of a population is not known to be correlated with events whose sequence is definitely known, it is best represented by a* *curve which proceeds from one horizontal straight line as asymptotic origin, passes through a point of inflection, and approaches a second horizontal straight line as asymptotic terminus.* The rate of growth of such a curve may be characterized as proceeding from a minimum to a maximum and then decreasing toward zero as limit, a characterization which is in full accord with our decrease in knowledge concerning the rate of growth as time goes on. It will be noted, in our statement of the logistic hypothesis, that it is not necessary to place any restriction on the chronological direction of growth, interpretation of growth as proceeding forward or backward being equally permissible.

It is, of course, true that the particular function fitted to the Ohio population does not conform rigorously to the logistic type; for at $x = 90$, the ordinates begin to decline in value But this is no detriment in the application of the function in the particular case, since no one would place any reliance on a forecast of such date when projected several centuries into the future. The use of such a function as we have employed seems far preferable to fitting the Verhulst function to subpopulations on the ground that the sum of logistics cannot be executed itself to be a logistic, a procedure which is strictly valid only when the growth changes of the subpopulations are mutually independent.[1]

When the growth of a population is known to be definitely correlated with an observed sequence of events, the logistic hypothesis must be modified accordingly. In a region where the population could not be recruited from without, an abrupt increase in the death rate, a decrease in the birth rate, or an emigration to regions outside would necessitate a modification of the growth formula.

NOTE.—In presenting this memoir to the public, the writer desires to make grateful acknowledgement of the invaluable assistance given by his wife, Hazel J Will, in the preparation of the manuscript.

[1] cf Pearl and Reed "The Population of an Area Around Chicago and the Logistic Curve" J A A S, March, 1929

Harry S. Will

TABLES OF PEARSON'S TYPE III FUNCTION

By

LUIS R. SALVOSA

SECTION I.—*Development of Pearson's Type III Function*
SECTION II.—*Areas*
SECTION III.—*Ordinates*
SECTION IV.—*Derivatives**

It is well recognized that the normal curve of error has played a prominent role in the development of the theory of Mathematical Statistics. Although it can describe more or less accurately many frequency distributions possessing a limited degree of skewness, there are many others in which it fails. To meet this situation two important methods of representing frequency functions have been devised.

One of these methods is due to the English biometrician Karl Pearson, who developed a system of generalized "probability" curves. Among the simplest of these curves is Type III, whose equation is $y=y_0\left(1+\frac{\alpha_3}{2}t\right)^{\frac{4}{\alpha_3^2}-1}e^{-\frac{2}{\alpha_3}t}$, where y_0 and α_3 are constants. It will be shown later that this curve approaches the normal curve of error as a limit when α_3 approaches zero. Pearson, realizing the importance of this generalized curve, published in 1922 his "Tables of the Incomplete Γ Function," from which the areas under the curve can be obtained. Unfortunately, these tables, unlike those of the normal curve of error, are not tabulated with the standard unit as the ordinate. Moreover, they do not contain the ordinates, which are useful in plotting frequency curves and essential to the solution of many problems in the theory of probability.

My object, therefore, is to provide tables of areas and ordinates of Pearson's Type III curve that will enable one to obtain readily an isolated frequency or the sum of the frequencies between any two limits of a theoretical distribution that can be represented by the Type III curve. Furthermore, there will be furnished tables of derivatives of this more effective type of frequency function which will permit its utilization as a generating function in an expansion corresponding to the Gram-Charlier series. In general, the more closely the generating function approximates the function to be graduated, the more rapid the convergence of the derivative series. In the following section there are given two separate developments of the Pearson Type III Function. It is believed that these developments will be of value to those desiring to use this function.

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*Section IV will appear in the August issue of the ANNALS.

SECTION I.—*Development of Pearson's Type III Function*

(*a*) *By Means of Bernoulli's Series*

One may consider first the Bernoulli series

$$(1) \qquad (q+p)^r = \sum_{x=0}^{r} \binom{r}{x} q^{r-x} p^x,$$

where p denotes the probability that an event will happen in a single trial and $q=1-p$, the probability that it will fail. Representing by $y_x=\binom{r}{x} q^{r-x} p^x$ the ordinate corresponding to x successes (x assuming the values of 0, 1, 2 . . . .), one may plot the $(r+1)$ points (x, y_x). Through these $(r+1)$ points one may imagine a curve that can be represented by an analytic function.

Since $\qquad y_x = \binom{r}{x} q^{r-x} p^x$

and hence $\qquad y_{x+1}=\binom{r}{x+1} q^{r-x-1} p^{x+1}$

one has

$$(2) \qquad \frac{y_{x+1}}{y_x} = \frac{rp-px}{qx+q}$$

This is the difference equation of the continuous curve.

From equation (2) it follows that

$$(3) \qquad \frac{y_{x+1}-y_x}{y_{x+1}+y_x} = \frac{rp-q-x}{rp+q+(q-p)x}$$

The mean of any two ordinates (y_x and y_{x+1}) will be considered as approximately equal to the ordinate ($y_{x+\frac{1}{2}}$) midway between them. The slope of the line joining any two points (x, y_x) and ($x+1$, $-y_{x+1}$) is also approximately equal to the tangent at the point midway between these two on the continuous curve and the error resulting from this approximation would be zero if the curve were a parabola. Under these two assumptions, equation (3) may be written as

$$(4) \qquad \frac{D_x\, y_{x+\frac{1}{2}}}{y_{x+\frac{1}{2}}} = \frac{2(rp-q-x)}{rp+q+(q-p)x}$$

The right hand member of this equation is, therefore, the derivative of log y at the point ($x+\frac{1}{2}$, $y_{x+\frac{1}{2}}$). At any point (x, y_x) this derivative is

$$\frac{d}{dx}\log y = \frac{2\left[rp-q-(x-\frac{1}{2})\right]}{rp+q+(q-p)(x-\frac{1}{2})} \;;$$

i. e.

$$(5) \qquad \frac{d}{dx}\log y = \frac{\frac{q-p}{2}+(x-rp)}{rpq+\frac{1}{4}+\frac{(x-rp)(q-p)}{2}} .$$

If one sets

$$\frac{q-p}{\sqrt{rpq}} = \alpha,$$

$$\frac{x-rp}{\sqrt{rpq}} = t ,$$

the above equation reduces to

$$(6) \qquad \frac{d}{dt}\log y = -\frac{\frac{\alpha_3}{2}+t}{1+\frac{\alpha_3}{2}t+\frac{1}{4rpq}}$$

If rpq is so large that $\frac{1}{4rpq}$ is relatively insignificant and may consequently be neglected, equation (6) becomes

$$(7) \qquad \frac{d}{dt} \log y = -\frac{\frac{\alpha_3}{2}+t}{1+\frac{\alpha_3}{2}t} ,$$

which upon integration yields Pearson's Type III frequency curve,

$$(8) \qquad y = y_0 \left(1+\frac{\alpha_3}{2}t\right)^{\frac{4}{\alpha_3^2}-1} e^{-\frac{2}{\alpha_3}t}$$

Thus, equation (8) has been obtained without resorting to the method of moments

(b) *By Means of Differential Equation*

Another method* of developing Pearson's Type III curve is by means of the differential equation

$$(9) \qquad \frac{dy}{dt} = \frac{a-t}{f(t)} y ,$$

which is suggested by certain characteristics of unimodal frequency distributions. Equation (9) is capable of representing a frequency function of this type, since

(a) As y approaches zero, the first derivative must also approach zero, and

(b) If the frequency distribution be unimodal between the limits of the distribution, there must be only one value, say $t = a$, for which the derivative is zero.

In equation (9) t denotes the abscissae in units of the standard

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*Carver's "Frequency Curves," p. 92 *Handbook of Mathematical Statistics*, by Rietz and Others.

deviation, and $f(t)$ is a function that does not vanish at any point within the range of the curve. If it be assumed that $f(t)$ is expressible as a power series, $b_0 + b_1 t + b_2 t^2 + \ldots\ldots$, then equation (9) may take the following form:

$$(10) \qquad \frac{dy}{dt} = \frac{(a-t)y}{b_0 + b_1 t + b_2 t^2 + \ldots} ,$$

On clearing equation (10) of fractions, multiplying both sides by t^n, and integrating both members with respect to t between the limits $(-l_1, l_2)$, one obtains

$$(11) \qquad \left[y_0(b_0 t^n + b_1 t^{n+1} + b_2 t^{n+2} + \cdots)\right]_{-l_1}^{l_2} - \int_{-l_1}^{l_2} y\left[n b_0 t^{n-1} + (n+1) b_1 t^n + (n+2) b_2 t^{n+1} + \cdots\right] dt = \int_{-l_1}^{l_2} (ayt^n - yt^{n+1}) dt ,$$

which can be written in the form

$$(12) \qquad a\alpha_n + n b_0 \alpha_{n-1} + (n+1) b_1 \alpha_n + (n+2) b_2 \alpha_{n+1} + \ldots = \alpha_{n+1}$$

since the expression in the first bracket vanishes at the limits, and $\int_{-l_1}^{l_1} t^n y\,dt = \frac{\mu_n}{\sigma^n} = \alpha_n$. It will be noted here that

$\alpha_0 = 1$, $\alpha_1 = 0$, and $\alpha_2 = 1$.

If the series $f(t)$ be so rapidly convergent that it would be sufficient to retain the first term only, then from equation (12), on placing successively $n = 0, 1, 2 \ldots$ one has

$$a = 0$$
$$b_0 = 1$$

On substituting these values in equation (10) and then integrating, one obtains

$$(13) \qquad y = y_0 e^{-\frac{t^2}{2}} ,$$

which is the normal curve of error, t being expressed in standard units.

If in the series $f(t)$ the first two terms be retained instead of b_0 only, then equation (12) gives

$$a = -b_1 = -\frac{\alpha_3}{2} \tag{14}$$

$$b_0 = 1 .$$

If the values given in equations (14) be substituted in equation (10), then upon integration one obtains

$$y = y_0 \left(1 + \frac{\alpha_3}{2} t\right)^{\frac{4}{\alpha_3^2} - 1} e^{-\frac{2}{\alpha_3} t} \tag{15}$$

This equation is the same as equation (8) and satisfies the conditions imposed, provided $\alpha_3 < \sqrt{2}$.

The substitution of the constants (14) in equation (12) yields the following recurrence relation of the functional Type III moments:

$$\alpha_{n+1} = n \left(\alpha_{n-1} + \frac{\alpha_3 \alpha_n}{2}\right) . \tag{16}$$

To find the recurrence relation of the functional moments for the normal curve of error, α_3 is set equal to zero and hence relation (16) reduces to the simple relation

$$\alpha_{n+1} = n \alpha_{n-1} \tag{17}$$

since $\alpha_1 = 0$

it follows from relation (17) that

$$\alpha_{2n+1} = 0 \tag{18}$$

For the moments of even order one writes from relation (17)

$$\alpha_2 = 1$$

$$\alpha_4 = 3$$

$$\alpha_6 = 5 \alpha_4$$

$$\cdot$$

$$\alpha_{2n} = 1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n \, n!}$$

The normal curve of error has been derived by retaining the first term only in the series $f(t)$. However, it can be shown that this curve is the limit of Type III curve as the skewness approaches zero. For, taking the logarithms of both sides of equation (15), one has

$$\log_e y = \log_e y_0 + \left(\frac{4}{\alpha_3^2} - 1\right) \log_e\left(1 + \frac{\alpha_3}{2} t\right) - \frac{2}{\alpha_3} t \ ,$$

or

$$\log_e\left(\frac{y}{y_0}\right) = \left(\frac{4}{\alpha_3^2} - 1\right)\left(\frac{\alpha_3}{2} t - \frac{1}{2} \cdot \frac{\alpha_3^2 t^2}{4} + \frac{1}{3} \cdot \frac{\alpha_3^3 t^3}{8} - \cdots\cdots\right) - \frac{2}{\alpha_3} t$$
$$= -\frac{1}{2} t^2 + f(\alpha_3 t) \ .$$

Therefore,

$$\lim_{\alpha_3 \to 0} \log_e\left(\frac{y}{y_0}\right) = -\frac{1}{2} t^2 ,$$

or

$$y = y_0 e^{-\frac{1}{2} t^2} ,$$

which agrees with equation (13).

The constant y_0 in equations (13) and (15) is determined imposing the condition that

$$(20) \qquad y_0 \int_{-\frac{2}{\alpha_3}}^{\infty} \left(1 + \frac{\alpha_3}{2} t\right)^{\frac{4}{\alpha_3^2} - 1} e^{-\frac{2}{\alpha_3} t} \, dt = 1 \ .$$

On introducing z by the relation $z = \frac{2}{\alpha_3}\left(\frac{2}{\alpha_3} + t\right)$ equation (20) becomes

$$y_0 = \frac{1}{\frac{\alpha_3}{2}\left(\frac{4}{\alpha_3^2}\right)^{\frac{4}{\alpha_3^2} - 1} e^{\frac{4}{\alpha_3^2}} \int_0^{\infty} z^{\frac{4}{\alpha_3^2} - 1} e^{-z} \, dz}$$

Therefore,

$$(21) \qquad y_0 = \frac{\left(\frac{4}{\alpha_3^2}\right)^{\frac{4}{\alpha_3^2} - \frac{1}{2}}}{e^{-\frac{4}{\alpha_3^2}} \Gamma\left(\frac{4}{\alpha_3^2}\right)} = n \, \frac{n^{n - \frac{1}{2}}}{e^{n} \Gamma(n)}$$

where

$$n = \frac{4}{\alpha_3^2}$$

With the aid of Sterling's formula, equation (21) can be written as

$$(22) \qquad y_0 = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\alpha_3^2}{48} + \frac{\alpha_3^6}{23040} - \frac{\alpha_3^{10}}{1290240} + \cdots\cdots}$$

For the normal curve of error, that is, when $\alpha_3 = 0$, equation (22) reduces to

$$(23) \qquad y_0 = \frac{1}{\sqrt{2\pi}} ,$$

and hence for the foregoing specialization with respect to α_3, equation (15) becomes:

$$y = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}$$

The intention was to have the Tables correct to six decimal places. To this end all computations of areas, ordinates and derivatives were carried through to eight significant digits of accuracy, which frequently meant ten or twelve decimal places. The results were then cut down to the nearest sixth decimal place, and I believe they possess this degree of accuracy.

L. R. Salvosa

Manila, P. I.

TABLE I

AREAS OF THE STANDARDIZED TYPE III FUNCTION

	SKEWNESS						
t	0	.1	.2	3	.4	.5	t
-4.99							-4.99
-4.98							-4.98
-4.97							-4.97
-4.96							-4.96
-4.95							-4.95
-4.94							-4.94
-4.93							-4.93
-4.92							-4.92
-4.91							-4.91
-4.90							-4.90
-4.89	000001						-4.89
-4.88	.000001						-4.88
-4.87	000001						-4.87
-4.86	000001						-4.86
-4.85	000001						-4.85
-4.84	000001						-4.84
-4.83	000001						-4.83
-4.82	.000001						-4.82
-4.81	000001						-4.81
-4.80	000001						-4.80
-4.79	.000001						-4.79
-4.78	000001						-4.78
-4.77	000001						-4.77
-4.76	000001						-4.76
-4.75	000001						-4.75
-4.74	000001						-4.74
-4.73	.000001						-4.73
-4.72	.000001						-4.72
-4.71	.000001						-4.71
-4.70	000001						-4.70
-4.69	000001						-4.69
-4.68	000001						-4.68
-4.67	.000002						-4.67
-4.66	000002						-4.66
-4.65	000002						-4.65
-4.64	000002						-4.64
-4.63	000002						-4.63
-4.62	.000002						-4.62
-4.61	000002						-4.61
-4.60	000002						-4.60
-4.59	000002						-4.59
-4.58	000002						-4.58
-4.57	.000002						-4.57
-4.56	000003						-4.56
-4.55	.000003						-4.55
-4.54	000003						-4.54
-4.53	.000003						-4.53
-4.52	000003						-4.52
-4.51	.000003	000001					-4.51
-4.50	000003	000001					-4.50

	SKEWNESS						
t	6	.7	8	.9	1 0	1.1	t
-4.99							-4.99
-4.98							-4.98
-4.97							-4.97
-4.96							-4.96
-4.95							-4.95
-4.94							-4.94
-4.93							-4.93
-4.92							-4.92
-4.91							-4.91
-4.90							-4.90
-4.89							-4.89
-4.88							-4.88
-4.87							-4.87
-4.86							-4.86
-4.85							-4.85
-4.84							-4.84
-4.83							-4.83
-4.82							-4.82
-4.81							-4.81
-4.80							-4.80
-4.79							-4.79
-4.78							-4.78
-4.77							-4.77
-4.76							-4.76
-4.75							-4.75
-4.74							-4.74
-4.73							-4.73
-4.72							-4.72
-4.71							-4.71
-4.70							-4.70
-4.69							-4.69
-4.68							-4.68
-4.67							-4.67
-4.66							-4.66
-4.65							-4.65
-4.64							-4.64
-4.63							-4.63
-4.62							-4.62
-4.61							-4.61
-4.60							-4.60
-4.59							-4.59
-4.58							-4.58
-4.57							-4.57
-4.56							-4.56
-4.55							-4.55
-4.54							-4.54
-4.53							-4.53
-4.52							-4.52
-4.51							-4.51
-4.50							-4.50

t	SKEWNESS 0	.1	.2	.3	.4	5	t
–4 49	000004	.000001					–4.49
–4.48	.000004	000001					–4 48
–4 47	000004	000001					–4.47
–4.46	.000004	000001					–4.46
–4 45	.000004	000001					–4.45
–4 44	.000004	.000001					–4.44
–4 43	000005	000001					–4 43
–4.42	000005	000001					–4.42
–4.41	000005	.000001					–4.41
–4.40	000005	.000001					–4.40
–4 39	.000006	.000001					–4.39
–4 38	000006	000001					–4.38
–4 37	000006	.000001					–4 37
–4 36	000007	000001					–4 36
–4 35	000007	.000001					–4.35
–4 34	000007	.000001					–4 34
–4 33	000007	000001					–4.33
–4 32	.000008	.000002					–4 32
–4.31	.000008	.000002					–4.31
–4.30	000009	.000002					–4 30
–4 29	000009	.000002					–4 29
–4.28	.000009	000002					–4.28
–4.27	.000010	.000002					–4.27
–4.26	000010	000002					–4 26
–4 25	.000011	000002					–4 25
–4 24	000011	000003					–4.24
–4 23	000012	.000003					–4 23
–4 22	000012	.000003					–4.22
–4 21	000013	.000003					–4.21
–4 20	000013	000003					–4.20
–4.19	000014	000003					–4.19
–4.18	000015	.000003					–4 18
–4 17	000015	.000004					–4.17
–4.16	000016	000004					–4.16
–4.15	000017	000004	.000001				–4.15
–4 14	000017	000004	.000001				–4.14
–4.13	000018	000005	.000001				–4 13
–4 12	000019	.000005	000001				–4.12
–4 11	000020	.000005	.000001				–4 11
–4.10	000021	.000005	.000001				–4.10
–4 09	000022	000006	.000001				–4 09
–4 08	000023	.000006	.000001				–4.08
–4 07	.000023	000006	.000001				–4 07
–4.06	000025	.000007	.000001				–4.06
–4.05	.000026	000007	.000001				–4 05
–4 04	.000027	.000007	.000001				–4.04
–4 03	000028	.000008	.000001				–4.03
–4.02	000029	000008	000001				–4.02
–4 01	.000030	000009	.000001				–4.01
–4.00	000032	.000009	.000001				–4.00

	SKEWNESS						
t	.6	.7	.8	.9	1.0	1.1	t
-4.49							-4.49
-4.48							-4.48
-4.47							-4.47
-4.46							-4.46
-4.45							-4.45
-4.44							-4.44
-4.43							-4.43
-4.42							-4.42
-4.41							-4.41
-4.40							-4.40
-4.39							-4.39
-4.38							-4.38
-4.37							-4.37
-4.36							-4.36
-4.35							-4.35
-4.34							-4.34
-4.33							-4.33
-4.32							-4.32
-4.31							-4.31
-4.30							-4.30
-4.29							-4.29
-4.28							-4.28
-4.27							-4.27
-4.26							-4.26
-4.25							-4.25
-4.24							-4.24
-4.23							-4.23
-4.22							-4.22
-4.21							-4.21
-4.20							-4.20
-4.19							-4.19
-4.18							-4.18
-4.17							-4.17
-4.16							-4.16
-4.15							-4.15
-4.14							-4.14
-4.13							-4.13
-4.12							-4.12
-4.11							-4.11
-4.10							-4.10
-4.09							-4.09
-4.08							-4.08
-4.07							-4.07
-4.06							-4.06
-4.05							-4.05
-4.04							-4.04
-4.03							-4.03
-4.02							-4.02
-4.01							-4.01
-4.00							-4.00

	SKEWNESS						
t	0	.1	.2	.3	.4	.5	t
-3.99	.000033	.000010	.000002				-3.99
-3.98	.000034	.000010	.000002				-3.98
-3.97	.000036	.000011	.000002				-3.97
-3.96	.000037	.000011	.000002				-3.96
-3.95	.000039	.000012	.000002				-3.95
-3.94	.000041	.000012	.000002				-3.94
-3.93	.000042	.000013	.000002				-3.93
-3.92	.000044	.000014	.000003				-3.92
-3.91	.000046	.000014	.000003				-3.91
-3.90	.000048	.000015	.000003				-3.90
-3.89	.000050	.000016	.000003				-3.89
-3.88	.000052	.000017	.000003				-3.88
-3.87	.000054	.000018	.000004				-3.87
-3.86	.000057	.000019	.000004				-3.86
-3.85	.000059	.000020	.000004				-3.85
-3.84	.000062	.000021	.000004				-3.84
-3.83	.000064	.000022	.000005				-3.83
-3.82	.000067	.000023	.000005				-3.82
-3.81	.000069	.000024	.000005				-3.81
-3.80	.000072	.000025	.000006	.000001			-3.80
-3.79	.000075	.000026	.000006	.000001			-3.79
-3.78	.000078	.000028	.000006	.000001			-3.78
-3.77	.000082	.000029	.000007	.000001			-3.77
-3.76	.000085	.000031	.000007	.000001			-3.76
-3.75	.000088	.000032	.000008	.000001			-3.75
-3.74	.000092	.000034	.000008	.000001			-3.74
-3.73	.000096	.000035	.000009	.000001			-3.73
-3.72	.000100	.000037	.000009	.000001			-3.72
-3.71	.000104	.000039	.000010	.000001			-3.71
-3.70	.000108	.000041	.000010	.000001			-3.70
-3.69	.000112	.000043	.000011	.000001			-3.69
-3.68	.000117	.000045	.000012	.000002			-3.68
-3.67	.000121	.000047	.000012	.000002			-3.67
-3.66	.000126	.000049	.000013	.000002			-3.66
-3.65	.000131	.000052	.000014	.000002			-3.65
-3.64	.000136	.000054	.000015	.000002			-3.64
-3.63	.000142	.000057	.000016	.000002			-3.63
-3.62	.000147	.000060	.000017	.000002			-3.62
-3.61	.000153	.000062	.000018	.000003			-3.61
-3.60	.000159	.000065	.000019	.000003			-3.60
-3.59	.000165	.000068	.000020	.000003			-3.59
-3.58	.000172	.000072	.000021	.000003			-3.58
-3.57	.000178	.000075	.000023	.000004			-3.57
-3.56	.000185	.000079	.000024	.000004			-3.56
-3.55	.000193	.000082	.000025	.000004			-3.55
-3.54	.000200	.000086	.000027	.000005			-3.54
-3.53	.000208	.000090	.000028	.000005			-3.53
-3.52	.000216	.000094	.000030	.000005			-3.52
-3.51	.000224	.000099	.000032	.000006			-3.51
-3.50	.000233	.000103	.000034	.000006			-3.50

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
–3.99							–3.99
–3.98							–3.98
–3.97							–3.97
–3.96							–3.96
–3.95							–3.95
–3.94							–3.94
–3.93							–3.93
–3.92							–3.92
–3.91							–3.91
–3.90							–3.90
–3.89							–3.89
–3.88							–3.88
–3.87							–3.87
–3.86							–3.86
–3.85							–3.85
–3.84							–3.84
–3.83							–3.83
–3.82							–3.82
–3.81							–3.81
–3.80							–3.80
–3.79							–3.79
–3.78							–3.78
–3.77							–3.77
–3.76							–3.76
–3.75							–3.75
–3.74							–3.74
–3.73							–3.73
–3.72							–3.72
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–3.70							–3.70
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–3.62							–3.62
–3.61							–3.61
–3.60							–3.60
–3.59							–3.59
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–3.57							–3.57
–3.56							–3.56
–3.55							–3.55
–3.54							–3.54
–3.53							–3.53
–3.52							–3.52
–3.51							–3.51
–3.50							–3.50

t	SKEWNESS						t
	0	.1	.2	.3	.4	.5	
−2.99	.001395	.000854	.000450	.000187	.000051	.000006	−2.99
−2.98	.001441	.000888	.000471	.000198	.000055	.000006	−2.98
−2.97	.001489	.000922	.000493	.000209	.000059	.000007	−2.97
−2.96	.001538	.000957	.000516	.000222	.000064	.000008	−2.96
−2.95	.001589	.000994	.000540	.000234	.000069	.000009	−2.95
−2.94	.001641	.001032	.000564	.000248	.000074	.000010	−2.94
−2.93	.001695	.001071	.000590	.000262	.000080	.000011	−2.93
−2.92	.001750	.001111	.000617	.000277	.000086	.000012	−2.92
−2.91	.001807	.001153	.000644	.000292	.000092	.000014	−2.91
−2.90	.001866	.001197	.000673	.000309	.000099	.000016	−2.90
−2.89	.001926	.001242	.000703	.000326	.000107	.000017	−2.89
−2.88	.001988	.001288	.000735	.000344	.000115	.000019	−2.88
−2.87	.002052	.001336	.000767	.000363	.000123	.000021	−2.87
−2.86	.002118	.001385	.000801	.000383	.000132	.000024	−2.86
−2.85	.002186	.001437	.000836	.000403	.000141	.000026	−2.85
−2.84	.002256	.001489	.000873	.000425	.000152	.000029	−2.84
−2.83	.002327	.001544	.000911	.000448	.000162	.000032	−2.83
−2.82	.002401	.001600	.000950	.000472	.000174	.000036	−2.82
−2.81	.002477	.001659	.000991	.000497	.000186	.000039	−2.81
−2.80	.002555	.001719	.001034	.000523	.000199	.000043	−2.80
−2.79	.002635	.001781	.001078	.000551	.000212	.000048	−2.79
−2.78	.002718	.001845	.001124	.000580	.000227	.000052	−2.78
−2.77	.002803	.001911	.001171	.000610	.000242	.000057	−2.77
−2.76	.002890	.001980	.001220	.000641	.000258	.000063	−2.76
−2.75	.002980	.002050	.001272	.000674	.000276	.000069	−2.75
−2.74	.003072	.002123	.001325	.000709	.000294	.000076	−2.74
−2.73	.003167	.002198	.001380	.000744	.000313	.000083	−2.73
−2.72	.003264	.002275	.001437	.000782	.000334	.000090	−2.72
−2.71	.003364	.002355	.001496	.000821	.000355	.000099	−2.71
−2.70	.003467	.002437	.001557	.000862	.000378	.000108	−2.70
−2.69	.003573	.002522	.001621	.000905	.000402	.000117	−2.69
−2.68	.003681	.002610	.001687	.000949	.000427	.000128	−2.68
−2.67	.003793	.002700	.001755	.000996	.000454	.000139	−2.67
−2.66	.003907	.002793	.001826	.001044	.000482	.000151	−2.66
−2.65	.004025	.002889	.001899	.001095	.000512	.000164	−2.65
−2.64	.004145	.002987	.001975	.001148	.000543	.000177	−2.64
−2.63	.004269	.003089	.002053	.001203	.000576	.000192	−2.63
−2.62	.004396	.003194	.002134	.001260	.000611	.000208	−2.62
−2.61	.004527	.003302	.002218	.001320	.000647	.000225	−2.61
−2.60	.004661	.003413	.002305	.001382	.000686	.000244	−2.60
−2.59	.004799	.003528	.002395	.001446	.000726	.000263	−2.59
−2.58	.004940	.003645	.002488	.001514	.000768	.000284	−2.58
−2.57	.005085	.003767	.002584	.001584	.000813	.000306	−2.57
−2.56	.005234	.003892	.002683	.001656	.000860	.000330	−2.56
−2.55	.005386	.004020	.002786	.001732	.000909	.000356	−2.55
−2.54	.005543	.004152	.002892	.001811	.000960	.000383	−2.54
−2.53	.005703	.004289	.003001	.001892	.001014	.000411	−2.53
−2.52	.005868	.004429	.003115	.001977	.001071	.000442	−2.52
−2.51	.006037	.004573	.003232	.002066	.001130	.000475	−2.51
−2.50	.006210	.004721	.003352	.002157	.001192	.000509	−2.50

	SKEWNESS						
t	.6	.7	.8	.9	1.0	1.1	t
-2.99							-2.99
-2.98							-2.98
-2.97							-2.97
-2.96							-2.96
-2.95							-2.95
-2.94							-2.94
-2.93							-2.93
-2.92							-2.92
-2.91							-2.91
-2.90							-2.90
-2.89							-2.89
-2.88							-2.88
-2.87	.000001						-2.87
-2.86	.000001						-2.86
-2.85	.000001						-2.85
-2.84	.000001						-2.84
-2.83	.000001						-2.83
-2.82	.000002						-2.82
-2.81	.000002						-2.81
-2.80	.000002						-2.80
-2.79	.000003						-2.79
-2.78	.000003						-2.78
-2.77	.000004						-2.77
-2.76	.000004						-2.76
-2.75	.000005						-2.75
-2.74	.000006						-2.74
-2.73	.000007						-2.73
-2.72	.000008						-2.72
-2.71	.000010						-2.71
-2.70	.000011						-2.70
-2.69	.000013						-2.69
-2.68	.000015						-2.68
-2.67	.000017						-2.67
-2.66	.000020						-2.66
-2.65	.000023						-2.65
-2.64	.000026						-2.64
-2.63	.000029						-2.63
-2.62	.000033						-2.62
-2.61	.000038	.000001					-2.61
-2.60	.000042	.000001					-2.60
-2.59	.000048	.000001					-2.59
-2.58	.000054	.000001					-2.58
-2.57	.000061	.000002					-2.57
-2.56	.000068	.000002					-2.56
-2.55	.000076	.000003					-2.55
-2.54	.000085	.000003					-2.54
-2.53	.000095	.000004					-2.53
-2.52	.000106	.000005					-2.52
-2.51	.000117	.000007					-2.51
-2.50	.000130	.000008					-2.50

t	SKEWNESS 0	.1	.2	.3	.4	.5	t
-2.49	.006387	.004873	.003477	.002253	.001258	.000546	-2.49
-2.48	.006569	.005030	.003606	.002351	.001326	.000585	-2.48
-2.47	.006756	.005191	.003739	.002454	.001397	.000626	-2.47
-2.46	.006947	.005356	.003876	.002560	.001471	.000670	-2.46
-2.45	.007143	.005527	.004017	.002671	.001549	.000716	-2.45
-2.44	.007344	.005701	.004163	.002785	.001631	.000765	-2.44
-2.43	.007549	.005881	.004314	.002904	.001716	.000817	-2.43
-2.42	.007760	.006066	.004469	.003027	.001805	.000872	-2.42
-2.41	.007976	.006256	.004629	.003154	.001898	.000930	-2.41
-2.40	.008198	.006450	.004794	.003286	.001994	.000992	-2.40
-2.39	.008424	.006651	.004964	.003423	.002095	.001056	-2.39
-2.38	.008656	.006856	.005140	.003565	.002201	.001124	-2.38
-2.37	.008894	.007067	.005320	.003711	.002310	.001196	-2.37
-2.36	.009137	.007284	.005507	.003863	.002425	.001272	-2.36
-2.35	.009387	.007506	.005698	.004020	.002544	.001352	-2.35
-2.34	.009642	.007735	.005896	.004183	.002668	.001435	-2.34
-2.33	.009903	.007969	.006099	.004351	.002797	.001524	-2.33
-2.32	.010170	.008209	.006308	.004525	.002932	.001616	-2.32
-2.31	.010444	.008456	.006523	.004704	.003072	.001714	-2.31
-2.30	.010724	.008709	.006745	.004890	.003217	.001816	-2.30
-2.29	.011011	.008969	.006973	.005082	.003368	.001923	-2.29
-2.28	.011304	.009235	.007208	.005280	.003526	.002036	-2.28
-2.27	.011604	.009508	.007449	.005485	.003689	.002154	-2.27
-2.26	.011911	.009788	.007698	.005697	.003859	.002277	-2.26
-2.25	.012224	.010075	.007953	.005915	.004035	.002407	-2.25
-2.24	.012545	.010369	.008215	.006140	.004218	.002542	-2.24
-2.23	.012874	.010670	.008485	.006373	.004407	.002684	-2.23
-2.22	.013209	.010979	.008762	.006613	.004604	.002832	-2.22
-2.21	.013553	.011296	.009047	.006860	.004808	.002987	-2.21
-2.20	.013903	.011621	.009340	.007116	.005020	.003149	-2.20
-2.19	.014262	.011953	.009641	.007379	.005239	.003318	-2.19
-2.18	.014629	.012293	.009950	.007650	.005466	.003494	-2.18
-2.17	.015003	.012642	.010267	.007930	.005702	.003679	-2.17
-2.16	.015386	.012999	.010593	.008218	.005945	.003871	-2.16
-2.15	.015778	.013365	.010927	.008515	.006197	.004071	-2.15
-2.14	.016177	.013739	.011271	.008821	.006458	.004279	-2.14
-2.13	.016586	.014122	.011623	.009135	.006728	.004496	-2.13
-2.12	.017003	.014515	.011984	.009460	.007007	.004722	-2.12
-2.11	.017429	.014916	.012355	.009793	.007296	.004958	-2.11
-2.10	.017864	.015327	.012736	.010136	.007594	.005202	-2.10
-2.09	.018309	.015747	.013126	.010490	.007903	.005457	-2.09
-2.08	.018763	.016177	.013526	.010853	.008221	.005721	-2.08
-2.07	.019226	.016617	.013936	.011227	.008550	.005996	-2.07
-2.06	.019699	.017067	.014357	.011611	.008890	.006281	-2.06
-2.05	.020182	.017527	.014788	.012006	.009240	.006577	-2.05
-2.04	.020675	.017997	.015230	.012412	.009602	.006885	-2.04
-2.03	.021178	.018478	.015683	.012829	.009975	.007203	-2.03
-2.02	.021692	.018970	.016147	.013258	.010359	.007534	-2.02
-2.01	.022216	.019472	.016622	.013698	.010756	.007876	-2.01
-2.00	.022750	.019986	.017108	.014150	.011165	.008231	-2.00

	SKEWNESS						
t	6	7	.8	.9	1 0	1.1	t
-2 49	000144	000010					-2 49
-2.48	.000160	.000012					-2.48
-2 47	000176	000015					-2 47
-2.46	.000195	.000018					-2.46
-2 45	000214	.000021					-2 45
-2.44	.000236	.000025					-2.44
-2.43	.000259	.000030					-2 43
-2 42	.000284	.000036					-2.42
-2.41	.000311	.000042					-2.41
-2.40	.000340	.000049					-2.40
-2 39	.000372	.000057					-2 39
-2.38	.000406	000066					-2.38
-2.37	.000442	.000076	.000001				-2 37
-2.36	.000481	.000087	000001				-2.36
-2 35	.000523	000100	.000001				-2.35
-2 34	.000569	000115	.000002				-2 34
-2.33	.000617	.000131	.000003				-2 33
-2.32	.000668	.000149	000004				-2 32
-2 31	.000724	.000169	.000005				-2 31
-2 30	.000783	.000191	.000007				-2.30
-2.29	.000845	000215	000010				-2.29
-2 28	000912	.000242	.000013				-2 28
-2.27	.000984	000271	000017				-2.27
-2.26	.001060	.000304	000021				-2.26
-2.25	.001140	.000339	.000027				-2 25
-2 24	.001226	000378	.000034				-2.24
-2 23	.001317	.000421	000042				-2.23
-2 22	.001413	000467	.000051				-2 22
-2 21	001515	.000517	.000062				-2.21
-2.20	001623	.000571	.000075				-2.20
-2.19	.001737	000631	000091				-2 19
-2.18	001857	000695	.000108				-2.18
-2.17	.001985	000764	.000128				-2.17
-2.16	002119	.000838	.000151				-2 16
-2 15	.002261	000918	.000178	000001			-2 15
-2.14	002410	001005	000207	000002			-2.14
-2.13	.002567	001097	.000241	000003			-2 13
-2 12	002732	.001197	.000279	.000005			-2 12
-2.11	002906	001303	.000321	.000008			-2 11
-2.10	.003088	001417	.000368	.000012			-2 10
-2 09	.003280	.001539	.000421	000017			-2 09
-2 08	.003481	.001669	000479	000024			-2.08
-2.07	.003692	.001807	.000543	.000033			-2.07
-2 06	003912	001954	.000614	000045			-2 06
-2 05	.004144	002111	.000692	.000059			-2 05
-2.04	.004386	.002277	.000777	.000076			-2.04
-2.03	.004639	.002453	.000870	.000097			-2.03
-2 02	004903	.002640	.000972	.000123			-2 02
-2.01	.005179	002838	001083	.000153			-2 01
-2 00	.005468	.003047	.001203	000189			-2 00

	SKEWNESS						
t	0	.1	2	.3	.4	5	t
-1.99	023295	020511	.017607	014615	.011586	.008598	-1.99
-1.98	.023852	021047	.018117	.015091	.012020	.008979	-1.98
-1.97	024419	021595	018639	015581	012467	009372	-1.97
-1.96	024998	.022155	.019174	.016083	012927	.009780	-1.96
-1.95	.025588	.022727	.019721	.016598	.013401	.010201	-1.95
-1.94	026190	023310	.020281	017126	.013888	.010636	-1.94
-1.93	026803	023907	.020854	017668	.014390	.011086	-1.93
-1.92	027429	.024515	.021439	018223	014906	.011550	-1.92
-1.91	.028067	.025137	.022039	.018793	.015436	.012030	-1.91
-1.90	028717	025771	.022652	.019377	.015982	.012525	-1.90
-1.89	029379	026419	023278	.019975	.016542	.013036	-1.89
-1.88	030054	027080	.023919	020588	017118	.013564	-1.88
-1.87	030742	.027754	.024573	.021216	.017710	.014107	-1.87
-1.86	031443	.028442	025243	.021859	.018318	.014668	-1.86
-1.85	032157	.029144	.025926	022517	.018942	.015245	-1.85
-1.84	.032884	029859	.026625	.023192	.019582	.015840	-1.84
-1.83	033625	.030590	027339	023882	020240	.016453	-1.83
-1.82	034379	.031344	028068	024588	.020914	017084	-1.82
-1.81	.035148	.032093	.028812	.025311	.021606	.017734	-1.81
-1.80	.035930	032867	029572	026050	.022315	.018402	-1.80
-1.79	.036727	.033656	030349	.026806	.023043	019089	-1.79
-1.78	037538	.034461	.031141	.027580	.023789	.019796	-1.78
-1.77	038364	035280	.031949	028371	.024553	020522	-1.77
-1.76	039204	036116	.032775	029179	.025336	.021269	-1.76
-1.75	.040059	.036967	.033617	.030005	026138	022036	-1.75
-1.74	040930	037834	.034476	030850	026960	.022823	-1.74
-1.73	.041815	038717	.035352	031713	027801	.023632	-1.73
-1.72	.042716	039617	036245	032594	.028662	.024462	-1.72
-1.71	043633	.040533	.037157	.033494	029543	.025314	-1.71
-1.70	044565	.041466	.038086	034414	.030445	.026188	-1.70
-1.69	045514	042416	039033	.035352	.031368	.027084	-1.69
-1.68	046479	.043384	.039999	.036310	.032311	.028002	-1.68
-1.67	.047460	044368	.040983	037288	.033276	.028944	-1.67
-1.66	048457	.045371	041985	038287	.034262	.029909	-1.66
-1.65	049471	046390	.043007	.039305	.035270	.030897	-1.65
-1.64	.050503	047428	.044048	040344	.036300	.031909	-1.64
-1.63	051551	048484	.045108	041403	.037353	.032946	-1.63
-1.62	052616	.049559	.046188	042484	.038428	.034006	-1.62
-1.61	053699	050652	.047288	.043586	039525	.035092	-1.61
-1.60	.054799	.051763	048407	.044709	.040646	.036202	-1.60
-1.59	055917	.052893	.049547	.045853	.041791	.037338	-1.59
-1.58	.057053	.054043	050707	047020	.042958	038499	-1.58
-1.57	.058208	.055212	.051887	048209	.044150	.039687	-1.57
-1.56	059380	.056400	.053089	.049420	045365	.040900	-1.56
-1.55	060571	057608	.054311	.050653	.046605	.042139	-1.55
-1.54	.061780	.058835	.055555	.051909	.047870	.043405	-1.54
-1.53	.063008	.060083	056820	.053189	.049159	.044698	-1.53
-1.52	064255	061351	.058106	.054491	.050473	.046018	-1.52
-1.51	065522	.062639	.059414	.055816	.051812	.047366	-1.51
-1.50	066807	.063947	.060744	.057165	.053176	.048740	-1.50

t	SKEWNESS						t
	.6	7	.8	.9	1.0	1.1	
–1.99	.005769	.003269	.001333	000230			–1.99
–1.98	.006083	003502	.001474	.000279			–1.98
–1.97	.006410	003748	.001625	000334	000001		–1.97
–1.96	.006751	.004007	.001789	.000397	000002		–1 96
–1.95	.007105	.004280	.001965	000469	.000004		–1.95
–1.94	.007475	.004567	.002153	.000551	.000008		–1.94
–1 93	.007859	.004869	.002355	.000642	000014		–1.93
–1.92	.008258	.005185	.002571	.000745	.000024		–1.92
–1.91	.008672	.005518	.002801	.000859	.000038		–1.91
–1 90	009103	.005866	.003047	.000986	000057		–1.90
–1 89	009550	006230	.003309	.001126	.000082		–1 89
–1.88	.010013	.006612	.003587	.001280	.000114		–1.88
–1.87	010494	.007011	.003882	.001449	.000155		–1 87
–1.86	.010992	.007428	004195	.001634	.000205		–1.86
–1 85	.011509	.007864	004526	.001836	.000266		–1 85
–1.84	012043	.008318	.004876	.002055	.000339		–1 84
–1.83	012596	.008792	.005246	002293	.000425		–1.83
–1.82	.013168	.009286	.005637	.002550	.000526		–1.82
–1.81	.013760	.009800	.006048	.002827	.000642		–1.81
–1.80	014372	.010334	.006480	003126	.000776	000001	–1.80
–1.79	.015003	.010891	.006935	.003446	.000929	.000006	–1.79
–1.78	015656	.011469	.007413	.003789	.001101	.000016	–1.78
–1.77	.016329	.012070	.007914	.004156	.001295	.000033	–1.77
–1 76	.017024	.012693	.008439	.004548	.001511	.000062	–1 76
–1.75	.017741	013340	.008989	.004965	.001752	.000103	–1.75
–1.74	018480	.014011	009564	.005408	.002017	.000159	–1.74
–1.73	019242	.014706	.010165	.005879	.002309	.000233	–1.73
–1.72	.020026	.015425	.010793	.006377	.002629	.000328	–1.72
–1 71	.020834	016170	.011448	.006904	.002979	.000446	–1 71
–1.70	.021666	.016941	.012130	.007461	.003358	.000590	–1.70
–1.69	022522	.017737	.012841	.008048	.003769	000761	–1.69
–1 68	.023402	.018560	.013580	.008667	.004213	.000962	–1.68
–1 67	.024307	.019410	.014349	.009317	004691	001195	–1.67
–1.66	.025238	.020288	.015148	.010001	.005204	.001462	–1.66
–1.65	026194	021193	.015978	010717	.005753	.001766	–1.65
–1.64	.027175	.022127	.016838	.011468	.006340	.002109	–1.64
–1 63	.028183	023090	.017731	.012254	.006965	.002491	–1.63
–1 62	.029218	.024081	.018655	.013075	.007629	.002916	–1.62
–1 61	.030280	.025102	019612	.013933	.008334	.003385	–1.61
–1.60	.031368	.026153	020602	.014827	.009080	.003899	–1 60
–1 59	.032485	027235	.021625	.015759	.009868	.004460	–1.59
–1.58	.033629	.028346	.022683	.016729	.010699	.005069	–1.58
–1 57	.034801	.029489	.023775	.017738	.011574	.005729	–1.57
–1.56	.036002	.030664	.024902	.018786	.012494	.006439	–1.56
–1.55	037232	.031870	.026064	.019873	.013459	.007202	–1.55
–1 54	038490	033108	.027262	.021001	.014470	.008018	–1.54
–1 53	039778	.034378	.028496	.022170	.015528	.008889	–1.53
–1.52	.041096	.035681	.029766	.023380	.016633	.009816	–1.52
–1.51	042444	.037018	.031074	024632	.017786	.010799	–1.51
–1.50	.043821	038387	.032418	.025926	.018988	.011839	–1.50

t	SKEWNESS						t
	0	1	2	3	4	5	
–1 49	068112	065277	062096	058538	.054566	050143	–1.49
–1 48	.069437	066627	063471	.059935	055982	051574	–1 48
–1.47	.070781	067998	064867	061355	057424	053033	–1 47
–1 46	072145	069390	066287	062800	.058892	.054520	–1.46
–1 45	073529	.070804	067729	064270	060387	.056036	–1 45
–1.44	074934	072239	.069194	.065764	061908	057580	–1 44
–1 43	.076359	073696	070682	067283	.063455	.059154	–1 43
–1 42	077804	075174	072194	068827	065030	.060757	–1 42
–1 41	.079270	.076675	.073729	.070398	.066631	.062389	–1 41
–1 40	080757	.078197	075288	071990	.068260	.064051	–1.40
–1 39	082264	079742	076870	073609	.069916	065742	–1 39
–1.38	083793	.081310	.078476	075254	.071600	.067463	–1.38
–1 37	.085343	082899	080106	076925	073312	.069214	–1.37
–1 36	.086915	.084512	081761	.078622	.075051	.070995	–1.36
–1 35	088508	.086147	.083439	080345	.076818	.072807	–1.35
–1.34	090123	.087806	085143	082094	078613	074649	–1 34
–1.33	.091759	.089487	086870	.083869	080436	076521	–1 33
–1.32	093418	091192	088623	.085670	.082288	078423	–1.32
–1.31	095098	.092920	090400	087498	.084168	080357	–1.31
–1 30	.096800	.094671	.092202	089353	086077	082321	–1.30
–1 29	098525	096446	094029	091234	.088014	.084316	–1 29
–1 28	.100273	098245	095881	093142	.089980	.086341	–1 28
–1.27	.102042	.100068	.097759	.095076	.091974	.088398	–1 27
–1 26	103835	.101914	.099662	097038	.093997	090485	–1 26
–1.25	105650	103785	.101590	099027	096050	092604	–1.25
–1 24	.107488	.105679	103544	101043	.098131	094754	–1.24
–1 23	.109349	.107598	.105523	.103086	.100241	.096934	–1.23
–1 22	111232	.109541	.107528	105156	.102380	.099146	–1.22
–1 21	.113139	111508	.109559	107254	.104548	.101389	–1.21
–1 20	.115070	113500	.111616	109379	.106746	.103663	–1.20
–1.19	.117023	.115516	.113698	111531	.108972	.105968	–1.19
–1 18	.119000	117557	115806	.113711	.111228	108304	–1 18
–1 17	.121000	.119622	.117941	115918	.113512	.110671	–1.17
–1 16	.123024	121713	.120101	118153	.115826	.113069	–1.16
–1 15	125072	123828	122288	120415	118169	.115498	–1.15
–1 14	127143	.125968	.124500	122705	.120541	.117958	–1 14
–1.13	.129238	128132	.126739	125022	.122942	.120448	–1.13
–1.12	.131357	130322	129004	127367	.125172	.122970	–1 12
–1 11	.133500	132537	131294	.129739	127831	.125522	–1.11
–1 10	135666	.134776	133612	132139	.130319	.128104	–1 10
–1.09	.137857	.137041	135955	134566	.132836	.130717	–1 09
–1 08	140071	.139330	.138324	137021	.135381	133361	–1.08
–1 07	.142310	141645	.140720	.139503	.137955	.136034	–1.07
–1 06	.144572	143985	143142	.142012	.140558	138738	–1.06
–1 05	.146859	146350	.145590	144549	.143190	.141472	–1 05
–1.04	.149170	.148740	.148064	.147112	.145850	.144235	–1 04
–1.03	151505	151155	150564	149703	.148538	.147029	–1 03
–1 02	153864	.153595	.153090	152322	.151255	.149851	–1 02
–1 01	156248	156060	.155643	.154967	.154000	.152703	–1.01
–1 00	.158655	158551	158221	.157639	.156773	.155584	–1 00

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
−1.49	045229	.039790	033800	.027262	.020239	.012937	−1.49
−1.48	.046668	.041227	.035220	.028641	.021539	.014094	−1.48
−1.47	.048138	.042698	.036678	.030063	.022890	.015311	−1.47
−1.46	.049638	.044203	038174	.031529	.024290	.016587	−1.46
−1.45	.051170	.045743	.039709	.033039	.025742	.017923	−1.45
−1.44	.052734	.047317	.041282	.034592	.027244	.019321	−1.44
−1.43	.054329	.048927	.042895	.036190	.028798	.020779	−1.43
−1.42	.055956	.050571	.044547	.037833	.030403	.022299	−1.42
−1.41	.057615	.052252	.046239	.039520	.032060	.023880	−1.41
−1.40	.059306	.053967	.047970	041252	033769	.025523	−1.40
−1.39	.061030	.055718	.049741	043029	.035530	.027228	−1.39
−1.38	.062786	.057506	.051551	.044852	.037343	.028995	−1.38
−1.37	.064575	.059329	.053402	.046719	.039208	.030824	−1.37
−1.36	.066397	.061188	.055293	.048632	.041126	.032714	−1.36
−1.35	.068251	.063083	.057225	.050591	.043095	.034667	−1.35
−1.34	.070139	.065015	.059196	.052595	.045117	.036681	−1.34
−1.33	.072060	.066983	.061208	.054644	.047191	.038756	−1.33
−1.32	074014	.068987	.063260	.056738	.049318	.040893	−1.32
−1.31	076001	.071028	.065353	.058878	.051496	.043090	−1.31
−1.30	.078021	.073105	.067486	.061064	.053725	.045348	−1.30
−1.29	.080075	.075219	.069659	.063294	.056007	.047667	−1.29
−1.28	.082163	.077370	.071873	.065570	.058339	.050045	−1.28
−1.27	.084284	.079557	.074127	.067891	.060723	.052482	−1.27
−1.26	.086438	.081780	.076422	.070256	.063157	.054978	−1.26
−1.25	.088626	.084040	.078756	.072666	.065642	.057533	−1.25
−1.24	.090847	.086336	.081130	.075121	.068178	.060146	−1.24
−1.23	.093102	.088669	.083545	.077620	.070763	.062816	−1.23
−1.22	.095391	.091038	.085999	.080163	.073397	.065543	−1.22
−1.21	.097712	.093444	.088493	.082749	.076081	.068325	−1.21
−1.20	.100068	.095885	.091026	.085380	.078813	.071164	−1.20
−1.19	.102456	.098363	.093598	.088053	.081594	.074057	−1.19
−1.18	.104878	.100876	.096210	.090770	.084422	.077004	−1.18
−1.17	.107333	.103426	.098860	.093529	.087298	.080004	−1.17
−1.16	.109821	.106011	.101549	.096330	.090221	.083057	−1.16
−1.15	.112342	.108631	.104277	.099173	.093189	.086162	−1.15
−1.14	.114897	.111287	.107042	.102058	.096204	.089318	−1.14
−1.13	.117484	.113978	.109846	.104984	.099264	.092525	−1.13
−1.12	.120103	.116704	.112687	.107951	.102368	.095781	−1.12
−1.11	.122756	.119464	.115565	.110958	.105517	.099086	−1.11
−1.10	.125440	.122259	.118480	.114005	.108708	.102438	−1.10
−1.09	.128157	.125089	.121433	.117091	.111943	.105838	−1.09
−1.08	.130906	.127952	.124421	.120216	.115220	.109283	−1.08
−1.07	.133687	.130849	.127445	.123380	.118539	.112774	−1.07
−1.06	.136499	.133780	.130505	.126582	.121898	.116310	−1.06
−1.05	.139344	.136744	.133600	.129822	.125298	.119888	−1.05
−1.04	.142219	.139741	.136731	.133099	.128737	.123510	−1.04
−1.03	.145125	.142771	.139895	.136412	.132215	.127173	−1.03
−1.02	.148063	.145833	.143094	.139761	.135732	.130877	−1.02
−1.01	.151031	.148927	.146326	.143146	.139286	.134621	−1.01
−1.00	.154029	.152053	.149592	.146565	.142877	.138404	−1.00

t	SKEWNESS						t
	0	1	2	.3	.4	.5	
- .99	161087	161066	160825	.160338	159573	158494	- 99
- 98	163543	163606	163455	163064	162402	161433	- 98
- 97	.166023	.166172	166111	.165816	165258	164401	- .97
- .96	168528	.168762	168792	168595	.168141	.167396	- 96
- .95	171056	171377	171500	171400	.171052	.170420	- .95
- 94	173609	174017	174232	174232	.173989	173472	- .94
- 93	176186	.176682	176990	177090	.176954	176551	- .93
- 92	178786	179371	179774	.179974	.179945	.179658	- 92
- 91	.181411	182085	182583	182883	.182963	.182792	- .91
- 90	184060	.184823	185416	185819	.186007	.185953	- .90
- .89	186733	.187587	188275	188780	.189077	.189140	- .89
- 88	.189430	190374	191159	191766	.192173	.192354	- 88
- .87	192150	.193186	.194068	.194778	195295	195594	- 87
- 86	194895	196022	.197001	.197814	198442	198860	- .86
- .85	197663	198882	.199959	200876	201614	.202151	- 85
- .84	200454	201766	.202941	203962	204812	.205467	- 84
- .83	203269	204674	205947	207073	208034	208808	- 83
- .82	.206108	.207606	.208977	210208	.211280	.212174	- 82
- .81	.208970	210561	212031	.213366	.214551	.215564	- 81
- 80	.211855	.213540	.215109	.216549	.217845	.218978	- .80
- .79	214764	.216542	.218210	219756	.221163	222415	- 79
- 78	.217695	.219568	.221335	.222985	224504	.225876	- 78
- 77	.220650	.222616	.224483	.226238	.227869	.229359	- 77
- .76	223627	.225687	227653	229514	231256	.232865	- .76
- 75	.226627	228782	.230847	232812	.234666	.236393	- .75
- .74	229650	231898	234063	236133	.238097	239943	- .74
- 73	232695	.235038	.237301	.239476	241551	.243514	- .73
- 72	.235762	.238199	.240561	242840	245026	247106	- .72
- 71	238852	241382	.243843	246227	248522	.250719	- 71
- .70	.241964	244588	.247147	.249634	252039	.254353	- .70
- 69	.245097	247815	.250472	.253063	255577	.258006	- .69
- .68	248252	.251063	253819	.256512	.259135	261678	- .68
- .67	251429	.254333	257186	.259982	.262712	.265370	- .67
- 66	254627	257624	260574	.263471	266310	.269080	- 66
- 65	.257846	260935	.263982	.266981	269926	.272809	- .65
- .64	.261086	.264267	.267410	.270510	.273561	276555	- .64
- .63	264347	.267620	270858	.274058	.277214	.280319	- 63
- 62	267629	.270993	274326	.277625	.280885	.284100	- .62
- 61	270931	274385	.277813	.281211	.284574	.287897	- .61
- 60	274253	.277797	281319	.284815	.288281	.291711	- 60
- .59	277595	281229	.284844	.288437	292004	.295540	- .59
- .58	.280957	.284680	.288387	.292076	295743	.299385	- .58
- 57	.284339	.288149	.291948	.295732	299499	.303244	- 57
- 56	.287740	.291638	.295527	.299406	303270	307118	- .56
- 55	.291160	.295144	.299123	.303095	.307057	.311006	- .55
- .54	294599	.298669	.302737	306801	.310859	.314908	- .54
- 53	298056	.302211	306368	.310523	314675	.318822	- .53
- .52	301532	.305771	.310015	.314260	318506	.322749	- .52
- .51	305026	309349	313678	318012	322350	.326689	- .51
- 50	308538	.312943	.317357	.321779	326207	330640	- .50

t	SKEWNESS						t
	6	7	8	.9	1 0	1 1	
– .99	.157058	155211	.152890	150019	146503	.142224	– 99
– .98	.160116	.158399	.156221	153507	.150166	.146082	– 98
– .97	.163204	.161618	.159584	157028	.153862	.149976	– .97
– 96	166321	.164868	.162978	.160582	157593	.153905	– .96
– .95	169468	168147	166403	.164168	.161357	.157869	– .95
– .94	.172643	171457	.169859	.167785	.165153	.161866	– .94
– .93	.175846	.174795	.173345	.171433	.168981	.165895	– .93
– 92	.179078	178162	.176860	.175111	172840	.169956	– .92
– 91	182337	.181558	.180405	.178819	.176729	.174047	– .91
– .90	.185624	.184981	.183978	.182556	.180648	.178168	– 90
– .89	.188938	.188433	.187578	.186321	.184595	.182317	– .89
– .88	192279	.191911	.191207	.190114	.188569	.186495	– 88
– .87	195646	.195416	.194862	.193934	.192571	.190699	– .87
– .86	.199040	.198948	.198544	.197781	.196599	.194929	– .86
– .85	.202459	.202505	.202252	201653	.200653	.199184	– 85
– .84	205903	.206088	205985	.205551	.204731	.203463	– 84
– 83	.209372	.209695	.209743	.209473	.208834	.207765	– .83
– 82	.212866	.213328	.213525	213419	212959	.212089	– 82
– 81	.216384	.216984	.217331	.217388	.217107	216434	– .81
– 80	.219926	.220664	.221160	.221379	.221277	.220801	– .80
– 79	.223491	.224366	.225012	225393	.225468	225186	– .79
– 78	.227079	.228092	228886	229428	229678	.229590	– .78
– .77	230690	.231839	.232781	233483	.233909	234012	– .77
– 76	234323	.235608	.236697	237558	.238157	.238451	– .76
– .75	.237977	239399	.240633	.241653	.242424	.242906	– .75
– .74	.241653	.243210	244589	.245766	.246708	.247376	– .74
– .73	.245350	.247040	.248565	.249897	.251008	.251861	– 73
– .72	.249067	.250891	.252558	254045	.255323	.256359	– .72
– 71	.252804	.254761	.256570	258210	.259654	.260869	– .71
– .70	.256561	.258649	.260599	.262391	.263998	.265392	– .70
– .69	.260336	.262555	.264645	.266587	268356	.269926	– .69
– .68	264131	.266479	.268708	270798	.272727	.274469	– .68
– 67	267943	270420	.272786	275022	.277110	.279023	– .67
– .66	.271773	.274377	.276879	.279260	.281503	.283585	– .66
– 65	.275621	.278351	.280986	.283511	285908	288154	– 65
– .64	279485	.282340	.285107	287774	.290322	.292731	– 64
– 63	283365	.286343	.289242	292048	294745	.297315	– 63
– .62	287261	.290361	.293389	.296333	299177	.301903	– .62
– 61	.291173	.294394	.297549	300628	.303616	.306497	– .61
– .60	295099	.298439	.301720	304932	.308063	.311095	– .60
– .59	.299040	.302497	.305902	309246	312515	315697	– .59
– .58	.302995	.306568	.310095	313567	316974	.320301	– .58
– .57	306963	.310650	.314297	317897	321437	.324907	– 57
– .56	.310944	.314743	.318509	322233	325905	.329514	– 56
– .55	.314938	.318848	.322729	.326575	.330377	.334122	– .55
– .54	.318944	.322962	.326958	.330924	.334851	.338730	– .54
– .53	.322961	.327086	.331194	.335277	.339328	.343338	– .53
– .52	.326989	.331219	.335437	.339635	.343807	.347944	– 52
– .51	.331027	.335361	.339686	.343997	.348287	.352548	– .51
– .50	.335076	.339511	343942	.348363	.352768	.357150	– .50

t	SKEWNESS						t
	0	.1	.2	.3	.4	.5	
- .49	.312067	.316554	.321051	.325560	.330077	.334603	- .49
- .48	.315614	320181	.324761	329354	.333960	338576	- .48
- .47	.319178	.323824	.328486	.333163	.337854	342560	- .47
- 46	.322758	.327483	.332225	.336984	.341761	.346553	- 46
- .45	.326355	.331157	.335978	340819	.345678	.350557	- .45
- 44	.329969	.334846	.339745	344665	.349607	.354569	- .44
- .43	.333598	.338550	.343525	.348524	.353545	.358590	- .43
- 42	.337243	.342269	.347319	352394	.357494	362619	- .42
- 41	340903	.346001	.351125	.356275	.361452	.366655	- .41
- .40	.344578	.349747	.354944	.360167	.365419	.370699	- .40
- .39	.348268	.353507	.358774	.364070	.369395	.374749	- .39
- .38	.351973	.357280	.362616	367982	373379	378806	- .38
- .37	.355691	.361065	.366469	.371904	.377371	382869	- .37
- .36	.359424	.364863	.370333	.375835	381370	386937	- .36
- .35	.363169	.368673	.374208	.379775	.385376	.391010	- .35
- .34	366928	.372494	.378092	.383723	.389388	395087	- .34
- .33	370700	376327	.381986	.387680	393407	.399169	- .33
- .32	374484	.380171	.385890	391643	.397431	.403254	- .32
- .31	.378280	.384025	389802	395614	.401460	.407342	- .31
- .30	382089	.387889	393723	.399591	.405494	.411432	- .30
- .29	.385908	.391763	397652	403575	.409532	.415525	- .29
- .28	.389739	.395647	.401589	407564	.413575	419620	- .28
- .27	.393580	.399540	.405533	.411559	417620	.423716	- .27
- 26	.397432	.403441	.409483	.415559	.421669	.427813	- .26
- 25	401294	.407351	.413441	.419563	425720	.431910	- 25
- 24	405165	.411269	.417404	.423572	.429773	436008	- .24
- .23	.409046	.415194	.421373	.427584	.433828	.440105	- .23
- .22	412936	.419126	.425348	.431600	.437884	.444201	- .22
- .21	.416834	.423066	.429327	.435619	441941	448296	- .21
- .20	.420740	.427011	.433311	.439640	.445999	.452389	- .20
- .19	424655	.430963	437298	.443663	.450056	.456479	- .19
- .18	.428576	.434920	.441290	.447688	.454113	.460568	- .18
- .17	.432505	.438882	.445284	.451714	.458170	.464653	- .17
- .16	436441	.442849	.449282	.455740	462225	.468735	- .16
- 15	.440382	.446820	.453282	.459768	.466278	.472813	- .15
- .14	.444330	.450796	.457284	.463795	.470329	476887	- .14
- .13	.448283	.454775	.461288	.467822	.474378	480956	- .13
- 12	.452242	458757	465293	.471848	.478424	.485020	- .12
- .11	.456205	.462742	.469298	.475873	482466	489079	- .11
- 10	460172	.466730	.473304	.479896	.486505	.493132	- .10
- .09	.464144	.470719	.477310	483917	490540	497178	- .09
- 08	.468119	.474711	481316	487936	.494570	501218	- .08
- .07	472097	.478703	.485321	.491952	498595	.505251	- .07
- 06	.476078	.482696	.489325	.495964	.502615	509277	- .06
- .05	.480061	.486690	.493327	.499973	.506629	.513295	- .05
- .04	.484047	.490683	.497327	.503978	.510637	517310	- 04
- .03	.488034	.494676	.501324	.507978	.514639	.521306	- .03
- .02	.492022	.498668	.505319	.511974	.518634	.525298	- .02
- .01	496011	.502660	.509311	.515965	.522621	.529282	- .01
.00	.500000	.506649	.513299	.519949	.526602	.533255	.00

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
- .49	.339134	.343668	.348202	.352731	.357249	.361748	- .49
- .48	.343201	.347833	.352468	.357102	.361729	.366343	- .48
- .47	.347277	.352004	.356737	.361474	.366208	.370934	- .47
- .46	.351361	.356181	.361011	.365847	.370685	.375519	- .46
- .45	.355452	.360363	.365287	.370221	.375160	.380099	- .45
- .44	.359551	.364551	.369566	.374595	.379632	.384673	- .44
- .43	.363656	.368742	.373847	.378968	.384101	.389240	- .43
- .42	.367767	.372938	.378130	.383340	.388565	.393800	- .42
- .41	.371884	.377137	.382414	.387711	.393025	.398353	- .41
- .40	.376006	.381340	.386698	.392079	.397480	.402897	- .40
- .39	.380133	.385544	.390982	.396445	.401930	.407432	- .39
- .38	.384264	.389751	.395266	.400808	.406373	.411958	- .38
- 37	.388398	.393959	.399549	.405167	.410810	.416475	- .37
- .36	.392536	.398168	.403830	.409522	.415240	.420981	- .36
- .35	.396677	.402377	.408109	.413872	.419662	.425476	- .35
- .34	.400820	.406587	.412386	.418217	.424076	.429961	- .34
- .33	.404965	.410796	.416660	.422556	.428482	.434434	- .33
- .32	.409111	.415004	.420930	.426889	.432878	.438895	- .32
- .31	.413259	.419211	.425197	.431216	.437266	.443343	- .31
- .30	.417406	.423415	.429459	.435535	.441643	.447779	- .30
- .29	.421554	.427618	.433716	.439848	.446010	.452201	- .29
- .28	.425701	.431817	.437968	.444152	.450366	.456610	- .28
- .27	.429847	.436014	.442214	.448448	.454712	.461004	- .27
- .26	.433992	.440206	.446454	.452735	.459046	.465384	- .26
- .25	.438136	.444395	.450687	.457012	.463367	.469750	- .25
- .24	.442276	.448579	.454914	.461281	.467677	.474100	- .24
- .23	.446415	.452758	.459133	.465539	.471974	.478435	- .23
- .22	.450550	.456931	.463344	.469786	.476257	.482754	- .22
- .21	.454681	.461099	.467546	.474023	.480527	.487056	- .21
- .20	.458809	.465260	.471740	.478249	.484784	.491343	- .20
- .19	.462932	.469414	.475925	.482463	.489026	.495612	- .19
- .18	.467051	.473562	.480100	.486665	.493254	.499864	- .18
- .17	.471164	.477702	.484266	.490855	.497467	.504099	- .17
- .16	.475271	.481834	.488421	.495032	.501665	.508316	- .16
- .15	.479373	.485957	.492566	.499196	.505847	.512515	- .15
- .14	.483468	.490072	.496699	.503347	.510013	.516696	- .14
- .13	.487556	.494178	.500821	.507484	.514164	.520858	- .13
- .12	.491637	.498275	.504932	.511607	.518298	.525002	- .12
- .11	.495711	.502362	.509030	.515715	.522415	.529126	- .11
- .10	.499776	.506438	.513116	.519809	.526515	.533232	- .10
- .09	.503833	.510504	.517189	.523888	.530598	.537317	- .09
- .08	.507882	.514559	.521249	.527952	.534664	.541383	- .08
- .07	.511921	.518603	.525296	.532000	.538712	.545429	- .07
- .06	.515951	.522635	.529329	.536032	.542742	.549455	- .06
- .05	.519970	.526655	.533348	.540048	.546753	.553461	- .05
- .04	.523980	.530663	.537353	.544048	.550746	.557446	- .04
- .03	.527979	.534659	.541343	.548031	.554721	.561410	- .03
- .02	.531968	.538641	.545318	.551997	.558676	.565354	- .02
- .01	.535945	.542610	.549278	.555946	.562613	.569276	- .01
.00	.539910	.546566	.553222	.559878	.566530	.573177	.00

t	SKEWNESS						t
	0	1	.2	.3	.4	.5	
.00	500000	506649	513299	519949	526602	533255	.00
01	503989	510637	517283	523928	530574	537219	.01
02	507978	514622	.521263	.527901	534537	.541172	.02
.03	511966	.518604	.525237	.531867	.538493	.545115	.03
04	.515953	522583	529207	.535825	.542439	.549047	04
05	519939	526559	.533171	539777	.546375	552967	.05
06	523922	.530531	.537130	543720	550302	.556876	.06
07	527903	.534498	541082	.547655	554219	.560774	.07
08	531881	.538460	.545027	551582	.558126	.564658	.08
.09	.535856	.542418	.548965	555500	562021	.568531	.09
.10	539828	.546370	.552896	.559408	.565906	572390	.10
11	543795	.550316	556819	563307	569779	576236	.11
.12	.547758	554255	560734	567196	.573641	580069	.12
.13	551717	.558189	564641	.571075	577490	583888	.13
14	.555670	562115	.568539	.574943	.581327	587693	.14
15	.559618	.566033	.572427	578800	.585152	.591484	.15
16	563559	.569944	.576306	.582645	588963	.595260	.16
17	567495	.573847	580175	.586479	.592761	599022	.17
18	571424	.577741	584033	.590301	.596546	.602768	.18
.19	575345	581626	.587881	.594111	600317	606499	19
20	.579260	585503	591719	.597909	.604073	.610214	20
.21	583166	589369	595545	601693	.607816	.613913	.21
22	.587064	.593226	.599359	.605464	.611543	.617596	.22
23	.590954	597072	.603161	.609222	.615256	621263	.23
.24	594835	.600908	606951	.612966	618953	.624913	24
.25	598706	.604733	.610729	616696	.622635	.628546	25
.26	602568	.608546	614493	620411	.626301	.632163	.26
27	.606420	612348	618245	624112	.629951	.635762	.27
.28	610261	616138	.621983	.627798	.633585	.639343	.28
29	.614092	619915	.625707	.631469	.637202	642907	.29
30	617911	623680	629417	.635125	.640803	646453	.30
.31	.621720	.627432	.633113	.638764	644386	649981	.31
.32	.625516	631170	636794	.642388	647953	653490	32
.33	629300	.634895	.640460	.645995	651502	.656981	33
34	633072	.638606	644111	.649586	.655034	.660454	.34
35	.636831	.642303	.647746	.653161	.658547	.663907	.35
36	.640576	.645986	.651366	656718	662043	667342	.36
37	.644309	649653	.654970	660258	665521	.670757	.37
38	.648027	.653306	.658557	663781	668980	.674153	.38
39	651732	.656943	.662128	667286	.672420	677530	39
40	.655422	660564	665682	670774	.675842	.680887	.40
.41	.659097	664170	669218	674243	.679245	.684224	.41
42	662757	667759	672738	.677694	.682628	.687541	.42
43	666402	.671332	.676240	.681127	.685993	.690838	.43
.44	670031	.674888	.679724	.684541	689338	.694115	.44
.45	673645	678427	.683191	.687936	.692663	.697372	.45
.46	.677242	681949	.686639	.691312	.695968	.700608	.46
.47	680822	685453	690069	.694669	699254	.703824	47
.48	684386	.688940	.693480	698006	.702519	.707019	.48
49	687933	.692408	696872	.701324	.705765	.710194	.49

	SKEWNESS						
t	.6	7	8	9	10	11	t
.00	.539910	.546566	.553222	.559878	.566530	.573177	.00
01	.543864	.550508	.557151	.563791	.570427	.577057	.01
02	.547805	.554436	.561064	567687	574305	580915	02
.03	.551734	.558349	.564960	.571565	.578163	584751	.03
.04	.555650	.562248	568840	.575425	582001	588566	.04
.05	.559553	.566132	.572703	.579266	.585818	592359	.05
.06	.563443	570001	.576549	.583088	589616	.596129	.06
07	.567318	.573854	.580378	.586892	.593392	599878	.07
.08	.571180	.577691	.584190	.590676	.597148	.603604	08
.09	.575028	.581512	.587983	.594441	.600883	.607307	.09
.10	.578861	.585317	.591759	.598186	.604597	.610988	.10
.11	.582679	.589106	.595517	601912	.608289	.614647	.11
.12	586481	592877	599257	.605618	611961	.618283	.12
.13	590269	596632	602977	.609304	.615611	.621896	13
.14	.594041	600370	606680	.612970	619239	625486	.14
.15	.597797	.604090	.610363	.616615	622846	629053	15
.16	.601537	.607792	614027	.620241	.626431	632598	.16
17	605260	.611477	.617672	.623845	629994	636119	17
.18	608967	615144	.621298	627429	.633536	.639617	.18
.19	612657	.618792	.624904	.630992	.637055	.643092	19
.20	.616330	.622422	.628490	.634534	.640552	646544	20
.21	619986	626034	.632057	.638055	.644027	.649972	21
.22	.623624	.629626	.635603	.641555	.647480	653377	.22
.23	.627244	.633200	.639130	.645033	.650910	.656759	23
.24	.630847	.636754	.642636	.648491	.654318	.660118	24
.25	634431	640289	646121	.651926	.657704	663453	.25
26	.637997	643805	.649586	.655341	.661067	666765	26
.27	.641545	.647301	653031	658733	.664408	670053	.27
.28	645074	650778	.656455	662104	.667726	673318	.28
.29	648584	654234	.659857	.665453	.671021	.676560	.29
.30	.652075	.657671	663239	.668780	674294	.679778	.30
.31	655547	661087	666600	.672086	.677544	.682973	.31
.32	.659000	.664483	.669939	675369	.680771	686145	.32
.33	.662433	667859	.673258	678630	.683976	689293	.33
.34	.665847	.671214	.676555	.681870	.687157	.692417	.34
.35	.669241	674548	.679830	.685087	.690316	695519	35
36	.672615	.677862	.683085	688282	693453	.698597	36
.37	.675969	.681155	.686317	691454	.696566	701652	.37
.38	.679302	.684427	689528	694605	699657	704683	38
.39	.682616	687678	692717	.697733	702725	.707692	.39
.40	.685909	690908	.695885	700839	.705770	.710677	40
.41	.689181	.694117	.699031	703923	.708793	.713639	.41
.42	.692433	.697304	.702155	706984	711792	.716577	.42
.43	.695664	700470	.705257	710023	.714769	.719493	.43
.44	.698875	.703615	.708337	713040	.717723	722386	.44
45	.702064	.706739	.711396	.716035	.720655	.725256	.45
.46	.705232	.709840	.714432	719007	.723564	.728103	.46
.47	.708380	.712921	.717446	.721957	.726450	.730927	.47
.48	.711506	.715979	.720439	.724884	729314	.733728	.48
.49	.714611	.719016	.723409	.727789	.732155	736506	49

t	SKEWNESS						t
	0	.1	.2	.3	.4	.5	
.50	.691462	.695859	.700245	.704622	.708990	.713347	.50
.51	.694974	699290	.703599	.707901	.712194	.716480	.51
.52	698468	.702704	.706934	.711159	.715378	.719591	.52
.53	.701944	.706098	.710249	.714397	.718541	.722681	.53
.54	.705401	.709473	.713544	.717615	.721684	.725751	.54
.55	.708840	.712829	.716820	.720812	.724805	.728798	.55
56	.712260	.716165	.720075	723989	.727906	.731825	.56
.57	.715661	.719481	.723310	727145	.730985	734830	.57
.58	.719043	722778	.726524	.730280	.734043	.737814	.58
.59	.722405	.726054	.729718	.733394	.737080	.740776	.59
.60	.725747	.729311	.732891	736487	.740096	.743716	.60
.61	.729069	.732546	.736044	.739559	.743090	.746635	.61
.62	.732371	.735761	739175	.742609	746062	.749532	.62
.63	.735653	.738955	.742285	.745638	749013	.752407	.63
.64	.738914	.742129	.745374	.748646	.751942	.755261	.64
.65	.742154	.745281	.748441	751632	754850	.758093	.65
.66	.745373	.748411	.751487	754596	.757736	.760902	.66
.67	.748571	.751521	.754511	.757539	760599	.763691	.67
.68	.751748	.754608	.757514	.760459	.763441	.766457	.68
.69	.754903	.757674	.760494	.763358	.766261	.769201	.69
.70	.758036	.760719	.763453	.766235	.769060	.771923	.70
.71	.761148	.763741	.766390	.769090	.771836	.774624	.71
.72	.764238	.766741	.769304	.771922	.774590	.777302	.72
.73	.767305	.769719	.772196	.774733	.777322	.779959	.73
.74	.770350	.772674	.775067	.777521	.780032	.782593	.74
.75	.773373	.775607	.777914	.780287	.782719	.785206	.75
.76	.776373	.778518	.780739	.783031	.785385	.787797	.76
.77	.779350	.781406	.783542	.785752	788029	.790366	.77
.78	.782305	.784271	.786322	.788451	.790650	.792913	.78
.79	.785236	.787113	.789080	.791128	.793249	.795438	.79
.80	.788145	.789933	.791815	.793782	.795826	.797941	.80
.81	.791030	.792729	.794527	.796414	.798382	.800423	.81
.82	.793892	.795503	.797217	.799023	.800914	.802883	.82
.83	.796731	.798254	.799883	.801610	.803425	.805321	.83
.84	.799546	.800981	.802527	.804175	.805914	.807737	.84
.85	.802337	.803685	.805148	.806717	.808381	.810132	85
.86	.805105	.806366	.807747	.809237	.810825	.812505	.86
.87	.807850	.809024	.810322	.811734	.813248	.814856	.87
.88	.810570	.811658	.812875	.814208	.815649	.817186	.88
.89	.813267	.814269	.815404	.816661	.818027	.819495	.89
.90	.815940	.816856	.817911	.819091	.820384	.821782	.90
.91	.818589	.819420	.820395	.821498	.822719	.824047	.91
.92	.821214	.821961	.822856	.823884	.825032	.826291	.92
.93	.823814	.824478	.825294	.826247	.827324	.828515	.93
.94	.826391	.826972	.827709	.828587	.829593	.830716	.94
.95	.828944	.829443	.830101	.830905	.831841	.832897	.95
.96	.831472	.831889	.832471	.833202	.834068	835057	.96
.97	.833977	.834313	.834818	.835476	.836272	.837196	.97
.98	.836457	.836713	.837142	.837727	.838456	.839314	.98
.99	.838913	.839089	.839443	.839957	.840618	.841411	.99

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
.50	717694	722031	726358	730672	.734974	.739262	.50
.51	.720757	.725025	.729284	.733533	.737770	.741995	.51
.52	723797	.727997	732188	.736371	.740544	.744706	.52
.53	.726817	.730947	.735070	.739187	.743296	.747394	.53
54	.729815	.733875	.737931	.741981	.746025	.750060	.54
.55	.732791	.736781	.740769	.744753	.748732	.752704	.55
56	.735745	.739666	.743585	.747503	751416	.755325	.56
.57	.738678	.742528	.746379	.750230	.754079	.757924	.57
.58	.741589	.745369	.749152	.752936	.756720	.760501	.58
59	.744479	.748188	.751902	.755619	.759338	.763056	.59
.60	.747346	.750985	.755130	.758281	.761935	.765590	.60
.61	.750192	.753760	.757337	.760920	.764509	.768101	.61
.62	.753016	.756513	.760021	.763538	.767062	.770591	.62
.63	.755818	759245	.762684	.766134	.769593	.773059	.63
.64	.758599	.761954	.765325	.768708	.772103	.775505	.64
.65	.761357	.764642	767944	.771261	.774590	.777930	.65
66	.764094	.767308	.770541	.773792	.777057	.780334	.66
.67	.766809	.769952	.773117	.776301	779502	.782717	.67
.68	769502	.772574	.775671	.778789	.781925	.785078	.68
.69	.772173	.775175	.778203	.781255	.784328	.787418	.69
.70	.774822	.777754	.780714	.783700	.786709	.789738	.70
.71	.777450	.780311	.783203	.786124	.789069	.792036	.71
.72	780056	.782847	.785671	.788526	.791408	.794314	.72
.73	.782640	.785361	.788118	.790908	.793727	.796571	.73
.74	.785202	.787853	.790543	.793268	.796024	.798808	.74
.75	787743	.790324	.792947	.795607	.798301	.801024	.75
76	.790262	.792774	.795330	.797926	.800557	.803220	.76
.77	.792759	.795202	.797692	.800223	.802793	.805396	.77
.78	.795234	797609	.800033	.802500	805008	.807551	.78
.79	.797688	.799995	.802352	.804757	.807203	.809687	.79
.80	.800121	.802359	.804651	.806992	.809378	.811803	.80
.81	.802532	.804703	.806929	.809208	.811532	.813899	.81
.82	.804922	.807025	.809187	.811403	.813667	.815975	.82
.83	.807290	.809326	.811424	.813577	.815782	.818032	.83
.84	809637	.811606	.813640	.815732	.817877	.820070	.84
.85	.811962	813866	.815835	.817866	.819952	.822088	.85
.86	.814267	.816104	.818011	.819980	.822008	.824087	.86
87	.816550	.818322	.820166	.822075	.824044	.826067	.87
.88	.818812	.820519	.822301	.824150	.826061	.828028	.88
.89	.821053	.822696	.824415	.826205	.828059	.829971	.89
90	823273	.824852	.826510	.828240	.830037	.831894	.90
.91	.825473	.826988	.828585	.830256	.831997	.833799	.91
.92	.827651	.829103	.830640	.832253	.833937	.835686	.92
.93	.829809	.831198	.832675	.834231	.835859	.837554	.93
.94	.831946	.833274	.834690	.836189	.837762	.839404	.94
.95	.834063	.835329	.836686	.838128	.839647	.841236	.95
.96	.836159	.837364	.838663	.840049	.841513	.843050	.96
.97	.838234	.839379	.840620	.841950	.843361	.844846	.97
.98	.840290	.841374	.842558	.843833	.845191	.846625	.98
.99	.842325	843350	.844477	.845697	.847002	.848386	.99

t	SKEWNESS 0	.1	.2	.3	.4	.5	t
1.00	.841345	.841442	841721	842165	.842758	.843487	1.00
1.01	843752	843772	.843977	.844351	.844877	.845543	1.01
1.02	.846136	846078	.846210	846515	.846975	847578	1.02
1.03	.848495	.848361	.848421	848657	.849052	.849592	1.03
1.04	850830	850620	.850609	.850777	.851108	.851586	1.04
1.05	853141	.852857	.852775	.852876	853142	853560	1.05
1.06	.855428	.855070	.854918	854952	855156	.855514	1.06
1.07	.857690	.857259	.857039	.857008	.857149	.857447	1.07
1.08	.859929	859426	.859137	.859042	.859121	.859361	1.08
1.09	862143	.861570	.861213	.861054	.861073	.861254	1.09
1.10	864334	.863690	863268	.863045	863004	863128	1.10
1.11	.866500	865788	.865300	865015	.864915	.864982	1.11
1.12	.868643	.867862	.867310	866964	.866805	.866816	1.12
1.13	.870762	.869914	869298	868891	.868675	.868631	1.13
1.14	872857	.871943	.871264	.870798	870524	.870427	1.14
1.15	.874928	.873949	873208	872683	872354	.872203	1.15
1.16	.876976	875933	.875131	.874548	.874164	.873960	1.16
1.17	.879000	877894	877032	.876393	.875954	.875697	1.17
1.18	.881000	.879832	878912	.878216	.877724	.877416	1.18
1.19	.882977	.881748	.880771	880019	.879474	.879116	1.19
1.20	884930	.883642	.882608	.881802	.881205	.880797	1.20
1.21	.886861	.885514	.884423	.883565	.882916	.882460	1.21
1.22	.888768	.887363	886218	885307	884609	.884104	1.22
1.23	.890651	.889191	.887992	.887029	.886281	.885729	1.23
1.24	.892512	.890997	.889745	.888732	.887935	.887336	1.24
1.25	.894350	.892780	.891477	.890414	.889570	888925	1.25
1.26	.896165	.894543	.893188	.892077	.891186	.890497	1.26
1.27	.897958	.896283	.894879	.893720	.892783	.892050	1.27
1.28	.899727	.898002	.896550	.895344	.894362	.893585	1.28
1.29	.901475	.899700	898200	.896948	.895922	.895102	1.29
1.30	.903200	901376	899030	.898533	.897464	.896602	1.30
1.31	.904902	.903032	.901440	.900099	898988	.898085	1.31
1.32	.906582	904666	.903029	901646	.900493	.899550	1.32
1.33	.908241	.906280	.904600	.903174	901980	.900998	1.33
1.34	.909877	.907873	.906150	.904683	.903450	.902429	1.34
1.35	.911492	909445	.907681	.906174	.904902	.903843	1.35
1.36	913085	.910996	.909192	.907646	.906336	905241	1.36
1.37	.914657	.912528	.910684	909100	.907753	.906621	1.37
1.38	.916207	.914039	.912157	.910536	909152	.907985	1.38
1.39	.917736	.915530	913611	911954	.910534	.909333	1.39
1.40	.919243	.917001	.915046	913353	911899	.910664	1.40
1.41	.920730	.918452	.916463	.914735	913247	911979	1.41
1.42	.922196	919884	.917860	.916099	.914579	.913278	1.42
1.43	923641	921296	919239	.917446	915893	.914561	1.43
1.44	.925066	.922689	.920600	.918775	.917191	.915828	1.44
1.45	.926471	.924062	.921943	.920087	.918473	.917080	1.45
1.46	.927855	.925417	.923267	921382	.919738	918316	1.46
1.47	.929219	.926752	.924574	.922660	.920987	.919537	1.47
1.48	.930563	.928069	.925863	.923921	.922221	.920743	1.48
1.49	931888	.929367	.927134	.925165	.923438	.921933	1.49

t	SKEWNESS						t
	6	7	8	9	1.0	1 1	
1.00	844340	845306	.846377	847543	848796	.850129	1 00
1 01	.846335	847243	848258	849370	850572	851855	1 01
1 02	848310	.849161	.850120	851179	852330	853564	1 02
1 03	850265	851059	851964	852970	854070	855256	1 03
1.04	.852200	852938	.853789	.854743	.855793	856931	1.04
1 05	.854116	.854798	.855595	.856498	.857499	858589	1.05
1 06	.856012	856639	.857383	.858236	859187	860230	1 06
1 07	.857889	.858461	859153	859955	860859	861854	1.07
1.08	.859746	860264	.860905	.861657	.862513	863463	1.08
1.09	861584	.862049	.862639	863342	.864150	865054	1 09
1.10	863403	863815	.864354	.865009	.865771	.866630	1.10
1.11	865203	865563	.866052	.866659	.867375	868189	1.11
1.12	.866984	.867293	.867733	868293	.868962	869733	1 12
1.13	.868746	.869004	.869396	869909	870533	871260	1 13
1.14	.870489	.870698	.871041	871508	.872088	.872772	1 14
1.15	872214	.872373	872669	.873090	873626	.874268	1.15
1.16	873920	874031	.874280	.874656	.875149	875749	1.16
1.17	.875608	875671	.875874	876206	.876656	877214	1 17
1 18	.877277	877293	877451	877739	878146	878664	1 18
1 19	878929	.878898	879011	.879256	879622	.880099	1 19
1 20	.880562	880486	880554	880756	881081	881519	1 20
1.21	.882178	882056	.882081	882241	882525	.882924	1 21
1 22	883775	883609	.883591	883710	883954	.884314	1 22
1 23	885355	.885145	.885085	885163	885368	885690	1 23
1 24	886918	.886665	.886563	886601	886767	.887051	1.24
1.25	.888463	.888167	888025	.888023	888150	888397	1.25
1.26	889991	889653	889470	889429	889519	.889730	1 26
1.27	.891501	891123	890900	890821	890873	891048	1 27
1.28	.892995	892576	.892314	892197	892213	.892352	1 28
1.29	894471	894012	893712	893558	893538	.893642	1 29
1 30	.895931	895433	895095	894904	.894849	894919	1 30
1.31	.897374	.896838	.896462	896236	.896146	896181	1 31
1.32	.898800	.898226	.897815	897553	897428	897430	1.32
1.33	.900210	899599	899152	898855	898697	898666	1 33
1 34	901603	.900956	900474	.900143	.899951	.899888	1.34
1 35	902981	.902298	901781	901416	901192	901098	1 35
1 36	904342	.903625	903073	902676	.902419	902294	1.36
1 37	905687	.904936	904351	903921	903633	903477	1 37
1 38	907017	906232	.905614	905152	.904833	.904647	1 38
1.39	908331	907512	906863	906370	906020	.905804	1 39
1.40	909629	.908778	908098	907574	.907194	906949	1 40
1 41	.910912	910030	909318	908764	.908355	.908081	1 41
1.42	.912179	.911266	.910524	.909941	.909503	909201	1.42
1 43	913431	912488	911717	.911104	.910638	910308	1.43
1 44	914669	913696	912895	.912254	911761	.911403	1.44
1.45	.915891	.914889	914060	.913391	.912870	912486	1.45
1.46	.917098	.916068	915212	914516	913968	.913557	1.46
1.47	.918291	.917233	.916350	915627	.915053	.914617	1 47
1 48	.919469	.918385	917474	.916725	.916125	915664	1 48
1.49	920633	.919522	.918586	.917811	.917186	916700	1.49

t	SKEWNESS						t
	0	.1	.2	.3	.4	.5	
1.50	.933193	.930647	.928368	.926393	924639	.923108	1.50
1.51	.934478	.931908	.929625	.927604	.925825	.924269	1.51
1.52	.935745	.933151	.930844	.928799	.926996	.925415	1.52
1.53	.936992	.934376	.932046	929978	.928151	.926546	1.53
1.54	.938220	.935584	.933232	.931141	.929291	.927663	1.54
1.55	.939429	936773	.934400	.932288	.930416	.928765	1.55
1.56	.940620	.937945	.935552	.933419	.931526	.929853	1.56
1.57	.941792	.939100	.936688	.934535	.932621	.930927	1.57
1.58	.942947	.940238	.937808	.935635	.933701	.931987	1.58
1.59	.944083	.941358	.938911	.936720	.934767	.933034	1.59
1.60	.945201	942462	.939998	937790	935819	.934066	1.60
1.61	.946301	.943548	941070	.938845	.936856	.935086	1.61
1.62	.947384	.944619	.942126	.939885	.937879	.936091	1.62
1.63	.948449	.945673	.943166	.940910	.938888	.937084	1.63
1.64	.949497	.946710	.944191	.941921	.939884	.938063	1.64
1.65	.950529	947732	.945201	.942918	.940866	.939029	1.65
1.66	.951543	948737	.946195	.943900	.941834	.939982	1.66
1.67	.952540	.949727	.947175	.944868	.942789	.940923	1.67
1.68	.953521	.950701	948140	.945822	.943730	.941851	1.68
1.69	954486	.951660	.949191	.946762	944658	.942766	1.69
1.70	.955435	.952604	.950027	.947688	.945574	.943669	1.70
1.71	956367	953532	.950948	948601	946476	.944559	1.71
1.72	.957284	.954446	.951856	949500	.947366	.945438	1.72
1.73	.958185	955345	.952750	.950387	.948243	.946304	1.73
1.74	.959070	.956229	.953629	.951260	.949107	.947159	1.74
1 75	.959941	957098	.954495	.952120	.949959	.948002	1.75
1.76	.960796	.957954	.955348	.952967	950799	.948833	1.76
1.77	.961636	.958795	.956187	.953802	.951627	.949652	1.77
1.78	.962462	.959623	.957013	.954624	.952443	.950461	1.78
1.79	.963273	.960436	.957826	.955433	.953247	.951258	1.79
1 80	.964070	.961236	.958626	.956230	.954040	.952044	1.80
1.81	.964852	.962023	.959413	.957015	.954820	.952818	1.81
1.82	.965621	.962796	.960188	.957788	.955590	953582	1.82
1.83	.966375	.963556	.960950	.958550	.956348	.954335	1.83
1.84	.967116	.964303	.961699	.959299	.957095	.955078	1.84
1.85	.967843	.965037	.962437	.960037	.957831	.955810	1.85
1.86	.968557	.965759	.963162	.960763	.958555	.956531	1.86
1.87	.969258	.966468	.963876	.961479	.959270	.957242	1.87
1.88	.969946	.967165	.964578	.962182	.959973	.957943	1.88
1.89	.970621	.967849	.965268	.962875	.960666	.958634	1.89
1.90	971283	.968522	.965947	.963557	.961348	.959315	1.90
1.91	.971933	.969183	966615	.964228	.962020	.959986	1.91
1.92	.972571	.969832	.967271	.964889	.962682	.960647	1.92
1.93	.973197	.970469	.967916	.965539	.963334	.961299	1.93
1.94	.973810	.971095	.968551	.966178	.963976	.961941	1.94
1.95	.974412	.971710	.969174	.966808	.964608	.962573	1.95
1.96	.975002	.972313	.969787	.967427	.965231	.963197	1.96
1.97	.975581	.972906	.970390	.968036	.965844	.963811	1.97
1.98	.976148	.973488	.970982	.968635	.966447	.964416	1.98
1.99	.976705	.974059	.971564	969224	.967041	.965013	1.99

t	SKEWNESS .6	.7	.8	9	1.0	1.1	t
1.50	.921782	.920646	.919684	.918884	.918235	.917724	1.50
1.51	.922918	.921756	.920769	919945	.919271	.918737	1.51
1.52	.924039	.922853	.921842	.920994	.920296	919738	1.52
1.53	.925146	923936	922902	.922030	.921309	.920729	1.53
1.54	.926240	.925006	.923949	.923054	.922311	921708	1.54
1.55	.927319	.926064	.924983	924066	.923301	.922676	1.55
1.56	.928386	.927108	.926006	.925067	.924280	.923633	1.56
1.57	.929439	.928139	.927016	.926056	.925247	.924579	1.57
1.58	.930478	.929158	.928014	.927033	.926203	.925515	1.58
1.59	.931504	.930164	.929000	.927998	.927149	.926440	1.59
1.60	.932518	.931158	.929973	.928952	.928083	.927355	1.60
1.61	.933518	.932139	.930936	.929895	.929006	.928259	1.61
1.62	.934506	.933108	.931886	.930827	.929919	.929153	1.62
1 63	.935481	.934065	.932825	.931748	.930821	.930037	1.63
1.64	.936443	.935010	.933752	.932657	931713	.930910	1.64
1.65	.937393	.935943	.934668	.933555	932594	.931774	1.65
1 66	.938330	936865	.935573	.934443	.933465	.932627	1 66
1 67	.939256	.937775	.936467	.935321	.934325	.933471	1.67
1.68	.940169	.938673	.937349	.936187	.935176	.934306	1.68
1.69	.941071	.939560	.938221	937043	.936016	.935130	1.69
1.70	.941960	.940435	.939082	.937889	.936847	.935945	1.70
1.71	.942838	.941300	.939932	.938725	.937668	.936751	1.71
1.72	.943704	.942153	940772	.939551	.938479	.937547	1.72
1.73	.944559	.942995	.941601	.940366	939280	.938334	1.73
1.74	.945403	.943827	.942420	.941172	940072	.939112	1.74
1.75	.946235	.944647	.943228	.941967	.940855	.939881	1.75
1.76	.947056	.945457	.944027	.942753	.941628	.940641	1.76
1.77	.947866	.946257	.944815	.943530	.942392	.941392	1.77
1.78	.948665	.947046	.945593	.944296	943146	.942134	1.78
1.79	.949454	.947825	.946362	.945054	.943892	.942868	1.79
1.80	.950232	.948594	.947121	.945802	.944629	.943593	1.80
1.81	.950999	.949353	.947870	.946541	.945357	.944309	1.81
1.82	.951756	.950101	.948609	.947270	.946076	.945017	1.82
1 83	952502	.950840	.949339	.947991	.946786	.945717	1.83
1.84	.953239	.951569	.950060	.948702	.947488	.946408	1.84
1.85	.953965	.952289	.950772	.949405	.948181	.947092	1.85
1 86	.954681	.952999	.951474	.950099	.948866	.947767	1.86
1.87	.955388	.953699	.952167	.950785	.949543	.948434	1.87
1.88	.956085	.954390	.952852	.951461	.950211	.949093	1.88
1.89	.956772	.955072	.953527	.952130	.950871	.949745	1.89
1.90	.957449	.955745	.954191	.952790	.951523	.950389	1.90
1.91	.958117	.956409	.954852	.953441	.952168	.951025	1.91
1.92	.958776	.957064	.955502	.954085	.952804	.951653	1.92
1.93	.959426	.957710	.956143	.954720	.953432	.952274	1.93
1 94	.960066	.958347	.956776	.955347	.954053	.952888	1.94
1.95	.960698	.958975	.957400	.955966	.954666	.953494	1.95
1.96	.961320	.959595	.958017	.956578	.955272	.954093	1.96
1.97	.961934	.960207	.958645	.957181	.955870	.954684	1.97
1.98	.962539	.960810	.959225	.957777	.956460	.955269	1.98
1.99	.963135	.961405	.959817	.958365	.957044	.955847	1.99

t	SKEWNESS						t
	0	.1	.2	3	.4	.5	
2.00	.977250	.974620	972136	.969804	.967626	965600	2.00
2.01	977784	975170	.972698	970375	.968202	966179	2.01
2.02	978308	.975711	973251	.970936	.968768	.966749	2.02
2.03	.978822	976241	.973793	.971487	969326	.967310	2.03
2.04	.979325	976761	.974326	.972030	969875	.967863	2.04
2.05	.979818	977271	974850	972563	.970415	968408	2.05
2.06	980301	977772	.975365	.973088	.970947	.968944	2.06
2.07	.980774	.978264	975870	.973604	971471	.969473	2.07
2.08	.981237	978745	.976366	.974111	.971986	.969993	2.08
2.09	.981691	.979218	.976854	.974609	972492	970505	2.09
2.10	.982136	979682	.977332	.975100	.972991	.971010	2.10
2.11	.982571	.980137	977802	.975581	973482	.971507	2.11
2.12	.982997	.980583	978264	976055	973964	971996	2.12
2.13	983414	.981020	.978717	976521	974439	.972478	2.13
2.14	983823	981449	.979162	976978	.974907	.972952	2.14
2.15	.984222	981869	.979591	977428	.975366	.973419	2.15
2.16	984614	982281	980028	977870	975818	.973879	2.16
2.17	.984997	.982685	.980449	978304	.976263	.974332	2.17
2.18	985371	.983081	980862	978731	976701	974777	2.18
2.19	.985738	983469	981267	.979151	977131	.975216	2.19
2.20	.986097	.983849	.981665	979563	977554	.975648	2.20
2.21	.986447	984222	982056	979967	977970	.976073	2.21
2.22	.986791	.984587	982439	980365	.978380	.976491	2.22
2.23	.987126	.984945	982815	.980756	978782	976903	2.23
2.24	987455	985296	.983184	.981140	979178	.977308	2.24
2.25	.987776	985639	983546	981517	.979567	.977707	2.25
2.26	988089	.985975	983901	.981887	979950	978099	2.26
2.27	.988396	.986305	.984249	982251	.980326	.978486	2.27
2.28	.988696	986627	984590	982608	980696	978866	2.28
2.29	.988989	986943	984925	.982959	981060	979240	2.29
2.30	989276	.987253	.985254	.983303	981417	979608	2.30
2.31	989556	987556	985576	983641	981768	.979970	2.31
2.32	989830	.987852	985892	983973	.982114	980326	2.32
2.33	.990097	.988143	.986202	.984299	982453	980677	2.33
2.34	990358	.988427	986506	984620	982787	981022	2.34
2.35	990613	.988705	986804	984934	983115	981361	2.35
2.36	990863	.988977	.987096	.985242	983438	.981695	2.36
2.37	.991106	989244	987382	985545	.983754	.982023	2.37
2.38	.991344	.989505	.987662	.985843	.984066	.982346	2.38
2.39	.991576	989760	.987937	.986134	.984372	982664	2.39
2.40	991802	990010	.988207	.986421	.984672	982976	2.40
2.41	.992024	990254	988471	.986702	.984968	983283	2.41
2.42	992240	.990493	.988730	986978	.985258	983586	2.42
2.43	.992451	990727	.988984	.987249	.985543	.983883	2.43
2.44	.992656	990955	989232	.987514	.985823	.984175	2.44
2.45	992857	.991179	989476	.987775	.986098	984463	2.45
2.46	.993053	991398	989714	.988031	.986369	.984746	2.46
2.47	.993244	991612	.989948	988281	.986534	.985024	2.47
2.48	993431	.991821	.990177	.988528	.986895	.985297	2.48
2.49	.993613	.992025	990401	988769	.987152	.985566	2.49

	SKEWNESS						
t	.6	7	.8	.9	1 0	1.1	t
2.00	.963723	961992	.960402	.958946	.957620	.956417	2.00
2 01	.964303	.962571	960978	959519	.958189	956981	2.01
2 02	.964874	.963142	961547	960085	958751	.957538	2 02
2 03	.965437	963705	.962109	960644	959306	958088	2.03
2 04	965992	964260	962663	961196	.959854	958632	2 04
2 05	966539	.964808	.963209	.961740	960395	959169	2 05
2 06	967079	965348	.963749	.962277	.960929	959699	2.06
2.07	967610	.965880	.964281	.962808	.961457	.960223	2.07
2.08	.968133	.966405	964806	.963331	.961978	.960740	2 08
2 09	968649	.966923	.965324	.963848	.962492	.961252	2.09
2 10	.969158	.967433	.965834	.964358	.963000	.961757	2.10
2 11	.969659	.967937	.966338	.964861	.963502	962256	2 11
2 12	970153	.968433	.966836	.965358	.963997	.962748	2 12
2 13	970639	968922	.967326	965848	.964486	963235	2 13
2.14	.971118	969404	.967810	966332	.964969	.963716	2.14
2.15	.971590	.969880	.968287	.966810	965446	964191	2 15
2.16	.972056	.970349	968758	967281	.965916	.964660	2 16
2 17	972514	.970811	.969222	967746	.966381	965123	2.17
2 18	.972965	.971266	969680	968205	.966840	965580	2 18
2.19	.973410	.971715	.970132	.968658	.967293	.966032	2 19
2.20	.973848	972158	.970577	.969105	967740	966479	2.20
2 21	.974280	.972594	971017	.969546	.968181	966919	2.21
2 22	.974705	.973024	.971450	.969981	.968617	.967355	2 22
2.23	.975124	.973448	.971877	.970411	969047	.967785	2.23
2 24	.975536	973866	.972299	.970834	.969472	.968209	2.24
2 25	.975943	.974278	.972714	.971252	.969891	.968628	2.25
2 26	976343	.974684	.973124	971665	.970305	969043	2.26
2 27	.976737	.975084	973528	.972072	.970713	.969451	2 27
2 28	.977125	.975478	.973927	.972473	.971116	.969855	2 28
2.29	.977507	.975866	.974320	972869	.971515	.970254	2.29
2.30	.977883	.976249	.974708	973260	.971907	.970648	2.30
2.31	.978254	.976626	.975090	.973646	.972295	971037	2.31
2.32	.978619	976998	.975466	974026	.972678	.971421	2.32
2.33	.978978	.977364	.975838	.974402	.973056	.971800	2.33
2.34	.979332	.977725	.976204	974772	.973429	.972175	2.34
2 35	.979681	.978081	.976565	.975137	.973797	.972544	2.35
2.36	.980024	.978431	.976921	.975497	.974160	.972909	2 36
2.37	.980361	.978776	.977273	.975853	.974519	.973270	2.37
2.38	.980694	.979116	.977619	976203	.974872	.973626	2.38
2.39	.981021	.979451	.977960	.976549	.975222	973977	3 39
2.40	.981343	.979782	.978296	.976891	.975566	974324	2.40
2.41	.981660	.980107	.978628	.977227	.975907	.974667	2.41
2.42	.981973	.980427	.978955	977559	.976242	.975005	2.42
2.43	.982280	.980743	.979277	.977886	.976574	.975339	2.43
2.44	.982583	.981054	.979594	.978209	.976900	.975669	2.44
2.45	.982880	.981360	.979908	.978528	.977223	.975995	2.45
2.46	.983174	981662	.980216	.978842	.977541	.976316	2.46
2.47	.983462	.981959	.980520	.979152	.977856	.976634	2.47
2.48	.983746	.982251	.980820	.979457	978166	976947	2 48
2.49	.984026	.982540	.981116	.979759	.978472	.977257	2.49

t	SKEWNESS						t
	0	1	2	3	4	5	
2 50	993790	.992225	.990621	989006	.987403	985830	2 50
2.51	.993963	.992421	990836	989238	.987650	986090	2 51
2.52	994132	.992612	991047	.989466	987893	.986346	2.52
2 53	.994297	992799	991253	.989690	988132	.986597	2.53
2.54	.994457	.992981	.991455	989909	988366	986844	2.54
2.55	.994614	.993159	991653	.990124	988596	987087	2.55
2 56	.994766	993334	.991847	.990335	.988822	.987326	2 56
2 57	.994915	.993504	.992036	990542	989043	987560	2 57
2 58	995060	993671	.992222	990744	989261	.987791	2 58
2.59	995201	.993833	.992404	.990943	989475	988018	2.59
2.60	.995339	993992	.992582	991138	.989685	988240	2.60
2 61	995473	994147	.992756	991329	989891	.988459	2 61
2.62	.995604	.994299	992926	.991516	.990093	988675	2 62
2.63	995731	994447	993093	.991700	.990291	988886	2 63
2 64	995855	.994591	.993256	991880	.990486	989094	2.64
2 65	995975	.994732	.993416	.992057	.990677	989299	2.65
2.66	.996093	994870	993571	992230	.990865	.989499	2.66
2.67	.996207	.995005	993725	.992399	991050	.989697	2.67
2.68	.996319	995136	993875	.992565	.991230	.989891	2 68
2.69	.996427	.995264	.994021	.992728	.991408	990081	2 69
2.70	.996533	.995389	.994165	.992888	.991582	.990268	2 70
2 71	996636	995512	.994305	.993044	991753	990452	2.71
2.72	.996736	.995631	.994442	.993197	.991921	990633	2 72
2.73	.996833	.995747	.994576	.993347	.992086	990811	2.73
2.74	.996928	.995861	.994707	.993494	.992247	.990985	2 74
2 75	997020	995972	.994835	.993638	.992406	.991156	2.75
2.76	.997110	.996080	.994960	993780	.992561	.991325	2.76
2 77	.997197	.996185	995083	.993918	.992714	.991490	2.77
2.78	997282	.996288	.995203	994053	.992863	.991653	2.78
2.79	997365	996388	.995320	994186	.993010	.991812	2.79
2.80	.997445	.996486	995435	.994316	.993154	.991969	2.80
2.81	.997523	.996582	.995547	.994443	.993295	.992123	2.81
2.82	.997599	.996675	.995656	.994568	.993434	.992274	2.82
2.83	.997673	996766	.995763	.994690	993570	992422	2.83
2 84	.997744	996854	.995868	.994810	.993703	992568	2.84
2.85	997814	.996941	.995970	994927	.993834	.992712	2.85
2.86	.997882	997025	.996070	.995041	.993962	.992852	2 86
2.87	.997948	997107	996167	995154	.994088	992990	2.87
2 88	.998012	997187	.996263	.995264	.994212	.993126	2.88
2.89	.998074	.997265	996356	.995371	994333	.993259	2 89
2.90	.998134	.997341	996447	.995477	994451	.993390	2.90
2.91	998193	.997415	.996536	.995580	.994568	.993519	2.91
2 92	.998250	.997487	.996623	.995681	.994682	.993645	2.92
2.93	.998305	997557	.996708	.995780	994794	.993768	2 93
2.94	.998359	.997626	.996791	995876	.994903	.993890	2.94
2.95	.998411	.997693	.996872	.995971	.995011	.994009	2.95
2.96	998462	.997758	.996951	.996064	.995116	.994127	2.96
2.97	.998511	.997821	.997028	996154	.995219	.994242	2.97
2.98	.998559	.997883	.997104	.996243	.995321	.994355	2.98
2.99	.998605	.997943	.997178	.996330	995420	.994465	2.99

t	SKEWNESS						t
	6	.7	.8	.9	1.0	1.1	
2.50	.984301	.982824	.981407	.980056	.978774	.977562	2.50
2.51	.984571	.983103	.981694	.980349	.979071	.977864	2.51
2.52	.984838	.983379	.981977	.980638	.979365	.978161	2.52
2.53	985100	.983650	.982256	980923	.979655	978455	2.53
2.54	.985358	.983917	.982531	.981204	.979942	.978745	2.54
2 55	.985611	.984180	.982802	.981482	.980224	.979032	2.55
2 56	.985861	.984440	.983069	.981755	.980503	.979314	2.56
2.57	.986107	.984695	.983332	.982025	.980777	.979593	2.57
2 58	.986349	.984946	.983591	.982290	.981049	.979869	2.58
2.59	.986587	.985193	.983847	.982553	.981316	.980141	2.59
2.60	.986821	.985437	.984098	.982811	.981580	.980409	2 60
2 61	.987051	.985677	.984346	.983066	.981841	.980674	2.61
2.62	.987278	.985913	.984591	.983317	.982098	.980936	2.62
2 63	.987500	.986146	.984832	.983565	.982351	.981194	2.63
2.64	.987720	986375	.985069	.983810	.982602	.981449	2.64
2.65	987935	.986600	.985303	.984050	.982848	.981701	2.65
2.66	.988148	.986822	.985533	.984288	.983092	.981949	2 66
2.67	988356	.987041	.985760	.984522	.983332	.982194	2.67
2.68	.988562	.987256	.985984	.984753	.983569	982436	2.68
2.69	.988764	.987468	.986204	.984981	.983803	.982675	2.69
2 70	.988962	.987676	.986422	.985205	984034	.982911	2.70
2 71	.989157	987882	.986635	.985426	.984261	.983143	2.71
2 72	.989350	988084	.986846	.985645	.984485	.983373	2.72
2.73	.989539	988283	.987054	.985860	.984707	.983600	2.73
2.74	.989724	.988479	.987258	986072	.984925	.983823	2.74
2.75	.989907	.988671	.987460	.986281	.985140	.984044	2 75
2.76	.990087	.988861	987658	.986487	.985353	.984262	2 76
2.77	.990264	.989048	.987854	.986690	.985562	.984477	2.77
2.78	.990438	.989232	.988046	.986890	.985769	.984689	2.78
2.79	.990608	.989413	.988236	.987087	.985973	.984899	2.79
2.80	.990777	.989591	.988423	987282	.986174	.985105	2.80
2.81	990942	.989766	.988607	.987474	.986373	.985309	2.81
2.82	991104	.989939	.988789	987663	.986568	.985511	2.82
2 83	991264	.990109	.988967	.987849	.986761	.985709	2.83
2 84	.991421	.990276	.989143	.988033	.986951	.985906	2.84
2 85	.991576	.990440	.989316	988214	.987139	.986099	2.85
2 86	.991727	.990602	.989487	.988392	.987324	.986290	2.86
2.87	.991877	.990761	.989655	.988568	.987507	.986478	2.87
2 88	.992024	.990918	.989821	.988741	.987687	.986664	2.88
2.89	.992168	.991072	989984	.988912	.987865	.986848	2.89
2.90	.992310	.991224	.990144	.989080	.988040	.987029	2.90
2.91	.992449	.991373	.990302	.989246	.988213	.987208	2.91
2.92	.992586	.991520	.990458	.989410	.988383	.987384	2.92
2.93	.992721	.991665	.990612	.989571	.988551	.987558	2.93
2.94	.992853	.991807	.990763	.989730	.988717	.987730	2.94
2.95	.992984	.991947	.990911	.989886	.988880	.987899	2.95
2.96	.993112	.992085	.991058	.990041	.989041	.988066	2.96
2.97	.993237	.992220	.991202	.990193	.989200	.988231	2.97
2.98	.993361	.992353	.991344	.990343	.989357	.988394	2.98
2.99	.993482	.992485	.991484	.990490	.989512	.988555	2.99

t	SKEWNESS 0	.1	.2	3	.4	.5	t
3 00	.998650	998001	.997250	.996415	.995517	994574	3.00
3.01	.998694	998058	.997320	996498	.995613	.994681	3 01
3 02	.998736	998114	.997389	996580	.995706	994786	3.02
3.03	.998777	998168	997456	.996659	.995798	.994889	3.03
3 04	998817	.998220	.997521	.996737	995888	994990	3.04
3.05	998856	.998272	.997585	996813	995976	.995089	3 05
3.06	.998893	998322	.997647	996888	996062	.995186	3.06
3 07	998930	.998370	.997708	.996961	.996147	.995282	3.07
3 08	.998965	.998417	.997767	.997032	.996229	995376	3.08
3 09	998999	.998463	.997825	.997102	996311	.995467	3.09
3 10	999032	998508	997882	.997170	996390	995558	3.10
3.11	.999065	998552	997937	.997237	996468	995646	3.11
3 12	999096	998594	997991	997302	.996545	.995733	3.12
3 13	999126	998635	998044	.997366	996619	.995819	3.13
3 14	999155	998675	998095	997429	996693	.995902	3.14
3 15	999184	998714	.998145	997490	.996765	.995984	3 15
3.16	.999211	998752	998194	.997550	.996835	.996065	3.16
3 17	.999238	998789	998241	997608	.996904	.996144	3.17
3.18	.999264	998825	998288	997665	.996971	.996221	3.18
3.19	999289	.998860	.998333	.997721	.997038	.996298	3.19
3.20	999313	.998894	998377	997775	.997102	.996372	3.20
3 21	999336	998927	998420	997829	.997166	.996445	3.21
3.22	.999359	998959	.998462	.997881	.997228	.996517	3.22
3 23	999381	998990	998503	997932	.997289	996588	3.23
3 24	999402	999020	998543	.997982	.997348	.996657	3.24
3 25	.999423	.999050	.998582	.998031	997407	996725	3.25
3.26	.999443	999079	998620	998078	.997464	.996791	3.26
3 27	999462	999106	.998657	998125	.997520	.996856	3.27
3 28	.999481	999134	.998694	.998170	.997575	996920	3 28
3.29	.999499	.999160	.998729	998215	.997629	.996983	3.29
3.30	999517	.999185	.998763	998258	.997681	997045	3 30
3.31	999534	.999210	998797	.998300	.997733	.997105	3.31
3 32	.999550	.999234	.998829	.998342	.997783	997164	3 32
3.33	999566	999258	.998861	.998382	997832	997222	3.33
3 34	999581	.999281	.998892	.998422	997881	.997279	3.34
3.35	.999596	999303	.998922	998461	.997928	.997335	3.35
3.36	.999610	.999324	.998952	.998499	997974	997390	3 36
3 37	.999624	.999345	998980	.998535	.998020	.997443	3.37
3.38	.999638	.999365	.999008	998572	.998064	.997496	3.38
3.39	.999651	.999385	.999035	.998607	998107	997547	3 39
3 40	999663	999404	999062	998641	.998150	997598	3.40
3.41	.999675	999423	.999088	.998675	.998192	997648	3.41
3.42	999687	.999441	.999113	.998708	998232	.997696	3.42
3 43	.999698	.999458	.999138	.998740	.998272	.997744	3.43
3 44	.999709	999475	.999161	.998771	.998311	.997791	3.44
3.45	.999720	999492	999185	.998802	.998349	.997837	3.45
3 46	.999730	.999508	.999207	998831	.998387	.997881	3.46
3 47	999740	999523	.999229	998861	.998423	.997926	3.47
3.48	.999749	999538	999251	.998889	998459	.997969	3.48
3 49	.999759	.999553	999272	998917	998494	.998011	3.49

	SKEWNESS						
t	.6	.7	.8	.9	1.0	1.1	t
3.00	.993602	.992614	.991622	.990636	.989664	.988713	3.00
3.01	.993719	.992741	.991757	.990779	.989814	.988869	3.01
3.02	.993834	.992865	.991891	.990920	.989962	.989024	3.02
3.03	.993948	.992988	.992022	.991060	.990109	.989176	3.03
3.04	.994059	.993109	.992152	.991197	.990253	.989326	3.04
3.05	.994169	.993228	.992279	.991332	.990395	.989474	3.05
3.06	.994276	.993345	.992405	.991465	.990535	.989621	3.06
3.07	.994382	.993460	.992528	.991596	.990673	.989765	3.07
3.08	.994486	.993573	.992650	.991726	.990809	.989907	3.08
3.09	.994588	.993684	.992770	.991853	.990943	.990048	3.09
3.10	.994688	.993794	.992888	.991979	.991076	.990186	3.10
3.11	.994786	.993902	.993004	.992102	.991206	.990323	3.11
3.12	.994883	.994008	.993118	.992224	.991335	.990458	3.12
3.13	.994978	.994112	.993230	.992344	.991462	.990591	3.13
3.14	.995072	.994214	.993341	.992463	.991587	.990722	3.14
3.15	.995163	.994315	.993450	.992579	.991711	.990851	3.15
3.16	.995254	.994414	.993558	.992694	.991832	.990979	3.16
3.17	.995342	.994511	.993663	.992807	.991952	.991105	3.17
3.18	.995429	.994607	.993767	.992919	.992070	.991229	3.18
3.19	.995515	.994702	.993870	.993029	.992187	.991352	3.19
3.20	.995599	.994794	.993970	.993137	.992302	.991473	3.20
3.21	.995681	.994885	.994070	.993243	.992415	.991592	3.21
3.22	.995762	.994975	.994167	.993348	.992527	.991710	3.22
3.23	.995842	.995063	.994263	.993452	.992637	.991826	3.23
3.24	.995920	.995150	.994358	.993554	.992746	.991941	3.24
3.25	.995997	.995235	.994451	.993654	.992853	.992054	3.25
3.26	.996072	.995319	.994543	.993753	.992958	.992166	3.26
3.27	.996146	.995401	.994633	.993850	.993062	.992276	3.27
3.28	.996219	.995482	.994722	.993946	.993165	.992384	3.28
3.29	.996290	.995562	.994809	.994041	.993266	.992491	3.29
3.30	.996360	.995640	.994895	.994134	.993365	.992597	3.30
3.31	.996429	.995717	.994980	.994226	.993464	.992701	3.31
3.32	.996497	.995793	.995063	.994316	.993560	.992804	3.32
3.33	.996563	.995868	.995145	.994405	.993656	.992905	3.33
3.34	.996629	.995941	.995226	.994492	.993750	.993005	3.34
3.35	.996693	.996013	.995305	.994579	.993843	.993103	3.35
3.36	.996756	.996083	.995383	.994664	.993934	.993201	3.36
3.37	.996817	.996153	.995460	.994748	.994024	.993297	3.37
3.38	.996878	.996221	.995536	.994830	.994113	.993391	3.38
3.39	.996938	.996289	.995610	.994911	.994200	.993485	3.39
3.40	.996996	.996355	.995683	.994991	.994287	.993577	3.40
3.41	.997054	.996420	.995755	.995070	.994372	.993667	3.41
3.42	.997110	.996483	.995826	.995148	.994456	.993757	3.42
3.43	.997165	.996546	.995896	.995224	.994538	.993845	3.43
3.44	.997220	.996608	.995965	.995299	.994620	.993932	3.44
3.45	.997273	.996669	.996032	.995374	.994700	.994018	3.45
3.46	.997325	.996728	.996099	.995447	.994779	.994103	3.46
3.47	.997377	.996787	.996164	.995519	.994857	.994187	3.47
3.48	.997427	.996844	.996229	.995589	.994934	.994269	3.48
3.49	.997477	.996901	.996292	.995659	.995009	.994350	3.49

	SKEWNESS						
t	0	.1	.2	.3	.4	.5	t
3.50	.999767	.999567	.999292	.998944	.998528	.998053	3.50
3.51	.999776	.999581	.999312	.998971	.998562	.998093	3.51
3.52	.999784	.999594	.999331	.998996	.998595	.998133	3.52
3.53	.999792	.999607	.999350	.999022	.998627	.998172	3.53
3.54	.999800	.999620	.999368	.999046	.998658	.998210	3.54
3.55	.999807	.999632	.999386	.999070	.998689	.998248	3.55
3.56	.999815	.999644	.999404	.999094	.998719	.998285	3.56
3.57	.999822	.999655	.999420	.999117	.998748	.998321	3.57
3.58	.999828	.999666	.999437	.999139	.998777	.998356	3.58
3.59	.999835	.999677	.999453	.999161	.998805	.998391	3.59
3.60	.999841	.999688	.999468	.999183	.998833	.998425	3.60
3.61	.999847	.999698	.999484	.999203	.998860	.998458	3.61
3.62	.999853	.999708	.999498	.999224	.998886	.998490	3.62
3.63	.999858	.999717	.999513	.999243	.998912	.998522	3.63
3.64	.999864	.999726	.999527	.999263	.998937	.998554	3.64
3.65	.999869	.999735	.999540	.999282	.998962	.998584	3.65
3.66	.999874	.999744	.999553	.999300	.998986	.998614	3.66
3.67	.999879	.999752	.999566	.999318	.999009	.998644	3.67
3.68	.999883	.999760	.999579	.999336	.999032	.998673	3.68
3.69	.999888	.999768	.999591	.999353	.999055	.998701	3.69
3.70	.999892	.999776	.999603	.999369	.999077	.998728	3.70
3.71	.999896	.999783	.999614	.999386	.999098	.998756	3.71
3.72	.999900	.999791	.999626	.999402	.999119	.998782	3.72
3.73	.999904	.999798	.999636	.999417	.999140	.998808	3.73
3.74	.999908	.999804	.999647	.999432	.999160	.998834	3.74
3.75	.999912	.999811	.999657	.999447	.999180	.998859	3.75
3.76	.999915	.999817	.999667	.999461	.999199	.998883	3.76
3.77	.999918	.999823	.999677	.999475	.999218	.998907	3.77
3.78	.999922	.999829	.999686	.999489	.999236	.998930	3.78
3.79	.999925	.999835	.999696	.999502	.999254	.998953	3.79
3.80	.999928	.999840	.999705	.999515	.999272	.998976	3.80
3.81	.999931	.999846	.999713	.999528	.999289	.998998	3.81
3.82	.999933	.999851	.999722	.999540	.999306	.999020	3.82
3.83	.999936	.999856	.999730	.999553	.999322	.999041	3.83
3.84	.999938	.999861	.999738	.999564	.999338	.999061	3.84
3.85	.999941	.999866	.999746	.999576	.999354	.999082	3.85
3.86	.999943	.999870	.999753	.999587	.999369	.999102	3.86
3.87	.999946	.999875	.999761	.999598	.999384	.999121	3.87
3.88	.999948	.999879	.999768	.999608	.999399	.999140	3.88
3.89	.999950	.999883	.999775	.999619	.999413	.999159	3.89
3.90	.999952	.999887	.999781	.999629	.999427	.999177	3.90
3.91	.999954	.999891	.999788	.999639	.999441	.999195	3.91
3.92	.999956	.999895	.999794	.999648	.999454	.999212	3.92
3.93	.999958	.999898	.999800	.999658	.999467	.999230	3.93
3.94	.999959	.999902	.999806	.999667	.999480	.999246	3.94
3.95	.999961	.999905	.999812	.999676	.999493	.999263	3.95
3.96	.999963	.999909	.999818	.999684	.999505	.999279	3.96
3.97	.999964	.999912	.999823	.999693	.999517	.999295	3.97
3.98	.999966	.999915	.999829	.999701	.999528	.999310	3.98
3.99	.999967	.999918	.999834	.999709	.999540	.999325	3.99

	SKEWNESS						
t	6	.7	.8	.9	1.0	1.1	t
3.50	997526	.996957	.996354	.995728	.995084	.994431	3.50
3.51	.997573	.997011	.996416	.995795	.995128	.994510	3.51
3.52	.997620	997065	.996476	.995862	.995230	.994588	3 52
3.53	.997666	.997118	.996536	.995928	.995302	.994665	3.53
3.54	.997711	997170	.996594	.995992	.995372	.994740	3.54
3.55	.997756	.997221	996651	.996056	.995441	.994815	3 55
3 56	.997799	.997271	.996708	.996118	.995510	.994889	3.56
3.57	.997842	.997320	.996764	.996180	.995577	.994962	3.57
3.58	.997884	.997369	.996818	996241	.995644	.995034	3.58
3.59	.997925	.997416	.996872	.996301	.995709	.995104	3.59
3.60	.997965	.997463	.996925	.996359	.995774	.995174	3 60
3.61	.998005	.997509	.996977	.996417	.995837	.995243	3 61
3.62	.998044	997554	.997028	.996475	.995900	.995311	3 62
3.63	.998082	.997598	.997079	.996531	.995962	.995378	3.63
3.64	.998120	.997642	.997128	.996586	.996022	.995444	3.64
3 65	.998156	.997685	.997177	.996641	996082	.995509	3 65
3.66	.998192	.997727	.997225	.996694	.996141	.995573	3.66
3 67	.998228	997768	997272	996747	.996200	.995636	3 67
3 68	.998263	.997809	997319	.996799	.996257	.995699	3.68
3.69	.998297	.997849	997364	.996850	.996314	.995760	3.69
3 70	998330	.997888	.997409	.996901	.996369	.995821	3.70
3.71	.998363	.997927	.997453	.996950	.996424	.995881	3.71
3.72	998395	.997964	997497	.996999	.996478	.995940	3 72
3 73	.998427	.998002	997540	.997047	.996532	.995998	3.73
3.74	.998458	.998038	.997582	.997095	.996584	.996056	3.74
3.75	.998488	.998074	.997623	.997141	.996636	.996112	3.75
3.76	.998518	.998109	.997664	.997187	.996687	.996168	3 76
3.77	.998547	.998144	.997704	.997233	.996737	.996223	3 77
3.78	.998576	.998178	997743	.997277	.996786	.996277	3.78
3.79	998604	.998211	.997782	.997321	.996835	.996331	3.79
3.80	.998632	.998244	997820	.997364	.996883	.996383	3.80
3.81	998659	.998277	997857	.997406	.996931	.996435	3.81
3 82	.998685	.998308	997894	997448	.996977	.996487	3.82
3.83	.998712	.998339	997930	.997489	.997023	.996537	3.83
3.84	.998737	.998370	.997966	.997530	.997068	.996587	3.84
3.85	.998762	.998400	.998001	997570	.997113	.996636	3 85
3.86	.998787	.998430	.998035	.997609	997157	.996685	3.86
3.87	998811	.998459	.998069	.997648	.997200	.996733	3.87
3.88	998835	.998487	.998102	.997686	.997243	.996780	3.88
3 89	.998858	.998515	.998135	.997723	.997285	.996826	3.89
3.90	.998881	.998543	.998167	.997760	.997326	.996872	3.90
3 91	998903	.998570	998199	.997796	.997367	.996917	3.91
3 92	.998925	998596	.998230	.997832	.997407	.996962	3.92
3.93	.998947	.998622	.998261	.997867	.997447	.997006	3.93
3 94	.998968	998648	998291	.997902	.997486	.997049	3.94
3.95	.998988	.998673	.998320	.997936	.997524	.997092	3.95
3 96	.999009	.998697	998349	.997969	.997562	.997134	3.96
3.97	.999029	.998722	.998378	.998002	997600	.997175	3.97
3 98	999048	.998745	998406	.998035	.997636	.997216	3.98
3 99	.999067	.998769	.998434	.998067	.997673	.997256	3.99

t	SKEWNESS 0	.1	.2	.3	.4	.5	t
4.00	.999968	.999921	.999839	.999717	.999551	.999340	4.00
4.01	.999970	.999923	.999844	.999724	.999562	.999355	4.01
4.02	.999971	.999926	.999849	.999732	.999572	.999369	4.02
4.03	.999972	.999929	.999853	.999739	.999583	.999383	4.03
4.04	.999973	.999931	.999858	.999746	.999593	.999396	4.04
4.05	.999974	.999934	.999862	.999753	.999603	.999409	4.05
4.06	.999975	.999936	.999866	.999760	.999612	.999423	4.06
4.07	.999977	.999938	.999870	.999766	.999622	.999435	4.07
4.08	.999977	.999940	.999874	.999772	.999631	.999448	4.08
4.09	.999978	.999943	.999878	.999779	.999640	.999460	4.09
4.10	.999979	.999945	.999882	.999785	.999649	.999472	4.10
4.11	.999980	.999947	.999886	.999790	.999657	.999484	4.11
4.12	.999981	.999948	.999889	.999796	.999665	.999495	4.12
4.13	.999982	.999950	.999892	.999802	.999674	.999506	4.13
4.14	.999983	.999952	.999896	.999807	.999682	.999517	4.14
4.15	.999983	.999954	.999899	.999812	.999689	.999528	4.15
4.16	.999984	.999955	.999902	.999818	.999697	.999538	4.16
4.17	.999985	.999957	.999905	.999823	.999704	.999549	4.17
4.18	.999985	.999959	.999908	.999827	.999712	.999559	4.18
4.19	.999986	.999960	.999911	.999832	.999719	.999569	4.19
4.20	.999987	.999962	.999914	.999837	.999726	.999578	4.20
4.21	.999987	.999963	.999916	.999841	.999732	.999588	4.21
4.22	.999988	.999964	.999919	.999846	.999739	.999597	4.22
4.23	.999988	.999966	.999922	.999850	.999746	.999606	4.23
4.24	.999989	.999967	.999924	.999854	.999752	.999615	4.24
4.25	.999989	.999968	.999926	.999858	.999758	.999623	4.25
4.26	.999990	.999969	.999929	.999862	.999764	.999632	4.26
4.27	.999990	.999970	.999931	.999866	.999770	.999640	4.27
4.28	.999991	.999971	.999933	.999870	.999776	.999648	4.28
4.29	.999991	.999972	.999935	.999873	.999781	.999656	4.29
4.30	.999991	.999973	.999937	.999877	.999787	.999664	4.30
4.31	.999992	.999974	.999939	.999880	.999792	.999671	4.31
4.32	.999992	.999975	.999941	.999883	.999797	.999679	4.32
4.33	.999993	.999976	.999943	.999887	.999802	.999686	4.33
4.34	.999993	.999977	.999945	.999890	.999807	.999693	4.34
4.35	.999993	.999978	.999947	.999893	.999812	.999700	4.35
4.36	.999993	.999979	.999948	.999896	.999817	.999707	4.36
4.37	.999994	.999980	.999950	.999899	.999821	.999713	4.37
4.38	.999994	.999980	.999952	.999902	.999826	.999720	4.38
4.39	.999994	.999981	.999953	.999904	.999830	.999726	4.39
4.40	.999995	.999982	.999955	.999907	.999834	.999733	4.40
4.41	.999995	.999983	.999956	.999910	.999838	.999739	4.41
4.42	.999995	.999983	.999958	.999912	.999842	.999745	4.42
4.43	.999995	.999984	.999959	.999915	.999846	.999750	4.43
4.44	.999996	.999984	.999960	.999917	.999850	.999756	4.44
4.45	.999996	.999985	.999962	.999920	.999854	.999762	4.45
4.46	.999996	.999986	.999963	.999922	.999858	.999767	4.46
4.47	.999996	.999986	.999964	.999924	.999861	.999772	4.47
4.48	.999996	.999987	.999965	.999926	.999865	.999778	4.48
4.49	.999996	.999987	.999966	.999928	.999868	.999783	4.49

t	SKEWNESS						t
	6	.7	.8	.9	1.0	1 1	
4 00	.999086	.998792	998461	998098	997708	997296	4 00
4 01	.999105	.998814	998488	998129	997743	.997335	4.01
4 02	999123	.998836	998514	998160	.997778	997374	4 02
4.03	.999140	.998858	998540	998190	997812	997412	4 03
4.04	999158	.998879	998565	.998219	997846	997450	4.04
4.05	.999175	998900	.998590	.998248	997879	997487	4 05
4 06	999191	998921	998615	998277	997911	997523	4 06
4 07	.999208	.998941	.998639	998305	997943	997559	4 07
4 08	.999224	998961	.998663	998333	.997975	.997594	4.08
4.09	.999239	998980	998686	.998360	.998006	997629	4 09
4.10	999255	999000	.998709	.998387	998037	997664	4 10
4 11	999270	999018	998731	.998413	998067	997698	4 11
4 12	999285	.999037	998754	998439	998097	.997731	4 12
4 13	999299	999055	998775	.998465	.998126	.997764	4 13
4.14	999314	999073	998797	.998490	.998155	997797	4 14
4 15	999328	.999090	998818	998514	998183	997829	4 15
4.16	999341	999107	998839	998539	998211	.997861	4.16
4.17	999355	999124	998859	.998563	998239	.997892	4 17
4.18	.999368	.999141	998879	998586	998266	.997923	4 18
4.19	999381	.999157	998899	998610	998293	997953	4 19
4.20	999394	.999173	.998918	.998632	998319	.997983	4 20
4.21	999406	999188	998937	.998655	998345	998012	4 21
4 22	999418	999204	.998956	.998677	998371	.998041	4 22
4.23	999430	999219	.998974	998699	998396	998070	3.23
4 24	999442	.999234	.998992	.998720	998421	998098	4 24
4 25	.999453	999248	.999010	.998741	998445	998126	4 25
4 26	999464	.999262	.999027	.998762	998470	998153	4 26
4.27	.999475	.999276	.999045	.998782	.998493	998180	4 27
4 28	999486	.999290	999061	998803	998517	998207	4 28
4 29	.999497	.999304	999078	998822	.998540	998233	4 29
4 30	.999507	999317	.999094	998842	998562	998259	4 30
4.31	999517	.999330	.999110	998861	.998585	998285	4.31
4 32	999527	999343	.999126	998880	.998607	.998310	4 32
4.33	.999537	999355	999141	.998898	.998628	.998335	4 33
4 34	999547	.999367	.999157	998916	.998650	.998359	4 34
4 35	999556	.999379	999172	998934	.998670	.998383	4 35
4.36	999565	999391	.999186	998952	998691	.998407	4.36
4.37	.999574	.999403	.999201	.998969	998712	.998430	4 37
4 38	.999583	999414	.999215	998986	998732	.998453	4.38
4.39	999592	.999425	.999229	999003	998751	998476	4.39
4.40	.999600	.999436	.999242	999020	998771	998498	4.40
4.11	999608	.999447	999256	.999036	.998790	998520	4.41
4 42	.999616	.999458	999269	999052	.998809	.998542	4.42
4.43	.999624	999468	.999282	.999068	998827	998564	4.43
4.44	999632	.999478	.999295	.999083	.998846	998585	4.44
4 45	999640	.999488	.999307	999098	998864	.998606	4 45
4 46	.999647	999498	.999320	.999113	998881	.998626	4.46
4.47	999655	.999508	.999332	.999128	.998899	.998646	4.47
4.48	999662	999517	.999344	.999143	.998916	998666	4.48
4.49	.999669	.999527	.999355	.999157	.998933	.998686	4.49

t	SKEWNESS						t
	0	.1	.2	.3	.4	.5	
4.50	.999997	.999988	.999967	.999930	.999872	.999788	4.50
4.51	.999997	.999988	.999968	.999932	.999875	.999792	4.51
4.52	.999997	.999989	.999969	.999934	.999878	.999797	4.52
4.53	.999997	.999989	.999970	.999936	.999881	.999802	4.53
4.54	.999997	.999989	.999971	.999938	.999884	.999806	4.54
4.55	.999997	.999990	.999972	.999940	.999887	.999811	4.55
4.56	.999997	.999990	.999973	.999941	.999890	.999815	4.56
4.57	.999998	.999991	.999974	.999943	.999893	.999819	4.57
4.58	.999998	.999991	.999975	.999945	.999896	.999824	4.58
4.59	.999998	.999991	.999976	.999946	.999898	.999828	4.59
4.60	.999998	.999992	.999977	.999948	.999901	.999832	4.60
4.61	.999998	.999992	.999977	.999949	.999903	.999836	4.61
4.62	.999998	.999992	.999978	.999951	.999906	.999839	4.62
4.63	.999998	.999993	.999979	.999952	.999908	.999843	4.63
4.64	.999998	.999993	.999980	.999954	.999911	.999847	4.64
4.65	.999998	.999993	.999980	.999955	.999913	.999850	4.65
4.66	.999998	.999993	.999981	.999956	.999915	.999854	4.66
4.67	.999998	.999994	.999982	.999958	.999917	.999857	4.67
4.68	.999999	.999994	.999982	.999959	.999919	.999860	4.68
4.69	.999999	.999994	.999983	.999960	.999922	.999864	4.69
4.70	.999999	.999994	.999983	.999961	.999924	.999867	4.70
4.71	.999999	.999995	.999984	.999962	.999926	.999870	4.71
4.72	.999999	.999995	.999984	.999963	.999927	.999873	4.72
4.73	.999999	.999995	.999985	.999964	.999929	.999876	4.73
4.74	.999999	.999995	.999985	.999965	.999931	.999879	4.74
4.75	.999999	.999995	.999986	.999966	.999933	.999882	4.75
4.76	.999999	.999996	.999986	.999967	.999935	.999884	4.76
4.77	.999999	.999996	.999987	.999968	.999936	.999887	4.77
4.78	.999999	.999996	.999987	.999969	.999938	.999890	4.78
4.79	.999999	.999996	.999988	.999970	.999940	.999892	4.79
4.80	.999999	.999996	.999988	.999971	.999941	.999895	4.80
4.81	.999999	.999996	.999989	.999972	.999943	.999897	4.81
4.82	.999999	.999997	.999989	.999973	.999944	.999900	4.82
4.83	.999999	.999997	.999989	.999974	.999946	.999902	4.83
4.84	.999999	.999997	.999990	.999974	.999947	.999904	4.84
4.85	.999999	.999997	.999990	.999975	.999948	.999907	4.85
4.86	.999999	.999997	.999990	.999976	.999950	.999909	4.86
4.87	.999999	.999997	.999991	.999977	.999951	.999911	4.87
4.88	.999999	.999997	.999991	.999977	.999952	.999913	4.88
4.89	.999999	.999997	.999991	.999978	.999954	.999915	4.89
4.90		.999998	.999992	.999979	.999955	.999917	4.90
4.91		.999998	.999992	.999979	.999956	.999919	4.91
4.92		.999998	.999992	.999980	.999957	.999921	4.92
4.93		.999998	.999992	.999980	.999958	.999923	4.93
4.94		.999998	.999993	.999981	.999959	.999925	4.94
4.95		.999998	.999993	.999982	.999961	.999926	4.95
4.96		.999998	.999993	.999982	.999962	.999928	4.96
4.97		.999998	.999993	.999983	.999963	.999930	4.97
4.98		.999998	.999994	.999983	.999964	.999932	4.98
4.99		.999998	.999994	.999984	.999965	.999933	4.99

t	SKEWNESS						t
	.6	7	.8	9	1 0	1.1	
4.50	.999676	.999536	999367	999171	.998950	.998705	4.50
4.51	.999683	999545	.999378	.999185	.998966	.998724	4.51
4.52	.999689	999553	.999389	.999198	.998982	.998743	4.52
4.53	999696	.999562	.999400	.999212	.998998	998762	4.53
4.54	.999702	999570	999411	.999225	999014	998780	4.54
4.55	999708	.999579	999422	.999238	999029	998798	4.55
4.56	999715	999587	999432	999251	.999045	998816	4.56
4.57	.999721	.999595	999442	999263	999060	998833	4.57
4.58	.999726	999603	999452	999275	.999074	998850	4.58
4.59	999732	.999610	.999462	.999288	999089	.998867	4.59
4.60	.999738	.999618	.999472	999300	999103	.998884	4.60
4.61	999743	999625	999481	.999311	.999117	.998901	4.61
4.62	.999749	999632	999490	999323	.999131	998917	4.62
4.63	999754	999640	999499	.999334	999145	.998933	4.63
4.64	999759	.999647	.999508	999345	.999158	998949	4.64
4.65	999764	.999653	999517	.999356	.999171	998964	4.65
4.66	.999769	999660	999526	.999367	999184	998980	4.66
4.67	999774	999667	999535	.999378	999197	998995	4.67
4.68	.999779	.999673	.999543	.999388	999210	999010	4.68
4.69	.999784	.999680	999551	.999398	.999222	999024	4.69
4.70	999788	999686	999559	.999408	999235	999039	4.70
4.71	999793	.999692	.999567	.999418	999247	999053	4.71
4.72	.999797	.999698	.999575	999428	999258	.999067	4.72
4.73	.999801	.999704	999583	.999438	999270	999081	4.73
4.74	.999806	999710	.999590	999447	.999282	.999095	4.74
4.75	999810	999715	.999598	999457	.999293	999108	4.75
4.76	.999814	.999721	.999605	999466	999304	.999121	4.76
4.77	.999818	.999726	999612	.999475	999315	999134	4.77
4.78	.999822	.999732	.999619	.999484	999326	999147	4.78
4.79	.999825	999737	999626	999492	999337	999160	4.79
4.80	999829	.999742	.999633	.999501	.999347	999173	4.80
4.81	999833	999747	999639	999509	999357	999185	4.81
4.82	.999836	.999752	999646	.999518	999368	.999197	4.82
4.83	999840	999757	.999652	999526	999378	.999209	4.83
4.84	.999843	999762	999659	.999534	.999387	.999221	4.84
4.85	999847	.999766	999665	999542	.999397	999232	4.85
4.86	999850	.999771	999671	.999549	999407	999244	4.86
4.87	.999853	.999775	.999677	.999557	999416	999255	4.87
4.88	999856	999780	999683	999564	999425	.999266	4.88
4.89	.999859	.999784	999688	999572	.999434	.999277	4.89
4.90	999862	999788	999694	999579	999443	.999288	4.90
4.91	.999865	999793	999700	.999586	999452	999298	4.91
4.92	.999868	.999797	.999705	.999593	999461	.999309	4.92
4.93	999871	999801	.999711	.999600	.999469	999319	4.93
4.94	999874	.999805	999716	.999607	999478	999329	4.94
4.95	.999876	999808	.999721	.999614	.999486	.999339	4.95
4.96	.999879	999812	999726	.999620	.999494	999349	4.96
4.97	.999882	.999816	999731	999627	999502	999359	4.97
4.98	999884	999820	999736	999633	.999510	999369	4.98
4.99	.999887	.999823	.999741	.999639	999518	999378	4.99

t	SKEWNESS						t
	0	1	2	3	4	5	
5.00		.999998	.999994	.999984	.999965	.999935	5.00
5.01		.999998	.999994	.999985	.999966	.999936	5.01
5.02		.999998	.999994	.999985	.999967	.999938	5.02
5.03		.999999	.999995	.999986	.999968	.999939	5.03
5.04		.999999	.999995	.999986	.999969	.999941	5.04
5.05		.999999	.999995	.999986	.999970	.999942	5.05
5.06		.999999	.999995	.999987	.999971	.999943	5.06
5.07		.999999	.999995	.999987	.999971	.999945	5.07
5.08		.999999	.999996	.999988	.999972	.999946	5.08
5.09		.999999	.999996	.999988	.999973	.999947	5.09
5.10		.999999	.999996	.999988	.999974	.999949	5.10
5.11		.999999	.999996	.999989	.999974	.999950	5.11
5.12		.999999	.999996	.999989	.999975	.999951	5.12
5.13		.999999	.999996	.999989	.999976	.999952	5.13
5.14		.999999	.999996	.999990	.999976	.999953	5.14
5.15		.999999	.999997	.999990	.999977	.999955	5.15
5.16		.999999	.999997	.999990	.999978	.999956	5.16
5.17		.999999	.999997	.999991	.999978	.999957	5.17
5.18		.999999	.999997	.999991	.999979	.999958	5.18
5.19		.999999	.999997	.999991	.999979	.999959	5.19
5.20		.999999	.999997	.999991	.999980	.999960	5.20
5.21		.999999	.999997	.999992	.999980	.999961	5.21
5.22		.999999	.999997	.999992	.999980	.999962	5.22
5.23		.999999	.999997	.999992	.999981	.999963	5.23
5.24		.999999	.999997	.999992	.999982	.999963	5.24
5.25		.999999	.999998	.999993	.999982	.999964	5.25
5.26		.999999	.999998	.999993	.999983	.999965	5.26
5.27		.999999	.999998	.999993	.999983	.999966	5.27
5.28			.999998	.999993	.999984	.999967	5.28
5.29			.999998	.999994	.999984	.999968	5.29
5.30			.999998	.999994	.999985	.999968	5.30
5.31			.999998	.999994	.999985	.999969	5.31
5.32			.999998	.999994	.999986	.999970	5.32
5.33			.999998	.999994	.999986	.999971	5.33
5.34			.999998	.999994	.999986	.999971	5.34
5.35			.999998	.999995	.999987	.999972	5.35
5.36			.999998	.999995	.999987	.999973	5.36
5.37			.999998	.999995	.999987	.999973	5.37
5.38			.999998	.999995	.999988	.999974	5.38
5.39			.999999	.999995	.999988	.999975	5.39
5.40			.999999	.999995	.999988	.999975	5.40
5.41			.999999	.999996	.999989	.999976	5.41
5.42			.999999	.999996	.999989	.999977	5.42
5.43			.999999	.999996	.999989	.999977	5.43
5.44			.999999	.999996	.999990	.999978	5.44
5.45			.999999	.999996	.999990	.999978	5.45
5.46			.999999	.999996	.999990	.999979	5.46
5.47			.999999	.999996	.999990	.999979	5.47
5.48			.999999	.999996	.999991	.999980	5.48
5.49			.999999	.999997	.999991	.999980	5.49

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
5.00	.999889	.999827	.999746	.999645	.999526	.999387	5.00
5.01	.999892	.999830	.999750	.999651	.999533	.999397	5.01
5.02	.999894	.999833	.999755	.999657	.999541	.999406	5.02
5.03	.999896	.999837	.999759	.999663	.999548	.999414	5.03
5.04	.999898	.999840	.999764	.999669	.999555	.999423	5.04
5.05	.999901	.999843	.999768	.999674	.999562	.999432	5.05
5.06	.999903	.999846	.999772	.999680	.999569	.999440	5.06
5.07	.999905	.999849	.999776	.999685	.999576	.999449	5.07
5.08	.999907	.999852	.999781	.999691	.999583	.999457	5.08
5.09	.999909	.999855	.999785	.999696	.999590	.999465	5.09
5.10	.999911	.999858	.999788	.999701	.999596	.999473	5.10
5.11	.999913	.999861	.999792	.999706	.999603	.999481	5.11
5.12	.999915	.999864	.999796	.999711	.999609	.999489	5.12
5.13	.999917	.999866	.999800	.999716	.999615	.999497	5.13
5.14	.999918	.999869	.999804	.999721	.999621	.999504	5.14
5.15	.999920	.999872	.999807	.999726	.999627	.999512	5.15
5.16	.999922	.999874	.999811	.999731	.999633	.999519	5.16
5.17	.999924	.999877	.999814	.999735	.999639	.999526	5.17
5.18	.999925	.999879	.999818	.999740	.999645	.999533	5.18
5.19	.999927	.999882	.999821	.999744	.999651	.999540	5.19
5.20	.999928	.999884	.999824	.999749	.999656	.999547	5.20
5.21	.999930	.999886	.999828	.999753	.999662	.999554	5.21
5.22	.999932	.999888	.999831	.999757	.999667	.999561	5.22
5.23	.999933	.999891	.999834	.999761	.999672	.999567	5.23
5.24	.999935	.999893	.999837	.999765	.999678	.999574	5.24
5.25	.999936	.999895	.999840	.999769	.999683	.999580	5.25
5.26	.999937	.999897	.999843	.999773	.999688	.999587	5.26
5.27	.999939	.999899	.999846	.999777	.999693	.999593	5.27
5.28	.999940	.999901	.999849	.999781	.999698	.999599	5.28
5.29	.999941	.999903	.999851	.999785	.999703	.999605	5.29
5.30	.999943	.999905	.999854	.999789	.999708	.999611	5.30
5.31	.999944	.999907	.999857	.999792	.999712	.999617	5.31
5.32	.999945	.999909	.999860	.999796	.999717	.999623	5.32
5.33	.999946	.999911	.999862	.999799	.999721	.999628	5.33
5.34	.999948	.999913	.999865	.999803	.999926	.999634	5.34
5.35	.999949	.999914	.999867	.999806	.999730	.999639	5.35
5.36	.999950	.999916	.999870	.999809	.999735	.999645	5.36
5.37	.999951	.999918	.999872	.999813	.999739	.999650	5.37
5.38	.999952	.999919	.999874	.999816	.999743	.999656	5.38
5.39	.999953	.999921	.999877	.999819	.999747	.999661	5.39
5.40	.999954	.999923	.999879	.999822	.999751	.999666	5.40
5.41	.999955	.999924	.999881	.999825	.999755	.999671	5.41
5.42	.999956	.999926	.999884	.999828	.999759	.999676	5.42
5.43	.999957	.999927	.999886	.999831	.999763	.999681	5.43
5.44	.999958	.999929	.999888	.999834	.999767	.999686	5.44
5.45	.999959	.999930	.999890	.999837	.999771	.999690	5.45
5.46	.999960	.999931	.999892	.999840	.999774	.999695	5.46
5.47	.999961	.999933	.999894	.999843	.999778	.999700	5.47
5.48	.999962	.999934	.999896	.999845	.999782	.999704	5.48
5.49	.999962	.999936	.999898	.999848	.999785	.999709	5.49

t	SKEWNESS						t
	0	1	.2	.3	.4	.5	
5 50			.999999	.999997	999991	.999981	5.50
5 51			999999	.999997	999991	.999981	5.51
5.52			999999	.999997	.999992	999982	5.52
5.53			.999999	999997	999992	.999982	5.53
5.54			999999	.999997	999992	999983	5.54
5.55			.999999	.999997	.999992	999983	5.55
5 56			999999	.999997	.999993	.999983	5.56
5.57			.999999	.999997	.999993	.999984	5.57
5 58			.999999	999997	.999993	999984	5 58
5 59			.999999	.999997	999993	.999985	5.59
5 60			.999999	999998	.999993	999985	5.60
5 61			.999999	.999998	.999994	999985	5 61
5.62			.999999	999998	.999994	999986	5 62
5.63			.999999	999998	.999994	999986	5 63
5 64			.999999	999998	.999994	999986	5.64
5.65			.999999	999998	.999994	999987	5 65
5 66			.999999	999998	999994	999987	5.66
5.67			.999999	.999998	999995	.999987	5.67
5 68				.999998	999995	999988	5.68
5.69				999998	.999995	.999988	5.69
5.70				.999998	.999995	999988	5.70
5.71				.999998	.999995	.999989	5.71
5 72				999998	999995	.999989	5.72
5.73				999998	.999995	.999989	5.73
5 74				999998	.999996	.999989	5.74
5.75				.999999	.999996	.999990	5.75
5 76				999999	999996	.999990	5.76
5 77				999999	.999996	.999990	5 77
5 78				.999999	999996	.999990	5.78
5.79				.999999	.999996	.999991	5.79
5 80				.999999	999996	999991	5.80
5 81				.999999	.999996	999991	5 81
5 82				999999	.999996	999991	5.82
5 83				999999	999997	.999992	5.83
5 84				.999999	.999997	.999992	5.84
5 85				.999999	.999997	.999992	5 85
5.86				.999999	.999997	.999992	5.86
5 87				.999999	.999997	999992	5 87
5 88				.999999	.999997	999993	5.88
5 89				.999999	999997	999993	5 89
5 90				.999999	.999997	.999993	5.90
5 91				999999	.999997	.999993	5.91
5.92				999999	999997	999993	5.92
5 93				999999	999997	.999993	5.93
5 94				999999	999997	.999994	5.94
5.95				.999999	.999998	.999994	5.95
5 96				999999	999998	.999994	5.96
5.97				999999	.999998	.999994	5.97
5.98				999999	.999998	.999994	5.98
5 99				999999	.999998	.999994	5.99

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
5.50	.999963	.999937	.999900	.999851	.999789	.999713	5.50
5.51	.999964	.999938	.999902	.999853	.999792	.999717	5.51
5.52	.999965	.999939	.999903	.999856	.999795	.999722	5.52
5.53	.999966	.999941	.999905	.999858	.999799	.999726	5.53
5.54	.999966	.999942	.999907	.999861	.999802	.999730	5.54
5.55	.999967	.999943	.999909	.999863	.999805	.999734	5.55
5.56	.999968	.999944	.999910	.999866	.999808	.999738	5.56
5.57	.999969	.999945	.999912	.999868	.999811	.999742	5.57
5.58	.999969	.999946	.999914	.999870	.999814	.999746	5.58
5.59	.999970	.999947	.999915	.999872	.999817	.999750	5.59
5.60	.999971	.999949	.999917	.999875	.999820	.999754	5.60
5.61	.999971	.999950	.999919	.999877	.999823	.999758	5.61
5.62	.999972	.999951	.999920	.999879	.999826	.999761	5.62
5.63	.999973	.999952	.999922	.999881	.999829	.999765	5.63
5.64	.999973	.999953	.999923	.999883	.999832	.999768	5.64
5.65	.999974	.999954	.999924	.999885	.999835	.999772	5.65
5.66	.999974	.999955	.999926	.999887	.999837	.999775	5.66
5.67	.999975	.999955	.999927	.999889	.999840	.999779	5.67
5.68	.999976	.999956	.999929	.999891	.999842	.999782	5.68
5.69	.999976	.999957	.999930	.999893	.999845	.999786	5.69
5.70	.999977	.999958	.999931	.999895	.999848	.999789	5.70
5.71	.999977	.999959	.999933	.999897	.999850	.999792	5.71
5.72	.999978	.999960	.999934	.999898	.999852	.999795	5.72
5.73	.999978	.999961	.999935	.999900	.999855	.999798	5.73
5.74	.999979	.999961	.999936	.999902	.999857	.999801	5.74
5.75	.999979	.999962	.999938	.999904	.999859	.999804	5.75
5.76	.999980	.999963	.999939	.999905	.999862	.999807	5.76
5.77	.999980	.999964	.999940	.999907	.999864	.999810	5.77
5.78	.999980	.999965	.999941	.999909	.999866	.999813	5.78
5.79	.999981	.999965	.999942	.999910	.999868	.999816	5.79
5.80	.999981	.999966	.999943	.999912	.999871	.999819	5.80
5.81	.999982	.999967	.999944	.999913	.999873	.999822	5.81
5.82	.999982	.999967	.999945	.999915	.999875	.999824	5.82
5.83	.999983	.999968	.999946	.999916	.999877	.999827	5.83
5.84	.999983	.999969	.999947	.999918	.999879	.999830	5.84
5.85	.999983	.999969	.999948	.999919	.999881	.999832	5.85
5.86	.999984	.999970	.999949	.999921	.999883	.999835	5.86
5.87	.999984	.999971	.999950	.999922	.999885	.999837	5.87
5.88	.999984	.999971	.999951	.999923	.999887	.999840	5.88
5.89	.999985	.999972	.999952	.999925	.999888	.999842	5.89
5.90	.999985	.999972	.999953	.999926	.999890	.999845	5.90
5.91	.999985	.999973	.999954	.999927	.999892	.999847	5.91
5.92	.999986	.999973	.999955	.999929	.999894	.999849	5.92
5.93	.999986	.999974	.999956	.999930	.999895	.999852	5.93
5.94	.999986	.999975	.999957	.999931	.999897	.999854	5.94
5.95	.999987	.999975	.999957	.999932	.999899	.999856	5.95
5.96	.999987	.999976	.999958	.999933	.999901	.999858	5.96
5.97	.999987	.999976	.999959	.999935	.999902	.999861	5.97
5.98	.999988	.999977	.999960	.999936	.999904	.999863	5.98
5.99	.999988	.999977	.999960	.999937	.999905	.999865	5.99

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
6.00				.999999	.999998	.999995	6.00
6.01				.999999	.999998	.999995	6.01
6.02				.999999	.999998	.999995	6.02
6.03				.999999	.999998	.999995	6.03
6.04				.999999	.999998	.999995	6.04
6.05				.999999	.999998	.999995	6.05
6.06				.999999	.999998	.999995	6.06
6.07				.999999	.999998	.999995	6.07
6.08				.999999	.999998	.999996	6.08
6.09					.999998	.999996	6.09
6.10					.999998	.999996	6.10
6.11					.999998	.999906	6.11
6.12					.999998	.999996	6.12
6.13					.999999	.999996	6.13
6.14					.999999	.999996	6.14
6.15					.999999	.999996	6.15
6.16					.999999	.999996	6.16
6.17					.999999	.999996	6.17
6.18					.999999	.999997	6.18
6.19					.999999	.999997	6.19
6.20					.999999	.999997	6.20
6.21					.999999	.999997	6.21
6.22					.999999	.999997	6.22
6.23					.999999	.999997	6.23
6.24					.999999	.999997	6.24
6.25					.999999	.999997	6.25
6.26					.999999	.999997	6.26
6.27					.999999	.999997	6.27
6.28					.999999	.999997	6.28
6.29					.999999	.999997	6.29
6.30					.999999	.999997	6.30
6.31					.999999	.999998	6.31
6.32					.999999	.999998	6.32
6.33					.999999	.999998	6.33
6.34					.999999	.999998	6.34
6.35					.999999	.999998	6.35
6.36					.999999	.999998	6.36
6.37					.999999	.999998	6.37
6.38					.999999	.999998	6.38
6.39					.999999	.999998	6.39
6.40					.999999	.999998	6.40
6.41					.999999	.999998	6.41
6.42					.999999	.999998	6.42
6.43					.999999	.999998	6.43
6.44					.999999	.999998	6.44
6.45					.999999	.999998	6.45
6.46					.999999	.999998	6.46
6.47					.999999	.999998	6.47
6.48					.999999	.999998	6.48
6.49					.999999	.999998	6.49

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
6.00	.999988	.999978	.999961	.999938	.999907	.999867	6.00
6.01	.999988	.999978	.999962	.999939	.999908	.999869	6.01
6.02	.999989	.999978	.999963	.999940	.999910	.999871	6.02
6.03	.999989	.999979	.999963	.999941	.999911	.999873	6.03
6.04	.999989	.999979	.999964	.999942	.999913	.999875	6.04
6.05	.999989	.999980	.999965	.999943	.999914	.999877	6.05
6.06	.999990	.999980	.999965	.999944	.999916	.999879	6.06
6.07	.999990	.999981	.999966	.999945	.999917	.999881	6.07
6.08	.999990	.999981	.999967	.999946	.999918	.999882	6.08
6.09	.999990	.999981	.999967	.999947	.999920	.999884	6.09
6.10	.999991	.999982	.999968	.999948	.999921	.999886	6.10
6.11	.999991	.999982	.999969	.999949	.999922	.999888	6.11
6.12	.999991	.999983	.999969	.999950	.999924	.999889	6.12
6.13	.999991	.999983	.999970	.999951	.999925	.999891	6.13
6.14	.999991	.999983	.999970	.999952	.999926	.999893	6.14
6.15	.999992	.999984	.999971	.999953	.999927	.999894	6.15
6.16	.999992	.999984	.999971	.999953	.999929	.999896	6.16
6.17	.999992	.999984	.999972	.999954	.999930	.999898	6.17
6.18	.999992	.999985	.999973	.999955	.999931	.999899	6.18
6.19	.999992	.999985	.999973	.999956	.999932	.999901	6.19
6.20	.999993	.999985	.999974	.999957	.999933	.999902	6.20
6.21	.999993	.999986	.999974	.999957	.999934	.999904	6.21
6.22	.999993	.999986	.999975	.999958	.999935	.999905	6.22
6.23	.999993	.999986	.999975	.999959	.999936	.999907	6.23
6.24	.999993	.999986	.999976	.999960	.999937	.999908	6.24
6.25	.999993	.999987	.999976	.999960	.999938	.999910	6.25
6.26	.999994	.999987	.999976	.999961	.999939	.999911	6.26
6.27	.999994	.999987	.999977	.999962	.999940	.999912	6.27
6.28	.999994	.999988	.999977	.999962	.999941	.999914	6.28
6.29	.999994	.999988	.999978	.999963	.999942	.999915	6.29
6.30	.999994	.999988	.999978	.999964	.999943	.999916	6.30
6.31	.999994	.999988	.999979	.999964	.999944	.999918	6.31
6.32	.999994	.999989	.999979	.999965	.999945	.999919	6.32
6.33	.999994	.999989	.999979	.999966	.999946	.999920	6.33
6.34	.999995	.999989	.999980	.999966	.999947	.999921	6.34
6.35	.999995	.999989	.999980	.999967	.999948	.999923	6.35
6.36	.999995	.999989	.999981	.999967	.999949	.999924	6.36
6.37	.999995	.999990	.999981	.999968	.999950	.999925	6.37
6.38	.999995	.999990	.999981	.999969	.999950	.999926	6.38
6.39	.999995	.999990	.999982	.999969	.999951	.999927	6.39
6.40	.999995	.999990	.999982	.999970	.999952	.999928	6.40
6.41	.999995	.999991	.999982	.999970	.999953	.999930	6.41
6.42	.999996	.999991	.999983	.999971	.999954	.999931	6.42
6.43	.999996	.999991	.999983	.999971	.999954	.999932	6.43
6.44	.999996	.999991	.999983	.999972	.999955	.999933	6.44
6.45	.999996	.999991	.999984	.999972	.999956	.999934	6.45
6.46	.999996	.999991	.999984	.999973	.999957	.999935	6.46
6.47	.999996	.999992	.999984	.999973	.999957	.999936	6.47
6.48	.999996	.999992	.999985	.999974	.999958	.999937	6.48
6.49	.999996	.999992	.999985	.999974	.999959	.999938	6.49

t	SKEWNESS						t
	0	1	2	3	4	5	
6.50						.999999	6.50
6.51						.999999	6.51
6.52						.999999	6.52
6.53						.999999	6.53
6.54						.999999	6.54
6.55						.999999	6.55
6.56						.999999	6.56
6.57						.999999	6.57
6.58						.999999	6.58
6.59						.999999	6.59
6.60						.999999	6.60
6.61						.999999	6.61
6.62						.999999	6.62
6.63						.999999	6.63
6.64						.999999	6.64
6.65						.999999	6.65
6.66						.999999	6.66
6.67						.999999	6.67
6.68						.999999	6.68
6.69						.999999	6.69
6.70						.999999	6.70
6.71						.999999	6.71
6.72						.999999	6.72
6.73						.999999	6.73
6.74						.999999	6.74
6.75						.999999	6.75
6.76						.999999	6.76
6.77						.999999	6.77
6.78						.999999	6.78
6.79						.999999	6.79
6.80						.999999	6.80
6.81						.999999	6.81
6.82						.999999	6.82
6.83						.999999	6.83
6.84						.999999	6.84
6.85						.999999	6.85
6.86						.999999	6.86
6.87						.999999	6.87
6.88						.999999	6.88
6.89						.999999	6.89
6.90						.999999	6.90
6.91						.999999	6.91
6.92							6.92
6.93							6.93
6.94							6.94
6.95							6.95
6.96							6.96
6.97							6.97
6.98							6.98
6.99							6.99

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
6 50	.999996	.999992	.999985	.999975	.999959	.999939	6 50
6.51	.999996	.999992	.999986	.999975	.999960	.999940	6.51
6.52	.999996	.999992	.999986	.999976	.999961	.999941	6.52
6.53	.999997	.999993	.999986	.999976	999961	.999942	6 53
6 54	.999997	.999993	.999986	.999976	.999962	.999942	6.54
6.55	.999997	.999993	.999987	.999977	.999963	999943	6 55
6.56	.999997	.999993	.999987	.999977	.999963	.999944	6.56
6.57	.999997	.999993	.999987	999978	.999964	.999945	6.57
6.58	.999997	.999993	.999987	.999978	.999964	.999946	6 58
6.59	.999997	.999994	.999988	.999978	.999965	999947	6.59
6.60	.999997	.999994	.999988	.999979	.999966	.999948	6.60
6.61	.999997	.999994	.999988	.999979	.999966	.999948	6.61
6 62	.999997	.999994	.999988	.999980	.999967	.999949	6.62
6.63	.999997	.999994	.999989	.999980	999967	999950	6 63
6 64	.999997	.999994	.999989	.999980	.999968	.999951	6.64
6.65	.999997	.999994	.999989	.999981	.999968	.999952	6.65
6.66	.999997	.999994	.999989	.999981	999969	.999952	6 66
6.67	.999998	.999995	.999989	.999981	999969	.999953	6.67
6.68	.999998	.999995	.999990	.999982	999970	999954	6.68
6.69	.999998	.999995	.999990	.999982	.999970	.999954	6 69
6 70	.999998	.999995	.999990	.999982	.999971	.999955	6.70
6.71	.999998	.999995	.999990	.999983	.999971	999956	6 71
6.72	.999998	.999995	.999990	999983	.999972	.999957	6 72
6.73	.999998	.999995	.999991	999983	.999972	.999957	6.73
6.74	.999998	.999995	.999991	.999984	.999973	.999958	6 74
6.75	.999998	.999995	.999991	.999984	.999973	.999959	6.75
6.76	.999998	.999996	.999991	.999984	.999974	.999959	6.76
6 77	.999998	.999996	.999991	999984	.999974	.999960	6 77
6 78	.999998	.999996	.999991	.999985	.999975	.999960	6 78
6.79	.999998	.999996	.999992	999985	.999975	.999961	6 79
6.80	.999998	.999996	.999992	.999985	.999975	.999962	6 80
6.81	.999998	.999996	.999992	.999985	.999976	.999962	6.81
6.82	.999998	.999996	.999992	999986	.999976	.999963	6 82
6.83	.999998	.999996	.999992	.999986	.999977	.999963	6 83
6.84	.999998	.999996	.999992	.999986	.999977	.999964	6.84
6.85	.999998	.999996	.999993	.999986	.999977	.999965	6.85
6.86	.999998	.999996	.999993	.999987	.999978	.999965	6.86
6 87	.999998	.999996	.999993	.999987	.999978	.999966	6.87
6 88	.999998	.999997	.999993	.999987	.999979	.999966	6.88
6.89	.999999	.999997	.999993	.999987	.999979	.999967	6.89
6.90	.999999	.999997	.999993	.999988	999979	.999967	6.90
6.91	.999999	.999997	.999993	.999988	.999980	.999968	6.91
6.92	.999999	.999997	.999994	.999988	.999980	.999968	6.92
6.93	.999999	.999997	.999994	.999988	.999980	.999969	6.93
6.94	.999999	.999997	.999994	.999989	.999981	999969	6.94
6.95	.999999	.999997	.999994	.999989	.999981	.999970	6.95
6.96	.999999	.999997	.999994	.999989	.999981	.999970	6.96
6.97	.999999	.999997	.999994	.999989	.999982	.999971	6.97
6.98	.999999	.999997	.999994	.999989	.999982	.999971	6.98
6.99	.999999	.999997	.999994	.999990	.999982	.999972	6.99

t	SKEWNESS						t
	.0	.1	.2	3	.4	5	
7.00							7.00
7.01							7.01
7.02							7.02
7.03							7.03
7.04							7.04
7.05							7.05
7.06							7.06
7.07							7.07
7.08							7.08
7.09							7.09
7.10							7.10
7.11							7.11
7.12							7.12
7.13							7.13
7.14							7.14
7.15							7.15
7.16							7.16
7.17							7.17
7.18							7.18
7.19							7.19
7.20							7.20
7.21							7.21
7.22							7.22
7.23							7.23
7.24							7.24
7.25							7.25
7.26							7.26
7.27							7.27
7.28							7.28
7.29							7.29
7.30							7.30
7.31							7.31
7.32							7.32
7.33							7.33
7.34							7.34
7.35							7.35
7.36							7.36
7.37							7.37
7.38							7.38
7.39							7.39
7.40							7.40
7.41							7.41
7.42							7.42
7.43							7.43
7.44							7.44
7.45							7.45
7.46							7.46
7.47							7.47
7.48							7.48
7.49							7.49

t	SKEWNESS						t
	6	7	8	9	1 0	1.1	
7.00	999999	999997	999994	999990	999982	999972	7 00
7.01	.999999	999997	999995	999990	999983	.999972	7 01
7 02	.999999	999997	999995	999990	.999983	999973	7 02
7.03	.999999	.999997	999995	999990	999983	.999973	7 03
7.04	.999999	999998	999995	999990	999984	999974	7.04
7.05	.999999	999998	999995	999991	999984	999974	7 05
7.06	.999999	999998	999995	.999991	.999984	999975	7 06
7.07	999999	999998	999995	999991	999984	999975	7 07
7.08	999999	.999998	.999995	999991	999985	999975	7 08
7 09	999999	999998	999995	999991	.999985	999976	7 09
7 10	999999	999998	.999995	.999991	.999985	.999976	7 10
7.11	999999	999998	999996	999992	999985	.999976	7.11
7 12	999999	999998	999996	999992	999986	999977	7 12
7 13	999999	999998	999996	999992	.999986	.999977	7 13
7.14	999999	.999998	999996	999992	999986	999978	7.14
7 15	999999	999998	999996	999992	.999986	.999978	7 15
7 16	.999999	999998	999996	999992	999987	999978	7.16
7.17	999999	999998	999996	999992	.999987	999979	7.17
7 18	999999	999998	.999996	999993	.999987	999979	7 18
7 19	999999	999998	.999996	999993	999987	999979	7 19
7 20	999999	.999998	999996	999993	999987	999980	7 20
7 21	999999	999998	999996	.999993	999988	999980	7 21
7.22	.999999	999998	.999996	.999993	.999988	999980	7 22
7 23	999999	999998	999996	999993	.999988	.999981	7 23
7.24	999999	999998	.999997	.999993	999988	999981	7 24
7 25	999999	.999998	999997	.999993	999988	999981	7.25
7.26	999999	999998	.999997	999994	.999989	999981	7 26
7 27	999999	.999999	999997	.999994	999989	999982	7 27
7.28	999999	999999	999997	.999994	999989	999982	7 28
7.29	999999	999999	.999997	999994	999989	.999982	7 29
7 30	999999	999999	.999997	999994	999989	.999983	7 30
7.31	.999999	999999	999997	999994	.999990	999983	7.31
7 32	999999	999999	999997	999994	999990	999983	7 32
7 33	.999999	999999	999997	.999994	999990	999983	7.33
7 34		999999	.999997	.999994	999990	999984	7 34
7 35		999999	.999997	.999995	.999990	999984	7.35
7.36		.999999	999997	.999995	999990	.999984	7 36
7.37		999999	999997	.999995	999991	999984	7 37
7 38		999999	999997	999995	999991	999985	7 38
7.39		999999	999997	999995	.999991	999985	7 39
7.40		999999	.999997	.999995	999991	999985	7 40
7.41		.999999	999998	.999995	999991	.999985	7.41
7.42		.999999	999998	.999995	999991	999986	7.42
7.43		999999	999998	999995	999992	.999986	7 43
7.44		999999	.999998	999995	.999992	999986	7 44
7 45		999999	.999998	999995	999992	999986	7 45
7 46		.999999	.999998	999996	999992	.999986	7.46
7.47		.999999	.999998	999996	.999992	.999987	7 47
7 48		999999	.999998	999996	999992	.999987	7 48
7 49		.999999	999998	999996	999992	999987	7 49

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
7.50							7.50
7.51							7.51
7.52							7.52
7.53							7.53
7.54							7.54
7.55							7.55
7.56							7.56
7.57							7.57
7.58							7.58
7.59							7.59
7.60							7.60
7.61							7.61
7.62							7.62
7.63							7.63
7.64							7.64
7.65							7.65
7.66							7.66
7.67							7.67
7.68							7.68
7.69							7.69
7.70							7.70
7.71							7.71
7.72							7.72
7.73							7.73
7.74							7.74
7.75							7.75
7.76							7.76
7.77							7.77
7.78							7.78
7.79							7.79
7.80							7.80
7.81							7.81
7.82							7.82
7.83							7.83
7.84							7.84
7.85							7.85
7.86							7.86
7.87							7.87
7.88							7.88
7.89							7.89
7.90							7.90
7.91							7.91
7.92							7.92
7.93							7.93
7.94							7.94
7.95							7.95
7.96							7.96
7.97							7.97
7.98							7.98
7.99							7.99

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
7.50		.999999	.999998	.999996	.999992	.999987	7.50
7.51		.999999	.999998	.999996	.999993	.999987	7.51
7.52		.999999	.999998	.999996	.999993	.999988	7.52
7.53		.999999	.999998	.999996	.999993	.999988	7.53
7.54		.999999	.999998	.999996	.999993	.999988	7.54
7.55		.999999	.999998	.999996	.999993	.999988	7.55
7.56		.999999	.999998	.999996	.999993	.999988	7.56
7.57		.999999	.999998	.999996	.999993	.999989	7.57
7.58		.999999	.999998	.999996	.999993	.999989	7.58
7.59		.999999	.999998	.999997	.999994	.999989	7.59
7.60		.999999	.999998	.999997	.999994	.999989	7.60
7.61		.999999	.999998	.999997	.999994	.999989	7.61
7.62		.999999	.999998	.999997	.999994	.999989	7.62
7.63		.999999	.999998	.999997	.999994	.999990	7.63
7.64		.999999	.999998	.999997	.999994	.999990	7.64
7.65		.999999	.999998	.999997	.999994	.999990	7.65
7.66		.999999	.999999	.999997	.999994	.999990	7.66
7.67		.999999	.999999	.999997	.999994	.999990	7.67
7.68		.999999	.999999	.999997	.999994	.999990	7.68
7.69		.999999	.999999	.999997	.999995	.999991	7.69
7.70		.999999	.999999	.999997	.999995	.999991	7.70
7.71		.999999	.999999	.999997	.999995	.999991	7.71
7.72		.999999	.999999	.999997	.999995	.999991	7.72
7.73		.999999	.999999	.999997	.999995	.999991	7.73
7.74		.999999	.999999	.999997	.999995	.999991	7.74
7.75		.999999	.999999	.999997	.999995	.999991	7.75
7.76		.999999	.999999	.999997	.999995	.999992	7.76
7.77			.999999	.999997	.999995	.999992	7.77
7.78			.999999	.999998	.999995	.999992	7.78
7.79			.999999	.999998	.999995	.999992	7.79
7.80			.999999	.999998	.999995	.999992	7.80
7.81			.999999	.999998	.999996	.999992	7.81
7.82			.999999	.999998	.999996	.999992	7.82
7.83			.999999	.999998	.999996	.999992	7.83
7.84			.999999	.999998	.999996	.999993	7.84
7.85			.999999	.999998	.999996	.999993	7.85
7.86			.999999	.999998	.999996	.999993	7.86
7.87			.999999	.999998	.999996	.999993	7.87
7.88			.999999	.999998	.999996	.999993	7.88
7.89			.999999	.999998	.999996	.999993	7.89
7.90			.999999	.999998	.999996	.999993	7.90
7.91			.999999	.999998	.999996	.999993	7.91
7.92			.999999	.999998	.999996	.999993	7.92
7.93			.999999	.999998	.999996	.999994	7.93
7.94			.999999	.999998	.999996	.999994	7.94
7.95			.999999	.999998	.999997	.999994	7.95
7.96			.999999	.999998	.999997	.999994	7.96
7.97			.999999	.999998	.999997	.999994	7.97
7.98			.999999	.999998	.999997	.999994	7.98
7.99			.999999	.999998	.999997	.999994	7.99

t	SKEWNESS 0	.1	2	.3	4	.5	t
8.00							8.00
8.01							8.01
8.02							8.02
8.03							8.03
8.04							8.04
8.05							8.05
8.06							8.06
8.07							8.07
8.08							8.08
8.09							8.09
8.10							8.10
8.11							8.11
8.12							8.12
8.13							8.13
8.14							8.14
8.15							8.15
8.16							8.16
8.17							8.17
8.18							8.18
8.19							8.19
8.20							8.20
8.21							8.21
8.22							8.22
8.23							8.23
8.24							8.24
8.25							8.25
8.26							8.26
8.27							8.27
8.28							8.28
8.29							8.29
8.30							8.30
8.31							8.31
8.32							8.32
8.33							8.33
8.34							8.34
8.35							8.35
8.36							8.36
8.37							8.37
8.38							8.38
8.39							8.39
8.40							8.40
8.41							8.41
8.42							8.42
8.43							8.43
8.44							8.44
8.45							8.45
8.46							8.46
8.47							8.47
8.48							8.48
8.49							8.49

t	SKEWNESS						t
	6	7	8	9	10	11	
8.00			.999999	.999998	.999997	999994	8 00
8 01			.999999	.999998	999997	999994	8 01
8 02			999999	999998	999997	999994	8 02
8.03			.999999	999998	999997	.999995	8 03
8 04			.999999	999998	.999997	.999995	8 04
8 05			999999	999999	.999997	.999995	8.05
8.06			.999999	999999	999997	999995	8.06
8 07			.999999	999999	.999997	.999995	8.07
8.08			999999	999999	999997	.999995	8 08
8.09			.999999	.999999	.999997	.999995	8.09
8 10			.999999	.999999	.999997	.999995	8 10
8.11			.999999	.999999	.999997	.999995	8 11
8 12			.999999	999999	.999997	.999995	8 12
8.13			.999999	.999999	999997	999995	8 13
8.14			999999	999999	999997	999995	8 14
8 15			999999	999999	999998	.999995	8 15
8.16			.999999	999999	999998	999996	8 16
8.17			.999999	999999	999998	.999996	8.17
8.18			.999999	.999999	.999998	.999996	8 18
8.19			.999999	999999	.999998	.999996	8 19
8 20				999999	.999998	999996	8 20
8 21				999999	999998	.999996	8 21
8 22				.999999	999998	999996	8 22
8.23				.999999	.999998	.999996	8 23
8 24				999999	.999998	999996	8.24
8 25				999999	.999998	.999996	8 25
8 26				.999999	.999998	999996	8 26
8.27				.999999	999998	.999996	8.27
8 28				999999	999998	.999996	8 28
8 29				999999	999998	999996	8 29
8.30				.999999	.999998	999996	8 30
8.31				999999	.999998	999997	8 31
8.32				999999	999998	999997	8.32
8 33				999999	.999998	999997	8 33
8 34				999999	.999998	999997	8 34
8 35				.999999	.999998	.999997	8 35
8 36				999999	.999998	.999997	8 36
8.37				999999	999998	.999997	8.37
8.38				.999999	.999998	999997	8 38
8 39				999999	.999998	.999997	8 39
8.40				.999999	.999998	999997	8.40
8.41				999999	999998	999997	8 41
8.42				.999999	.999998	.999997	8.42
8 43				999999	.999998	.999997	8 43
8.44				.999999	.999999	999997	8 44
8.45				999999	999999	999997	8 45
8 46				999999	.999999	999997	8 46
8.47				.999999	.999999	.999997	8 47
8.48				999999	.999999	999997	8.48
8.49				999999	999999	.999997	8.49

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
8.50							8.50
8.51							8.51
8.52							8.52
8.53							8.53
8.54							8.54
8.55							8.55
8.56							8.56
8.57							8.57
8.58							8.58
8.59							8.59
8.60							8.60
8.61							8.61
8.62							8.62
8.63							8.63
8.64							8.64
8.65							8.65
8.66							8.66
8.67							8.67
8.68							8.68
8.69							8.69
8.70							8.70
8.71							8.71
8.72							8.72
8.73							8.73
8.74							8.74
8.75							8.75
8.76							8.76
8.77							8.77
8.78							8.78
8.79							8.79
8.80							8.80
8.81							8.81
8.82							8.82
8.83							8.83
8.84							8.84
8.85							8.85
8.86							8.86
8.87							8.87
8.88							8.88
8.89							8.89
8.90							8.90
8.91							8.91
8.92							8.92
8.93							8.93
8.94							8.94
8.95							8.95
8.96							8.96
8.97							8.97
8.98							8.98
8.99							8.99

t	SKEWNESS						t
	.6	7	.8	.9	1.0	1.1	
8.50				.999999	.999999	.999997	8.50
8.51				.999999	.999999	.999997	8.51
8.52				.999999	.999999	.999998	8.52
8.53				.999999	.999999	.999998	8.53
8.54				.999999	.999999	.999998	8.54
8.55				.999999	.999999	.999998	8.55
8.56				.999999	.999999	.999998	8.56
8.57				.999999	.999999	.999998	8.57
8.58				.999999	.999999	.999998	8.58
8.59				.999999	.999999	.999998	8.59
8.60				.999999	.999999	.999998	8.60
8.61				.999999	.999999	.999998	8.61
8.62				.999999	.999999	.999998	8.62
8.63				.999999	.999999	.999998	8.63
8.64					.999999	.999998	8.64
8.65					.999999	.999998	8.65
8.66					.999999	.999998	8.66
8.67					.999999	.999998	8.67
8.68					.999999	.999998	8.68
8.69					.999999	.999998	8.69
8.70					.999999	.999998	8.70
8.71					.999999	.999998	8.71
8.72					.999999	.999998	8.72
8.73					.999999	.999998	8.73
8.74					.999999	.999998	8.74
8.75					.999999	.999998	8.75
8.76					.999999	.999998	8.76
8.77					.999999	.999998	8.77
8.78					.999999	.999998	8.78
8.79					.999999	.999998	8.79
8.80					.999999	.999998	8.80
8.81					.999999	.999998	8.81
8.82					.999999	.999998	8.82
8.83					.999999	.999998	8.83
8.84					.999999	.999999	8.84
8.85					.999999	.999999	8.85
8.86					.999999	.999999	8.86
8.87					.999999	.999999	8.87
8.88					.999999	.999999	8.88
8.89					.999999	.999999	8.89
8.90					.999999	.999999	8.90
8.91					.999999	.999999	8.91
8.92					.999999	.999999	8.92
8.93					.999999	.999999	8.93
8.94					.999999	.999999	8.94
8.95					.999999	.999999	8.95
8.96					.999999	.999999	8.96
8.97					.999999	.999999	8.97
8.98					.999999	.999999	8.98
8.99					.999999	.999999	8.99

	SKEWNESS						
t	0	1	2	.3	.4	5	t
9.00							9.00
9.01							9.01
9.02							9.02
9.03							9.03
9.04							9.04
9.05							9.05
9.06							9.06
9.07							9.07
9.08							9.08
9.09							9.09
9.10							9.10
9.11							9.11
9.12							9.12
9.13							9.13
9.14							9.14
9.15							9.15
9.16							9.16
9.17							9.17
9.18							9.18
9.19							9.19
9.20							9.20
9.21							9.21
9.22							9.22
9.23							9.23
9.24							9.24
9.25							9.25
9.26							9.26
9.27							9.27
9.28							9.28
9.29							9.29
9.30							9.30
9.31							9.31
9.32							9.32
9.33							9.33
9.34							9.34
9.35							9.35
9.36							9.36
9.37							9.37
9.38							9.38
9.39							9.39
9.40							9.40
9.41							9.41
9.42							9.42
9.43							9.43
9.44							9.44
9.45							9.45
9.46							9.46
9.47							9.47
9.48							9.48
9.49							9.49

t	SKEWNESS .6	.7	.8	.9	1.0	1.1	t
9.00					.999999	.999999	9.00
9.01					.999999	.999999	9.01
9.02					.999999	.999999	9.02
9.03					.999999	.999999	9.03
9.04					.999999	.999999	9.04
9.05					.999999	.999999	9.05
9.06					.999999	.999999	9.06
9.07					.999999	.999999	9.07
9.08						.999999	9.08
9.09						.999999	9.09
9.10						.999999	9.10
9.11						.999999	9.11
9.12						.999999	9.12
9.13						.999999	9.13
9.14						.999999	9.14
9.15						.999999	9.15
9.16						.999999	9.16
9.17						.999999	9.17
9.18						.999999	9.18
9.19						.999999	9.19
9.20						.999999	9.20
9.21						.999999	9.21
9.22						.999999	9.22
9.23						.999999	9.23
9.24						.999999	9.24
9.25						.999999	9.25
9.26						.999999	9.26
9.27						.999999	9.27
9.28						.999999	9.28
9.29						.999999	9.29
9.30						.999999	9.30
9.31						.999999	9.31
9.32						.999999	9.32
9.33						.999999	9.33
9.34						.999999	9.34
9.35						.999999	9.35
9.36						.999999	9.36
9.37						.999999	9.37
9.38						.999999	9.38
9.39						.999999	9.39
9.40						.999999	9.40
9.41						.999999	9.41
9.42						.999999	9.42
9.43						.999999	9.43
9.44						.999999	9.44
9.45						.999999	9.45
9.46						.999999	9.46
9.47						.999999	9.47
9.48						.999999	9.48
9.49						.999999	9.49

	SKEWNESS						
t	.0	1	2	.3	4	.5	t
9.50							9.50
9.51							9.51
9.52							9.52
9.53							9.53
9.54							9.54
9.55							9.55
9.56							9.56
9.57							9.57
9.58							9.58
9.59							9.59
9.60							9.60
9.61							9.61
9.62							9.62
9.63							9.63
9.64							9.64
9.65							9.65
9.66							9.66
9.67							9.67
9.68							9.68
9.69							9.69
9.70							9.70
9.71							9.71
9.72							9.72
9.73							9.73
9.74							9.74
9.75							9.75
9.76							9.76
9.77							9.77
9.78							9.78
9.79							9.79
9.80							9.80
9.81							9.81
9.82							9.82
9.83							9.83
9.84							9.84
9.85							9.85
9.86							9.86
9.87							9.87
9.88							9.88
9.89							9.89
9.90							9.90
9.91							9.91
9.92							9.92
9.93							9.93
9.94							9.94
9.95							9.95
9.96							9.96
9.97							9.97
9.98							9.98
9.99							9.99

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
9.50						.999999	9.50
9.51						.999999	9.51
9.52							9.52
9.53							9.53
9.54							9.54
9.55							9.55
9.56							9.56
9.57							9.57
9.58							9.58
9.59							9.59
9.60							9.60
9.61							9.61
9.62							9.62
9.63							9.63
9.64							9.64
9.65							9.65
9.66							9.66
9.67							9.67
9.68							9.68
9.69							9.69
9.70							9.70
9.71							9.71
9.72							9.72
9.73							9.73
9.74							9.74
9.75							9.75
9.76							9.76
9.77							9.77
9.78							9.78
9.79							9.79
9.80							9.80
9.81							9.81
9.82							9.82
9.83							9.83
9.84							9.84
9.85							9.85
9.86							9.86
9.87							9.87
9.88							9.88
9.89							9.89
9.90							9.90
9.91							9.91
9.92							9.92
9.93							9.93
9.94							9.94
9.95							9.95
9.96							9.96
9.97							9.97
9.98							9.98
9.99							9.99

TABLE II

ORDINATES OF THE STANDARDIZED TYPE III FUNCTION

$$y = y_0 \left(1 + \frac{\alpha_3}{2} t\right)^{\frac{4}{\alpha_3^2} - 1} e^{-\frac{2}{\alpha_3} t}$$

t	SKEWNESS 0	1	.2	.3	.4	5	t
-5.49							-5.49
-5.48							-5.48
-5.47							-5.47
-5.46							-5.46
-5.45							-5.45
-5.44							-5.44
-5.43							-5.43
-5.42							-5.42
-5.41							-5.41
-5.40							-5.40
-5.39							-5.39
-5.38							-5.38
-5.37							-5.37
-5.36							-5.36
-5.35							-5.35
-5.34							-5.34
-5.33							-5.33
-5.32							-5.32
-5.31							-5.31
-5.30							-5.30
-5.29							-5.29
-5.28							-5.28
-5.27							-5.27
-5.26							-5.26
-5.25							-5.25
-5.24							-5.24
-5.23							-5.23
-5.22							-5.22
-5.21	.000001						-5.21
-5.20	.000001						-5.20
-5.19	.000001						-5.19
-5.18	.000001						-5.18
-5.17	.000001						-5.17
-5.16	.000001						-5.16
-5.15	.000001						-5.15
-5.14	.000001						-5.14
-5.13	.000001						-5.13
-5.12	.000001						-5.12
-5.11	.000001						-5.11
-5.10	.000001						-5.10
-5.09	.000001						-5.09
-5.08	.000001						-5.08
-5.07	.000001						-5.07
-5.06	.000001						-5.06
-5.05	.000001						-5.05
-5.04	.000001						-5.04
-5.03	.000001						-5.03
-5.02	.000001						-5.02
-5.01	.000001						-5.01
-5.00	.000001						-5.00

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	t
-5.49							–5.49
-5.48							–5.48
-5.47							–5.47
-5.46							–5.46
-5.45							–5.45
-5.44							–5.44
-5.43							–5.43
-5.42							–5.42
-5.41							–5.41
-5.40							–5.40
-5.39							–5.39
-5.38							–5.38
-5.37							–5.37
-5.36							–5.36
-5.35							–5.35
-5.34							–5.34
-5.33							–5.33
-5.32							–5.32
-5.31							–5.31
-5.30							–5.30
-5.29							–5.29
-5.28							–5.28
-5.27							–5.27
-5.26							–5.26
-5.25							–5.25
-5.24							–5.24
-5.23							–5.23
-5.22							–5.22
-5.21							–5.21
-5.20							–5.20
-5.19							–5.19
-5.18							–5.18
-5.17							–5.17
-5.16							–5.16
-5.15							–5.15
-5.14							–5.14
-5.13							–5.13
-5.12							–5.12
-5.11							–5.11
-5.10							–5.10
-5.09							–5.09
-5.08							–5.08
-5.07							–5.07
-5.06							–5.06
-5.05							–5.05
-5.04							–5.04
-5.03							–5.03
-5.02							–5.02
-5.01							–5.01
-5.00							–5.00

t	SKEWNESS						t
	.0	.1	.2	3	.4	.5	
-4.99	.000002						-4.99
-4.98	.000002						-4.98
-4.97	.000002						-4.97
-4.96	.000002						-4.96
-4.95	.000002						-4.95
-4.94	.000002						-4.94
-4.93	.000002						-4.93
-4.92	.000002						-4.92
-4.91	.000002						-4.91
-4.90	.000002						-4.90
-4.89	.000003						-4.89
-4.88	.000003						-4.88
-4.87	.000003						-4.87
-4.86	.000003						-4.86
-4.85	.000003						-4.85
-4.84	.000003						-4.84
-4.83	.000003						-4.83
-4.82	.000004						-4.82
-4.81	.000004	.000001					-4.81
-4.80	.000004	.000001					-4.80
-4.79	.000004	.000001					-4.79
-4.78	.000004	.000001					-4.78
-4.77	.000005	.000001					-4.77
-4.76	.000005	.000001					-4.76
-4.75	.000005	.000001					-4.75
-4.74	.000005	.000001					-4.74
-4.73	.000006	.000001					-4.73
-4.72	.000006	.000001					-4.72
-4.71	.000006	.000001					-4.71
-4.70	.000006	.000001					-4.70
-4.69	.000007	.000001					-4.69
-4.68	.000007	.000001					-4.68
-4.67	.000007	.000001					-4.67
-4.66	.000008	.000001					-4.66
-4.65	.000008	.000001					-4.65
-4.64	.000008	.000001					-4.64
-4.63	.000009	.000002					-4.63
-4.62	.000009	.000002					-4.62
-4.61	.000010	.000002					-4.61
-4.60	.000010	.000002					-4.60
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-4.57	.000012	.000002					-4.57
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-4.53	.000014	.000003					-4.53
-4.52	.000015	.000003					-4.52
-4.51	.000015	.000003					-4.51
-4.50	.000016	.000003					-4.50

t	SKEWNESS						t
	6	7	8	.9	1.0	1.1	
–4.99							–4.99
–4.98							–4.98
–4.97							–4.97
–4.96							–4.96
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–4.85							–4.85
–4.84							–4.84
–4.83							–4.83
–4.82							–4.82
–4.81							–4.81
–4.80							–4.80
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–4.78							–4.78
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–4.76							–4.76
–4.75							–4.75
–4.74							–4.74
–4.73							–4.73
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–4.54							–4.54
–4.53							–4.53
–4.52							–4.52
–4.51							–4.51
–4.50							–4.50

t	SKEWNESS						t
	.0	.1	.2	3	.4	.5	
–4.49	.000017	000003					–4 49
–4 48	.000018	000004					–4.48
–4 47	.000018	000004					–4.47
–4.46	000019	000004					–4 46
–4 45	000020	.000004					–4.45
–4.44	000021	000005					–4.44
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–4.42	000023	.000005	.000001				–4.42
–4.41	000024	000005	000001				–4.41
–4.40	.000025	.000006	000001				–4.40
–4.39	000026	.000006	000001				–4.39
–4 38	.000027	.000006	.000001				–4.38
-4.37	000028	.000007	.000001				–4.37
–4.36	000030	000007	.000001				–4.36
–4 35	000031	000008	.000001				–4 35
–4 34	.000032	000008	000001				–4.34
–4.33	.000034	.000009	000001				–4.33
–4.32	000035	.000009	.000001				–4.32
–4 31	000037	.000010	.000001				–4.31
–4.30	.000039	.000010	.000001				–4.30
–4 29	000040	.000011	.000001				–4.29
–4.28	.000042	000011	.000001				–4 28
–4 27	.000044	000012	000002				–4.27
–4.26	.000046	.000012	.000002				–4.26
–4 25	.000048	000013	.000002				–4 25
–4 24	.000050	.000014	.000002				–4 24
–4.23	.000052	.000015	000002				–4.23
–4.22	.000054	.000015	000002				–4.22
–4 21	000057	.000016	.000002				–4 21
–4 20	000059	000017	000003				–4.20
–4.19	000062	.000018	.000003				–4.19
–4.18	.000064	.000019	000003				–4.18
–4.17	000067	.000020	.000003				–4.17
–4.16	000070	.000021	.000003				–4.16
–4 15	.000073	.000022	.000004				–4.15
–4.14	.000076	000023	.000004				–4.14
–4 13	000079	.000025	.000004				–4.13
–4.12	.000082	.000026	.000005				–4.12
–4.11	000086	.000027	000005				–4.11
–4 10	000089	000029	000005				–4.10
–4 09	.000093	000030	000006				–4 09
–4.08	000097	000032	000006				–4.08
–4 07	.000101	.000034	.000006				–4.07
–4.06	000105	.000035	000007				–4 06
–4.05	000109	000037	.000007				–4.05
–4 04	.000114	.000039	.000008	.000001			–4.04
–4.03	000119	.000041	.000008	000001			–4.03
–4.02	.000124	.000043	.000009	000001			–4.02
–4.01	.000129	.000045	.000010	000001			–4.01
–4.00	.000134	.000048	.000010	000001			–4.00

	SKEWNESS						
t	.6	.7	.8	.9	1.0	1.1	t
-4.49							-4.49
-4.48							-4.48
-4.47							-4.47
-4.46							-4.46
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-4.43							-4.43
-4.42							-4.42
-4.41							-4.41
-4.40							-4.40
-4.39							-4.39
-4.38							-4.38
-4.37							-4.37
-4.36							-4.36
-4.35							-4.35
-4.34							-4.34
-4.33							-4.33
-4.32							-4.32
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-4.25							-4.25
-4.24							-4.24
-4.23							-4.23
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-4.15							-4.15
-4.14							-4.14
-4.13							-4.13
-4.12							-4.12
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-4.10							-4.10
-4.09							-4.09
-4.08							-4.08
-4.07							-4.07
-4.06							-4.06
-4.05							-4.05
-4.04							-4.04
-4.03							-4.03
-4.02							-4.02
-4.01							-4.01
-4.00							-4.00

t	SKEWNESS						t
	0	.1	2	.3	4	5	
−3.99	.000139	000050	.000011	000001			−3 99
−3.98	.000144	000052	.000012	000001			−3 98
−3.97	000151	.000055	000012	000001			−3.97
−3.96	000157	000058	.000013	000001			−3.96
−3.95	000163	.000061	.000014	.000001			−3.95
−3 94	.000170	000064	.000015	.000001			−3 94
−3 93	000177	000067	.000016	.000002			−3.93
−3 92	000184	000070	.000017	000002			−3.92
−3.91	.000191	.000074	.000018	000002			−3.91
−3 90	000199	.000077	000019	000002			−3.90
−3.89	000207	000081	000021	000002			−3.89
−3 88	000215	.000085	000022	000002			−3 88
−3.87	.000223	.000089	000023	.000003			−3 87
−3.86	.000232	000094	000025	.000003			−3 86
−3 85	000241	000098	.000026	.000003			−3 85
−3 84	000251	000103	000028	000003			−3 84
−3 83	.000260	000108	.000030	.000004			−3 83
−3.82	000271	.000113	.000032	000004			−3.82
−3.81	000281	.000118	000033	.000004			−3 81
−3.80	000292	000124	.000036	.000005			−3.80
−3 79	.000303	000130	000038	000005			−3.79
−3 78	000315	000136	.000040	000006			−3.78
−3 77	.000327	.000142	.000042	.000006			−3.77
−3 76	000340	.000149	.000045	.000007			−3.76
−3 75	.000353	000156	000048	000007			−3.75
−3 74	.000366	000163	.000051	.000008			−3.74
−3.73	.000380	000171	.000054	.000009			−3 73
−3.72	.000394	000178	000057	.000009			−3.72
−3.71	000409	.000187	000060	000010			−3.71
−3.70	000425	.000195	.000064	.000011			−3 70
−3 69	.000441	000204	000067	000012			−3.69
−3.68	000457	000213	000071	.000013	.000001		−3.68
−3 67	000474	.000223	.000076	.000014	.000001		−3 67
−3 66	.000492	.000233	000080	.000015	.000001		−3.66
−3 65	.000510	000244	000085	.000016	000001		−3 65
−3.64	000529	000255	000089	.000017	000001		−3.64
−3.63	.000549	000266	000094	000019	000001		−3 63
−3 62	.000569	000278	000100	000020	000001		−3.62
−3.61	000590	000290	.000105	000022	.000001		−3.61
−3 60	000612	000303	.000111	000024	.000001		−3.60
−3 59	.000634	000317	000118	.000025	.000002		−3 59
−3.58	.999657	000331	000124	000027	000002		−3.58
−3 57	.000681	000345	.000131	.000029	000002		−3.57
−3.56	000706	.000360	.000138	000032	.000002		−3.56
−3 55	.000732	000376	.000146	.000034	000003		−3.55
−3 54	.000758	000392	.000154	.000037	.000003		−3.54
−3.53	.000785	.000409	000162	.000039	000003		−3.53
−3.52	000814	000427	.000171	.000042	000004		−3.52
−3 51	000843	.000445	.000181	000045	000004		−3 51
−3.50	000873	.000464	000190	.000049	000004		−3.50

	SKEWNESS						
t	6	.7	8	.9	1 0	1 1	t
-3.99							-3.99
-3.98							-3.98
-3.97							-3.97
-3.96							-3.96
-3.95							-3.95
-3.94							-3.94
-3.93							-3.93
-3.92							-3.92
-3.91							-3.91
-3.90							-3.90
-3.89							-3.89
-3.88							-3.88
-3.87							-3.87
-3.86							-3.86
-3.85							-3.85
-3.84							-3.84
-3.83							-3.83
-3.82							-3.82
-3.81							-3.81
-3.80							-3.80
-3.79							-3.79
-3.78							-3.78
-3.77							-3.77
-3.76							-3.76
-3.75							-3.75
-3.74							-3.74
-3.73							-3.73
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-3.71							-3.71
-3.70							-3.70
-3.69							-3.69
-3.68							-3.68
-3.67							-3.67
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-3.58							-3.58
-3.57							-3.57
-3.56							-3.56
-3.55							-3.55
-3.54							-3.54
-3.53							-3.53
-3.52							-3.52
-3.51							-3.51
-3.50							-3.50

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
-3.49	.000904	.000484	.000200	.000052	.000005		-3.49
-3.48	.000936	.000504	.000211	.000056	.000006		-3.48
-3.47	.000969	.000526	.000222	.000060	.000006		-3.47
-3.46	.001003	.000548	.000234	.000064	.000007		-3.46
-3.45	.001038	.000571	.000246	.000069	.000008		-3.45
-3.44	.001075	.000595	.000259	.000074	.000008		-3.44
-3.43	.001112	.000620	.000273	.000079	.000009		-3.43
-3.42	.001151	.000645	.000287	.000085	.000010		-3.42
-3.41	.001191	.000672	.000302	.000090	.000012		-3.41
-3.40	.001232	.000700	.000317	.000097	.000013		-3.40
-3.39	.001275	.000729	.000333	.000103	.000014		-3.39
-3.38	.001319	.000758	.000350	.000110	.000016		-3.38
-3.37	.001364	.000789	.000368	.000118	.000017		-3.37
-3.36	.001411	.000822	.000387	.000126	.000019		-3.36
-3.35	.001459	.000855	.000406	.000134	.000021		-3.35
-3.34	.001508	.000889	.000427	.000143	.000023		-3.34
-3.33	.001560	.000925	.000448	.000152	.000025	.000001	-3.33
-3.32	.001612	.000962	.000470	.000162	.000028	.000001	-3.32
-3.31	.001667	.001001	.000493	.000173	.000030	.000001	-3.31
-3.30	.001723	.001040	.000517	.000184	.000033	.000001	-3.30
-3.29	.001780	.001082	.000542	.000196	.000036	.000001	-3.29
-3.28	.001840	.001124	.000569	.000208	.000040	.000001	-3.28
-3.27	.001901	.001168	.000596	.000221	.000043	.000002	-3.27
-3.26	.001964	.001214	.000625	.000235	.000047	.000002	-3.26
-3.25	.002029	.001262	.000655	.000250	.000052	.000002	-3.25
-3.24	.002096	.001311	.000686	.000265	.000057	.000003	-3.24
-3.23	.002165	.001361	.000719	.000282	.000062	.000003	-3.23
-3.22	.002236	.001414	.000753	.000299	.000067	.000003	-3.22
-3.21	.002309	.001468	.000788	.000317	.000073	.000004	-3.21
-3.20	.002384	.001524	.000825	.000336	.000079	.000005	-3.20
-3.19	.002462	.001583	.000863	.000357	.000086	.000005	-3.19
-3.18	.002541	.001643	.000903	.000378	.000094	.000006	-3.18
-3.17	.002623	.001705	.000945	.000401	.000102	.000007	-3.17
-3.16	.002707	.001769	.000988	.000424	.000110	.000008	-3.16
-3.15	.002794	.001836	.001033	.000449	.000119	.000010	-3.15
-3.14	.002884	.001904	.001080	.000475	.000129	.000011	-3.14
-3.13	.002975	.001975	.001129	.000503	.000140	.000013	-3.13
-3.12	.003070	.002048	.001180	.000532	.000151	.000014	-3.12
-3.11	.003167	.002124	.001233	.000562	.000163	.000016	-3.11
-3.10	.003267	.002202	.001287	.000594	.000176	.000018	-3.10
-3.09	.003370	.002283	.001345	.000628	.000190	.000021	-3.09
-3.08	.003475	.002367	.001404	.000663	.000205	.000024	-3.08
-3.07	.003584	.002453	.001465	.000700	.000221	.000027	-3.07
-3.06	.003695	.002542	.001529	.000739	.000238	.000030	-3.06
-3.05	.003810	.002633	.001596	.000780	.000256	.000034	-3.05
-3.04	.003928	.002728	.001665	.000822	.000275	.000038	-3.04
-3.03	.004049	.002826	.001737	.000867	.000296	.000043	-3.03
-3.02	.004173	.002927	.001811	.000914	.000318	.000048	-3.02
-3.01	.004301	.003031	.001888	.000963	.000341	.000054	-3.01
-3.00	.004432	.003138	.001968	.001014	.000366	.000060	-3.00

	SKEWNESS						
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-3.49							-3.49
-3.48							-3.48
-3.47							-3.47
-3.46							-3.46
-3.45							-3.45
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-3.37							-3.37
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-3.33							-3.33
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-3.08							-3.08
-3.07							-3.07
-3.06							-3.06
-3.05							-3.05
-3.04							-3.04
-3.03							-3.03
-3.02							-3.02
-3.01	.000001						-3.01
-3.00	.000001						-3.00

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
−2.99	.004567	.003248	.002051	.001068	.000392	.000067	−2.99
−2.98	.004705	.003363	.002137	.001124	.000420	.000075	−2.98
−2.97	.004847	.003480	.002227	.001183	.000450	.000083	−2.97
−2.96	.004993	.003601	.002319	.001245	.000482	.000092	−2.96
−2.95	.005143	.003726	.002415	.001309	.000515	.000102	−2.95
−2.94	.005296	.003855	.002515	.001376	.000551	.000113	−2.94
−2.93	.005454	.003987	.002618	.001447	.000589	.000125	−2.93
−2.92	.005616	.004124	.002724	.001520	.000629	.000139	−2.92
−2.91	.005782	.004265	.002835	.001596	.000671	.000153	−2.91
−2.90	.005953	.004409	.002949	.001676	.000716	.000168	−2.90
−2.89	.006127	.004559	.003067	.001760	.000763	.000185	−2.89
−2.88	.006307	.004712	.003190	.001847	.000813	.000204	−2.88
−2.87	.006491	.004870	.003316	.001937	.000866	.000224	−2.87
−2.86	.006679	.005033	.003447	.002032	.000922	.000245	−2.86
−2.85	.006873	.005200	.003583	.002130	.000980	.000269	−2.85
−2.84	.007071	.005372	.003723	.002233	.001042	.000294	−2.84
−2.83	.007274	.005550	.003868	.002339	.001108	.000321	−2.83
−2.82	.007483	.005732	.004018	.002450	.001177	.000351	−2.82
−2.81	.007697	.005919	.004172	.002566	.001249	.000382	−2.81
−2.80	.007915	.006112	.004332	.002686	.001326	.000416	−2.80
−2.79	.008140	.006310	.004497	.002812	.001406	.000453	−2.79
−2.78	.008370	.006514	.004668	.002942	.001491	.000493	−2.78
−2.77	.008605	.006723	.004844	.003077	.001579	.000535	−2.77
−2.76	.008847	.006939	.005026	.003218	.001673	.000580	−2.76
−2.75	.009094	.007160	.005213	.003364	.001771	.000629	−2.75
−2.74	.009347	.007387	.005407	.003516	.001873	.000681	−2.74
−2.73	.009606	.007620	.005607	.003673	.001981	.000737	−2.73
−2.72	.009871	.007860	.005813	.003837	.002094	.000796	−2.72
−2.71	.010143	.008106	.006025	.004006	.002213	.000860	−2.71
−2.70	.010421	.008359	.006244	.004182	.002337	.000927	−2.70
−2.69	.010706	.008618	.006470	.004365	.002467	.001000	−2.69
−2.68	.010997	.008885	.006703	.004554	.002603	.001076	−2.68
−2.67	.011295	.009158	.006942	.004750	.002745	.001158	−2.67
−2.66	.011600	.009439	.007189	.004953	.002894	.001245	−2.66
−2.65	.011912	.009726	.007444	.005164	.003049	.001337	−2.65
−2.64	.012232	.010022	.007706	.005382	.003212	.001435	−2.64
−2.63	.012558	.010324	.007976	.005608	.003382	.001539	−2.63
−2.62	.012892	.010635	.008254	.005842	.003559	.001649	−2.62
−2.61	.013234	.010954	.008540	.006083	.003743	.001766	−2.61
−2.60	.013583	.011280	.008834	.006333	.003936	.001889	−2.60
−2.59	.013940	.011615	.009136	.006592	.004137	.002020	−2.59
−2.58	.014305	.011958	.009448	.006860	.004347	.002158	−2.58
−2.57	.014678	.012310	.009768	.007136	.004565	.002303	−2.57
−2.56	.015060	.012670	.010097	.007422	.004792	.002457	−2.56
−2.55	.015449	.013039	.010436	.007717	.005028	.002619	−2.55
−2.54	.015848	.013417	.010784	.008022	.005274	.002789	−2.54
−2.53	.016254	.013805	.011141	.008336	.005530	.002969	−2.53
−2.52	.016670	.014201	.011509	.008661	.005796	.003158	−2.52
−2.51	.017095	.014607	.011886	.008997	.006072	.003356	−2.51
−2.50	.017528	.015023	.012274	.009342	.006360	.003565	−2.50

t	SKEWNESS						t
	6	7	8	9	10	11	
-2.99	.000001						-2.99
-2.98	.000001						-2.98
-2.97	.000001						-2.97
-2.96	.000002						-2.96
-2.95	.000002						-2.95
-2.94	.000003						-2.94
-2.93	.000004						-2.93
-2.92	.000005						-2.92
-2.91	.000006						-2.91
-2.90	.000007						-2.90
-2.89	.000008						-2.89
-2.88	.000010						-2.88
-2.87	.000012						-2.87
-2.86	.000015						-2.86
-2.85	.000018						-2.85
-2.84	.000021						-2.84
-2.83	.000025						-2.83
-2.82	.000029						-2.82
-2.81	.000034						-2.81
-2.80	.000040						-2.80
-2.79	.000047						-2.79
-2.78	.000055						-2.78
-2.77	.000063						-2.77
-2.76	.000073						-2.76
-2.75	.000084						-2.75
-2.74	.000097						-2.74
-2.73	.000111						-2.73
-2.72	.000126						-2.72
-2.71	.000144						-2.71
-2.70	.000164						-2.70
-2.69	.000185	.000001					-2.69
-2.68	.000210	.000002					-2.68
-2.67	.000236	.000003					-2.67
-2.66	.000266	.000004					-2.66
-2.65	.000299	.000005					-2.65
-2.64	.000334	.000007					-2.64
-2.63	.000374	.000010					-2.63
-2.62	.000417	.000013					-2.62
-2.61	.000464	.000017					-2.61
-2.60	.000516	.000021					-2.60
-2.59	.000572	.000027					-2.59
-2.58	.000634	.000035					-2.58
-2.57	.000700	.000043					-2.57
-2.56	.000773	.000054					-2.56
-2.55	.000851	.000066					-2.55
-2.54	.000936	.000081					-2.54
-2.53	.001027	.000099					-2.53
-2.52	.001126	.000119					-2.52
-2.51	.001232	.000142					-2.51
-2.50	.001347	.000170					-2.50

t	SKEWNESS						t
	0	1	.2	3	4	.5	
-2.49	017971	015449	.012672	009699	.006658	003784	-2.49
-2.48	.018423	.015884	013080	010067	.006967	004014	-2.48
-2.47	.018885	016330	.013500	010446	.007288	004255	-2.47
-2.46	019356	016786	013930	.010837	.007621	.004508	-2.46
-2.45	019837	.017253	014372	.011240	.007966	004773	-2.45
-2.44	.020328	017730	.014825	.011655	008324	005050	-2.44
-2.43	.020829	.018218	.015290	.012082	008695	005341	-2.43
-2.42	.021341	018717	015766	.012522	.009079	005644	-2.42
-2.41	021862	019227	016255	.012974	009476	005961	-2.41
-2.40	.022395	.019749	.016756	.013440	009887	006292	-2.40
-2.39	.022937	.020282	.017269	.013919	.010313	.006637	-2.39
-2.38	023491	020827	.017795	.014412	.010753	.006998	-2.38
-2.37	024056	.021384	.018334	.014919	011208	007374	-2.37
-2.36	024631	.021953	.018886	.015440	.011678	007765	-2.36
-2.35	.025218	.022534	019451	.015976	012163	008173	-2.35
-2.34	.025817	.023127	020030	016526	.012664	008597	-2.34
-2.33	026426	.023733	.020623	.017091	013182	.009039	-2.33
-2.32	.027048	.024352	.021230	017672	013716	.009498	-2.32
-2.31	.027682	024984	021850	018267	.014267	.009975	-2.31
-2.30	.028327	025629	.022486	018879	014835	010471	-2.30
-2.29	028985	.026287	.023135	019507	.015421	.010986	-2.29
-2.28	.029655	.026959	023800	.020151	016025	.011520	-2.28
-2.27	.030337	027644	024480	020812	.016647	012074	-2.27
-2.26	031032	.028344	025174	021489	.017287	012648	-2.26
-2.25	031740	029057	025885	.022184	017947	.013242	-2.25
-2.24	.032460	029784	.026611	022896	.018625	013859	-2.24
-2.23	.033194	.030526	.027352	023625	019323	.014496	-2.23
-2.22	033941	031282	028110	.024372	.020041	015156	-2.22
-2.21	034701	032053	.028884	025138	020780	.015838	-2.21
-2.20	.035475	.032839	.029675	.025922	.021538	016544	-2.20
-2.19	036262	.033640	030482	.026724	.022318	.017272	-2.19
-2.18	037063	.034456	031307	027546	023119	018025	-2.18
-2.17	.037878	035287	032148	.028386	023942	018801	-2.17
-2.16	038707	.036134	033006	.029246	.024786	.019603	-2.16
-2.15	.039550	036997	.033883	030125	025652	020429	-2.15
-2.14	040408	037875	.034776	.031025	.026541	.021282	-2.14
-2.13	.041280	038770	.035688	.031944	027453	.022160	-2.13
-2.12	042166	.039680	.036617	032884	028387	.023064	-2.12
-2.11	.043067	040607	.037565	.033844	029345	.023995	-2.11
-2.10	043984	041551	.038531	034825	030327	.024953	-2.10
-2.09	044915	042511	039516	.035827	.031332	025939	-2.09
-2.08	.045861	.043488	040520	.036850	.032362	.026952	-2.08
-2.07	046823	044481	.041542	037894	033416	.027994	-2.07
-2.06	047800	045492	042584	038960	.034495	.029064	-2.06
-2.05	048792	046520	.043645	040048	.035598	.030164	-2.05
-2.04	049800	.047566	.044725	041158	.036727	.031292	-2.04
-2.03	.050824	.048628	045825	042290	037881	.032450	-2.03
-2.02	.051864	.049709	.046944	.043444	.039061	.033638	-2.02
-2.01	.052919	050807	048084	044621	.040267	.034856	-2.01
-2.00	053991	.051923	.049243	.045820	.041499	.036104	-2.00

	SKEWNESS						
t	.6	7	8	.9	1.0	1 1	t
–2.49	001470	000201					–2.49
–2.48	.001601	.000237					–2.48
–2 47	001742	000277					–2 47
–2.46	001893	000324					–2 46
–2 45	.002055	.000376					–2 45
–2.44	.002227	000435	000001				–2 44
–2.43	.002411	000501	.000001				–2.43
–2 42	002606	000574	.000002				–2.42
–2.41	002814	000656	000004				–2 41
–2 40	.003035	.000747	.000007				–2.40
–2 39	003270	.000848	.000012				–2 39
–2.38	003518	000959	.000018				–2 38
–2 37	.003781	001081	000027				–2 37
–2 36	004060	.001215	.000038				–2.36
–2 35	.004355	001362	000054				–2 35
–2 34	004665	.001523	000074				–2 34
–2 33	004993	001697	.000099				–2.33
–2 32	.005339	.001887	.000130				–2 32
–2.31	.005703	.002093	000169				–2.31
–2.30	.006086	002316	.000216				–2 30
–2.29	006489	002557	000272				–2 29
–2 28	006912	002816	000338				–2 28
–2 27	.007355	003095	000417				–2 27
–2 26	007820	003395	000508				–2.26
–2 25	008307	003716	.000614				–2 25
–2 24	008817	004060	.000736				–2.24
–2 23	009351	.004427	.000875				–2 23
–2.22	.009908	.004818	001032				–2.22
–2 21	010490	005235	.001211	.000001			–2.21
–2.20	011097	.005678	001411	000001			–2 20
–2 19	011730	006149	001634	.000003			–2 19
–2 18	012390	006648	001883	000008			–2 18
–2.17	.013077	007176	.002159	000019			–2.17
–2.16	013792	.007734	.002463	.000037			–2.16
–2 15	014535	008324	.002797	000064			–2 15
–2.14	.015307	008945	.003162	000105			–2 14
–2 13	016108	009600	.003561	000161			–2 13
–2.12	016940	010288	003995	000236			–2.12
–2.11	.017803	011012	004466	.000334			–2 11
–2.10	018696	.011771	.004975	.000457			–2 10
–2.09	.019622	012566	005524	.000609			–2.09
–2 08	.020580	.013400	006114	.000793			–2 08
–2.07	021570	.014271	.006747	.001014			–2.07
–2 06	.022595	.015182	.007425	001274			–2 06
–2 05	.023653	.016133	.008148	.001577			–2 05
–2.04	.024745	.017124	.008919	.001926			–2.04
–2 03	025872	018157	.009739	002325			–2 03
–2.02	.027035	.019231	.010608	.002777			–2.02
–2.01	.028233	.020349	.011529	003284			–2.01
–2.00	.029467	021510	.012503	.003851			–2 00

t	SKEWNESS						t
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−1.99	.055079	053057	050423	047043	.042757	037383	−1.99
−1.98	056183	054209	051623	048288	044042	038693	−1.98
−1.97	057304	055379	052843	.049557	045353	.040034	−1.97
−1.96	058441	056568	054084	050849	046691	041407	−1.96
−1.95	.059595	057775	.055346	052164	.048056	.042811	−1.95
−1.94	060765	059001	056628	.053503	049448	044247	−1.94
−1.93	.061952	060245	057931	054866	050868	045715	−1.93
−1.92	063157	061508	059255	056252	.052315	047215	−1.92
−1.91	064378	.062790	060601	057663	053790	048748	−1.91
−1.90	.065616	064090	061967	.059097	055292	.050313	−1.90
−1.89	066871	065410	063355	.060556	.056822	.051910	−1.89
−1.88	068144	066749	064764	.062038	058380	.053540	−1.88
−1.87	069433	068107	066194	.063545	059965	.055203	−1.87
−1.86	070740	.069484	067616	.065076	.061579	.056899	−1.86
−1.85	072065	070880	069119	066632	063221	.058628	−1.85
−1.84	073407	072295	070614	.068212	064891	.060390	−1.84
−1.83	074766	073730	.072131	.069817	066588	062184	−1.83
−1.82	076143	.075184	.073669	071446	068315	064012	−1.82
−1.81	077538	076658	075229	073099	070069	065872	−1.81
−1.80	078950	078151	076810	074777	071851	067766	−1.80
−1.79	080380	.079664	078414	076479	073661	069692	−1.79
−1.78	081828	081196	080039	078206	075500	071651	−1.78
−1.77	.083293	082748	081685	.079957	.077366	.073642	−1.77
−1.76	084776	.084319	.083354	081733	079260	075666	−1.76
−1.75	.086277	.085910	085044	083533	081182	077723	−1.75
−1.74	.087796	.087520	.086755	085358	083132	079811	−1.74
−1.73	089333	.089150	088489	087206	085109	081932	−1.73
−1.72	090887	090799	.090244	.089079	.087114	.084085	−1.72
−1.71	092459	.092468	092020	090976	089146	086269	−1.71
−1.70	.094049	.094156	.093818	092897	.091205	088485	−1.70
−1.69	.095657	095864	095637	.094841	093292	.090732	−1.69
−1.68	097282	.097591	.097478	.096810	.095405	093009	−1.68
−1.67	098925	099337	.099339	.098802	097545	095318	−1.67
−1.66	100586	101102	.101222	.100817	099711	097657	−1.66
−1.65	102265	.102887	.103126	.102856	101903	.100025	−1.65
−1.64	.103961	.104691	105051	104918	104121	.102424	−1.64
−1.63	105675	106514	106996	107002	106365	.104851	−1.63
−1.62	107406	108355	.108962	109110	.108635	107308	−1.62
−1.61	109155	.110216	.110949	111240	110929	109793	−1.61
−1.60	110921	.112095	112956	.113392	113248	112306	−1.60
−1.59	112704	.113993	114983	115567	.115592	114846	−1.59
−1.58	114505	.115909	117030	.117763	117960	.117414	−1.58
−1.57	116323	117844	.119097	119981	.120352	120009	−1.57
−1.56	118157	119797	.121184	122220	122767	.122629	−1.56
−1.55	120009	.121768	.123290	.124480	125205	.125275	−1.55
−1.54	.121878	123757	.125415	.126761	127666	127947	−1.54
−1.53	123763	.125763	.127559	.129062	.130150	130643	−1.53
−1.52	125665	127787	129722	.131384	.132655	.133363	−1.52
−1.51	127583	.129829	.131903	.133725	.135181	136106	−1.51
−1.50	.129518	131888	134103	136086	.137729	138872	−1.50

t	SKEWNESS						t
	6	7	8	9	1.0	1.1	
−1.99	.030738	.022715	.013530	.004479	.000003		−1.99
−1.98	.032045	.023965	.014612	.005171	.000020		−1.98
−1.97	.033390	.025260	.015750	.005931	.000068		−1.97
−1.96	.034772	.026601	.016945	.006761	.000158		−1.96
−1.95	.036192	.027988	.018198	.007662	.000302		−1.95
−1.94	.037650	.029422	.019510	.008638	.000511		−1.94
−1.93	.039146	.030903	.020881	.009690	.000795		−1.93
−1.92	.040681	.032432	.022313	.010820	.001163		−1.92
−1.91	.042255	.034008	.023805	.012030	.001624		−1.91
−1.90	.043868	.035633	.025359	.013320	.002183		−1.90
−1.89	.045520	.037306	.026975	.014694	.002848		−1.89
−1.88	.047211	.039027	.028654	.016152	.003625		−1.88
−1.87	.048941	.040797	.030395	.017694	.004517		−1.87
−1.86	.050711	.042617	.032200	.019322	.005530		−1.86
−1.85	.052521	.044485	.034068	.021037	.006667		−1.85
−1.84	.054370	.046403	.036000	.022839	.007931		−1.84
−1.83	.056259	.048370	.037995	.024729	.009325		−1.83
−1.82	.058187	.050385	.040054	.026706	.010850		−1.82
−1.81	.060155	.052450	.042177	.028771	.012508	.000041	−1.81
−1.80	.062162	.054564	.044364	.030925	.014300	.000251	−1.80
−1.79	.064209	.056727	.046614	.033166	.016226	.000677	−1.79
−1.78	.066295	.058938	.048927	.035494	.018287	.001339	−1.78
−1.77	.068420	.061198	.051302	.037910	.020482	.002249	−1.77
−1.76	.070584	.063506	.053740	.040412	.022811	.003411	−1.76
−1.75	.072787	.065862	.056241	.043000	.025272	.004829	−1.75
−1.74	.075028	.068265	.058802	.045673	.027865	.006502	−1.74
−1.73	.077307	.070715	.061424	.048430	.030587	.008427	−1.73
−1.72	.079624	.073212	.064107	.051270	.033438	.010600	−1.72
−1.71	.081979	.075755	.066848	.054192	.036414	.013018	−1.71
−1.70	.084371	.078344	.069649	.057194	.039514	.015674	−1.70
−1.69	.086800	.080978	.072507	.060276	.042736	.018562	−1.69
−1.68	.089266	.083656	.075422	.063435	.046076	.021675	−1.68
−1.67	.091767	.086379	.078393	.066671	.049531	.025005	−1.67
−1.66	.094304	.089144	.081419	.069981	.053099	.028545	−1.66
−1.65	.096877	.091953	.084499	.073364	.056776	.032287	−1.65
−1.64	.099484	.094803	.087632	.076819	.060560	.036221	−1.64
−1.63	.102125	.097694	.090817	.080342	.064446	.040341	−1.63
−1.62	.104800	.100626	.094052	.083933	.068431	.044638	−1.62
−1.61	.107507	.103597	.097336	.087589	.072512	.049102	−1.61
−1.60	.110248	.106606	.100668	.091308	.076685	.053726	−1.60
−1.59	.113020	.109653	.104046	.095088	.080947	.058501	−1.59
−1.58	.115823	.112737	.107470	.098927	.085292	.063419	−1.58
−1.57	.118656	.115857	.110938	.102824	.089718	.068472	−1.57
−1.56	.121520	.119012	.114448	.106774	.094221	.073650	−1.56
−1.55	.124412	.122201	.117999	.110777	.098796	.078946	−1.55
−1.54	.127333	.125422	.121590	.114830	.103441	.084352	−1.54
−1.53	.130282	.128676	.125218	.118930	.108150	.089859	−1.53
−1.52	.133257	.131960	.128883	.123076	.112920	.095461	−1.52
−1.51	.136258	.135273	.132582	.127264	.117747	.101149	−1.51
−1.50	.139285	.138615	.136315	.131493	.122626	.106916	−1.50

t	SKEWNESS						t
	.0	.1	.2	3	.4	.5	
−1.49	.131468	133964	136321	138466	140297	.141661	−1.49
−1.48	133435	.136057	138556	.140865	142885	.144471	−1 48
−1 47	.135418	.138166	.140809	.143282	145492	147302	−1 47
−1.46	137417	.140292	143079	.145718	148119	150153	−1 46
−1.45	139431	142434	.145366	.148171	.150764	.153024	−1 45
−1.44	.141460	144592	.147670	.150641	.153427	.155914	−1.44
−1 43	143505	.146765	149989	153128	.156108	158822	−1.43
−1.42	.145564	148954	.152325	.155631	158806	.161747	−1.42
−1.41	.147639	151158	154676	158636	161520	.164690	−1.41
−1 40	149727	.153377	.157043	.160686	164249	167648	−1.40
−1 39	.151831	.155611	.159424	.163235	.166994	.170621	−1.39
−1.38	153948	157859	.161820	.165800	.169754	173609	−1.38
−1.37	.156080	160121	164230	.168379	.172528	.176611	−1.37
−1.36	.158225	162397	.166653	.170971	.175315	.179626	−1.36
−1 35	160383	164686	169090	.173576	.178114	.182653	−1.35
−1.34	.162555	.166989	171540	.176194	.180926	.185691	−1 34
−1.33	164740	169304	.174002	178825	.183750	188740	−1.33
−1.32	.166937	171632	.176477	181466	.186584	191799	−1 32
−1 31	.169147	.173971	.178963	.184119	.189428	194866	−1 31
−1.30	.171369	176323	181461	186782	192282	.197941	−1.30
−1.29	173602	.178686	183969	189455	.195144	.201024	−1.29
−1 28	175847	.181060	186487	.192138	.198015	204113	−1 28
−1.27	178104	.183444	189016	194829	.200893	.207207	−1.27
−1.26	180371	185839	191554	.197529	.203777	.210306	−1 26
−1 25	182649	188244	.194100	.200236	206668	213408	−1.25
−1 24	.184937	190658	.196656	202951	.209564	.216514	−1 24
−1 23	187235	193081	.199219	205671	.212464	219621	−1.23
−1.22	.189543	195513	201789	.208398	.215368	.222729	−1.22
−1.21	191860	.197953	.204367	.211130	.218276	.225838	−1.21
−1 20	.194186	.200401	.206951	.213866	.221185	.228946	−1 20
−1.19	196520	.202857	.209540	.216607	.224096	.232052	−1.19
−1.18	.198863	.205319	212135	.219350	.227008	235155	−1 18
−1 17	201214	207788	.214735	.222097	.229919	238256	−1.17
−1 16	.203571	.210262	217339	.224845	232830	.241352	−1 16
−1 15	205936	.212743	.219947	227595	.235740	.244442	−1.15
−1.14	208308	215228	.222558	.230346	.238647	247527	−1.14
−1.13	210686	.217718	225172	.233097	.241551	.250604	−1.13
−1.12	.213069	.220212	227788	.235847	.244452	.253674	−1.12
−1.11	.215458	.222710	.230405	.238596	.247347	256734	−1 11
−1.10	217852	225211	.233023	241343	250238	.259785	−1.10
−1.09	.220251	.227715	.235641	.244087	.253122	262825	−1.09
−1 08	222653	.230220	238259	.246828	.255999	265854	−1.08
−1 07	225060	.232728	240876	.249566	.258868	268871	−1.07
−1.06	.227470	235236	.243492	.252298	261729	.271874	−1 06
−1.05	.229882	.237745	246105	255025	.264581	.274863	−1.05
−1.04	232297	.240254	248716	.257746	.267423	.277836	−1.04
−1.03	234714	.242763	.251324	.260461	270253	.280794	−1.03
−1 02	.237132	.245271	.253927	.263168	.273073	.283736	−1.02
−1 01	.239551	.247777	.256526	.265867	275879	.286659	−1.01
−1.00	.241971	250281	.259120	268557	.278673	.289564	−1.00

	SKEWNESS						
t	.6	7	.8	.9	1.0	1 1	t
-1.49	.142336	141985	140079	.135760	.127555	.112755	-1 49
-1.48	.145411	145380	143873	.140062	132530	118658	-1 48
-1.47	.148508	148801	.147695	.144398	137545	.124618	-1.47
-1 46	.151627	.152245	.151544	.148764	.142598	130629	-1.46
-1 45	.154767	155712	155418	153158	147684	.136684	-1 45
-1 44	.157927	.159201	.159314	157578	.152800	.142777	-1.44
-1 43	161106	162709	163233	162021	.157942	148900	-1.43
-1 42	164303	.166236	.167170	.166485	.163106	155049	-1.42
-1 41	167517	.169781	.171126	.170967	.168290	161218	-1 41
-1.40	.170748	173342	.175098	175465	.173488	.167399	-1.40
-1 39	173994	176918	.179085	.179977	.178698	173589	-1 39
-1.38	177254	180508	.183084	.184499	183916	179781	1 38
-1 37	.180527	184110	.187094	.189030	.189138	185970	-1 37
-1 36	.183812	187724	191113	.193568	194362	.192152	-1.36
-1 35	187109	191347	195140	198109	199584	198321	-1.35
-1 34	190416	194978	199173	202652	.204801	204473	-1 34
-1.33	.193732	.198616	.203210	207195	210009	210604	-1.33
-1 32	197056	.202261	.207249	.211734	.215206	.216708	-1 32
-1 31	200387	.205909	211288	216268	220389	222782	-1.31
-1.30	.203724	209561	215327	220795	225554	.228822	-1.30
-1 29	207066	213215	219362	225312	230699	.234824	-1 29
-1 28	210412	216869	.223394	229818	.235821	240785	-1.28
-1.27	.213761	220522	.227419	234309	.240917	.246700	-1.27
-1 26	217111	224173	231436	.238785	245985	252566	-1 26
-1 25	220462	.227820	.235444	243242	.251021	258381	-1.25
-1 24	223813	231463	239441	247680	.256025	264141	-1.24
-1 23	227162	.235099	.243425	252095	260992	269843	-1.23
-1 22	.230509	238728	247394	.256487	265921	.275484	-1.22
-1 21	233852	.242348	251348	.260853	270810	.281062	-1.21
-1 20	.237190	245958	255285	265191	275656	.286574	-1.20
-1.19	240522	249556	259203	269500	.280457	.292019	-1 19
-1.18	243848	253142	263100	273777	.285212	.297392	-1 18
-1 17	.247165	256715	266976	.278022	.289917	302694	-1.17
-1 16	250474	260272	270828	282232	294572	.307920	-1.16
-1 15	.253773	.263813	274655	286406	.299175	.313070	-1 15
-1.14	257060	267336	278457	290541	303723	.318142	-1 14
-1.13	260335	270840	282231	294638	308215	323134	-1.13
-1.12	263598	274325	285976	298694	312650	.328045	-1.12
-1 11	266846	277788	289690	.302707	317026	.332873	-1 11
-1.10	.270078	281229	293374	.306677	.321341	.337616	-1.10
-1.09	.273295	284647	.297024	310601	325594	.342274	-1.09
-1 08	.276494	288040	300641	.314479	.329785	.346846	-1 08
-1.07	.279675	291408	.304222	.318310	.333910	351330	-1 07
-1 06	.282837	294749	.307768	322091	337970	.355725	-1.06
-1.05	.285979	.298062	311275	.325823	341964	.360030	-1.05
-1.04	289099	301346	.314744	.329503	.345889	364246	-1.04
-1.03	292197	.304600	.318173	.333131	.349746	.368370	-1 03
-1 02	.295272	307823	.321562	336706	.353533	372403	-1.02
-1.01	.298324	.311015	324908	.340226	.357249	.376344	-1.01
-1 00	.301350	.314173	.328212	343691	.360894	.380193	-1.00

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
- 99	.244390	.252782	.261708	.271238	.281452	292450	- .99
- 98	.246809	.255281	.264290	.273908	.284217	.295316	- .98
- .97	.249228	.257775	.266865	.276568	.286967	.298161	- .97
- .96	.251644	.260265	.269432	.279216	289700	300984	- .96
- .95	.254059	.262750	.271991	.281852	.292416	.303784	- .95
- .94	.256471	.265230	.274541	.284475	.295115	.306561	- .94
- .93	.258881	.267704	.277082	.287084	.297795	.309314	- .93
- 92	.261286	.270172	.279612	.289679	.300456	.312042	- .92
- .91	.263688	.272632	.282132	.292259	.303097	.314744	- .91
- .90	.266085	.275084	.284640	.294824	.305717	.317420	- .90
- .89	268477	.277528	287136	.297372	.308316	320068	- .89
- 88	270864	279964	.289620	.299903	.310893	.322689	- .88
- 87	.273244	.282389	.292090	302416	313447	.325281	- 87
- 86	.275618	.284805	.294547	.304911	.315978	.327843	- 86
- 85	.277985	.287210	296988	307387	.318484	.330376	- .85
- .84	280344	.289604	299415	.309843	.320966	.332877	- .84
- 83	282694	291986	.301826	312279	.323422	335348	- .83
- 82	285036	294356	.304220	314694	.325852	.337786	- .82
- 81	.287369	.296713	.306598	.317087	328255	.340191	- 81
- 80	.289692	299057	308958	319458	.330631	.342563	- .80
- 79	.292004	.301386	.311299	.321806	.332978	.344901	- .79
- 78	.294305	.303700	.313622	324130	.335297	347205	- .78
- .77	296595	.306000	315925	.326431	.337586	.349473	- .77
- .76	.298872	.308283	.318208	328707	.339846	.351705	- .76
- .75	301137	.310550	.320471	.330957	.342075	353901	- .75
- .74	.303389	.312800	.322712	.333181	.344273	.356061	- .74
- 73	305627	.315032	.324931	.335379	.346439	.358183	- .73
- 72	307851	.317246	.327128	.337550	.348572	.360267	- .72
- 71	310060	319442	.329302	.339693	350673	362313	- .71
- .70	312254	.321618	331453	.341808	.352741	.364319	- .70
- 69	314432	.323774	.333579	343894	354775	.366287	- .69
- 68	316593	325910	335680	.345950	356774	.368214	- .68
- 67	.318737	328025	.337757	.347977	.358739	.370102	- .67
- .66	.320864	330119	339807	349974	.360668	.371949	- .66
- .65	.322972	.332190	.341831	.351939	.362561	.373754	- .65
- 64	.325062	.334239	343829	353873	.364419	375519	- .64
- .63	.327133	.336264	.345798	.355775	.366239	.377241	- 63
- .62	329184	.338266	347740	.357645	.368022	.378922	- .62
- 61	331215	.340244	.349654	.359481	369768	.380559	- .61
- .60	.333225	342196	351538	.361285	371475	.382155	- .60
- .59	.335213	.344124	.353393	.363054	.373145	.383707	- .59
- .58	.337180	.346026	.355218	.364790	.374775	.385215	- .58
- .57	.339124	.347901	.357013	.366490	.376367	.386680	- .57
- .56	.341046	.349750	.358777	.368156	.377919	.388101	- .56
- .55	.342944	351571	.360510	.369786	.379431	.389478	- .55
- .54	.344818	.353365	.362210	.371380	380903	.390810	- .54
- .53	.346668	.355130	.363879	.372938	.382334	.392098	- .53
- .52	.348493	.356867	.365515	374459	.383725	.393341	- .52
- .51	.350292	.358574	.367117	.375943	.385075	.394540	- .51
- .50	.352065	.360252	.368687	.377390	.386384	.395693	- .50

t	SKEWNESS						t
	6	7	8	.9	1.0	1.1	
- 99	.304350	.317298	331472	.347099	.364467	.383948	- 99
- 98	307324	320388	.334688	350451	.367967	.387610	- 98
- .97	.310270	323442	.337857	.353745	.371393	.391179	- .97
- 96	313188	.326460	.340981	.356979	374745	394654	- .96
- .95	.316076	.329440	344057	.360155	.378023	.398035	- .95
- .94	.318934	.332382	.347085	363270	.381226	401323	- .94
- .93	321762	335285	350064	.366325	.384353	.404516	- .93
- 92	324557	.338148	.352993	.369318	.387404	407616	- .92
- .91	327320	340970	.355872	.372249	.390379	410622	- .91
- .90	.330050	.343751	358700	.375117	393277	.413535	- .90
- .89	.332745	346490	361477	377922	396099	416354	- .89
- .88	.335406	.349186	.364201	.380664	398844	.419081	- .88
- 87	338031	351839	366872	383342	.401512	.421715	- .87
- 86	.340620	.354447	.369490	385956	.404103	.424257	- .86
- .85	.343173	357011	.372054	.388505	406616	426706	- .85
- .84	.345687	359529	.374563	.390989	409053	429065	- .84
- 83	348164	.362001	.377018	393408	.411412	.431332	- .83
- .82	350601	.364427	.379417	395762	.413695	433509	- 82
- 81	.352999	.366805	.381760	.398049	415900	435596	- .81
- .80	355357	.369136	.384048	.400271	.418028	.437594	- .80
- 79	.357675	.371419	.386278	.402427	.420080	439504	- .79
- .78	359951	373654	.388453	.404517	422056	.441325	- .78
- 77	362186	375839	.390569	406541	423955	.443059	- .77
- .76	364378	.377975	392629	408499	.425778	444707	- .76
- 75	366528	.380062	.394631	.410390	427526	.446269	- .75
- 74	.368635	382098	.396575	.412215	429199	447745	- .74
- .73	370698	.384084	.398462	413975	.430796	.449138	- .73
- 72	.372716	386019	400290	415668	432319	.450447	- .72
- .71	.374691	.387903	.402060	.417295	.433769	.451674	- .71
- 70	376620	.389735	.403771	.418857	.435144	452819	- .70
- .69	.378505	.391516	405424	.420352	436447	.453883	- .69
- 68	380343	.393245	407019	421783	.437676	.454867	- .68
- 67	.382136	.394922	.408555	.423148	.438834	455773	- .67
- 66	383882	.396547	.410033	.424449	.439920	.456600	- .66
- 65	.385582	.398119	.411452	.425684	.440935	457350	- 65
- 64	.387235	399639	.412813	.426855	.441880	.458024	- .64
- 63	.388841	.401106	.414116	427962	.442755	.458623	- .63
- .62	.390399	402520	.415360	.429006	.443561	.459147	- .62
- 61	.391910	.403882	.416546	.429986	.444298	459598	- .61
- 60	393373	405191	.417674	.430902	.444968	459977	- .60
- .59	.394789	.406446	418744	.431757	.445570	.460285	- .59
- .58	396156	.407649	.419757	432548	446105	.460523	- .58
- .57	.397474	408799	.420712	.433278	446575	.460691	- .57
- .56	.398745	.409896	.421610	.433947	.446980	.460792	- .56
- .55	.399966	.410940	.422451	.434555	447320	.460825	- .55
- .54	.401140	.411932	423235	.435102	.447597	.460792	- .54
- .53	.402264	.412871	.423962	.435590	.447811	.460694	- .53
- .52	.403340	.413757	.424634	.436018	.447963	.460532	- .52
- .51	.404367	.414590	.425249	.436387	.448054	.460307	- .51
- 50	.405345	.415372	.425809	.436698	.448084	.460021	- .50

t	SKEWNESS						t
	0	.1	.2	.3	.4	5	
- 49	353812	361899	.370222	.378799	.387651	396801	- .49
- 48	355533	363516	.371723	.380170	.388876	.397863	- .48
- 47	357225	365102	.373189	.381503	.390060	.398880	- 47
- .46	.358890	366657	.374621	382797	391201	.399852	- .46
- .45	.360527	.368179	.376017	.384052	.392300	.400778	- .45
- 44	.362135	369670	.377377	385268	.393357	.401658	- .44
- 43	363714	.371128	378701	.386444	.394370	.402493	- .43
- .42	.365263	.372552	.379988	.387581	.395341	.403282	- .42
- 41	.366782	373944	.381239	.388678	396269	.404025	- .41
- 40	.368270	375301	382453	.389735	.397154	404722	- .40
- 39	.369728	376624	.383630	.390751	.397996	.405374	- .39
- .38	371154	377913	.384768	.391727	.398795	.405980	- .38
- .37	372548	379167	385869	392662	399550	406540	- 37
- .36	373911	.380385	386932	.393556	400262	407055	- 36
- .35	375240	.381568	.387956	394409	.400930	.407525	- 35
- 34	.376537	.382715	388942	.395221	.401555	.407949	- .34
- 33	377801	383826	389889	.395991	.402136	.408327	- .33
- 32	379031	384901	.390797	396721	.402674	.408661	- .32
- 31	380226	.385938	.391665	397408	403169	.408949	- 31
- 30	381388	.386939	.392494	.398054	.403620	.409193	- .30
- .29	382515	.387902	393284	398659	.404028	409391	- .29
- 28	.383606	388828	.394033	399221	.404392	.409546	- .28
- 27	384663	.389716	.394743	.399742	404713	409656	- .27
- 26	385683	390566	.395413	.400221	404991	.409722	- 26
- .25	386668	391378	396042	400658	405226	409743	- .25
- 24	387617	.392152	396631	401054	.405418	.409722	- .24
- 23	.388529	392886	.397180	.401407	.405567	.409656	- .23
- 22	.389404	.393582	397688	.401719	405673	.409548	- .22
- 21	.390242	394239	.398156	401989	.405736	409396	- 21
- .20	391043	394857	.398583	402217	.405758	.409202	- 20
- 19	391806	.395436	398969	.402404	405736	.408966	- .19
- 18	392531	395975	399315	402548	.405673	.408687	- .18
- .17	.393219	.396475	399620	402652	405568	.408367	- .17
- .16	.393868	.396935	399884	402714	.405421	408005	- 16
- .15	394479	.397355	400107	.402734	.405233	.407602	- .15
- 14	.395052	.397735	400290	.402714	.405004	.407158	- .14
- 13	395585	.398076	.400432	.402652	.404733	406674	- .13
- 12	396080	398376	.400534	.402550	.404422	.406149	- .12
- 11	.396536	.398637	400594	402406	404070	.405585	- .11
- .10	396953	398857	.400615	402222	403678	404982	- .10
- 09	.397330	399038	400594	.401998	.403247	.404339	- .09
- .08	397668	399178	400534	.401733	402775	403658	- 08
- .07	.397966	.399278	400433	.401429	402264	.402939	- 07
- 06	398225	399338	400292	401084	.401715	402182	- 06
- .05	.398444	.399358	400111	.400700	401126	401388	- .05
- 04	398623	.399338	.399890	.400277	.400499	.400556	- .04
- 03	398763	399278	.399629	399814	399834	.399688	- .03
- .02	398863	.399179	399329	.399313	.399132	.398784	- .02
- .01	.398922	399039	398989	.398773	.398392	.397845	- .01
00	398942	.398859	.398610	.398195	.397615	.396870	.00

t	SKEWNESS						t
	.6	.7	8	.9	1.0	1.1	
– .49	406274	.416101	426314	436950	.448054	.459673	– 49
– 48	.407155	.416778	.426764	.437146	.447965	.459266	– .48
– .47	.407987	417403	.427159	.437285	447818	.458800	– .47
– .46	408770	417977	.427500	437368	.447614	.458276	– .46
– .45	409504	418500	.427788	437396	447354	.457696	– .45
– .44	410190	.418971	.428022	.437368	.447037	.457060	– .44
– 43	.410827	.419391	.428204	437287	.446666	.456370	– .43
– 42	411416	.419761	.428333	437152	.446241	.455626	– .42
– 41	.411957	.420080	428410	.436964	445763	.454830	– .41
– 40	.412450	.420349	428435	.436723	.445232	453982	– .40
– .39	.412895	.420569	428410	.436431	444650	453084	– .39
– 38	.413292	420739	.428334	.436088	.444017	452137	– .38
– .37	413641	.420860	.428208	.435695	.443335	451141	– .37
– 36	.413943	420933	.428032	435252	.442603	450098	– 36
– 35	414198	420957	.427808	434760	441823	459009	– 35
– 34	.414406	420933	427535	.434220	.440997	.447874	– 34
– 33	414567	.420861	427214	.433633	440124	446695	– 33
– .32	414682	420743	426846	.432998	439205	.445473	– 32
– 31	414751	.420577	.426431	.432318	.438242	.444209	– 31
– 30	414774	.420365	.425970	.431592	437234	442903	– .30
– .29	414751	420108	.425463	.430821	436184	441557	– 29
– 28	414683	419804	424912	430007	435092	.440171	– 28
– 27	414570	.419456	.424315	429149	433958	438748	– 27
– 26	414412	.419063	423675	428248	.432785	437286	– 26
– 25	.414211	.418626	422991	427306	431571	435788	– 25
– 24	.413965	.418146	422265	426323	430319	434255	– .24
– .23	.413675	417622	.421497	.425299	.429028	.432687	– 23
– 22	413343	.417056	.420687	.424235	.427701	.431085	– 22
– .21	412967	.416448	.419837	.423133	426338	.429450	– 21
– 20	412550	.415798	418946	421993	424939	427784	– .20
– 19	.412090	.415107	.418015	.420815	423505	426086	– .19
– 18	.411588	414375	417046	419600	.422037	.424358	– .18
– 17	411046	413603	416038	.418349	.420537	.422601	– 17
– .16	.410462	.412792	414993	.417064	419004	.420816	– 16
– .15	.409839	.411942	413910	415743	.417440	.419002	– 15
– 14	.409175	411053	412791	414389	.415846	.417162	– .14
– 13	.408472	410127	.411636	.413001	.414221	.415296	– .13
– .12	407730	.409163	410447	.411582	412568	413406	– 12
– 11	406949	408162	409222	.410130	410886	411490	– .11
– 10	406131	.407125	407964	.408648	409176	409551	– 10
– .09	.405275	.406053	.406673	.407135	407440	407590	– .09
– .08	.404382	.404945	.405349	.405593	.405678	.405607	– .08
– .07	.403452	.403803	403993	.404022	.403891	.403602	– .07
– .06	.402486	.402627	402606	402423	402079	401577	– 06
– 05	.401485	.401418	401188	.400796	400243	.399533	– 05
– 04	.400448	.400176	399740	.399143	.398385	397469	– .04
– .03	.399377	.398902	.398263	.397463	.396504	.395387	– 03
– 02	.398272	397596	396758	395758	394601	.393288	– 02
– 01	397134	396260	395224	394029	392677	391172	– 01
00	395962	394892	.393663	.392276	.390734	389040	.00

	SKEWNESS					
t	0	1	.2	3	.4	.5
00	.398942	.398859	398610	398195	397615	.396870
01	398922	398640	398192	397579	.396801	395861
02	.398863	398381	397735	.396925	.395952	394817
.03	.398763	.398083	397239	396233	395066	393739
04	398623	397745	396705	395505	394145	392629
05	.398444	.397368	.396133	394739	.393189	.391485
06	.398225	396952	395523	393938	392199	390309
07	.397966	.396498	394875	393100	391174	.389101
.08	397668	396004	394190	392226	390116	387862
.09	.397330	.395472	.393467	391317	.389024	386591
10	.396953	394902	.392708	.390373	.387900	.385291
11	396536	394294	391912	[illegible]	[illegible]	[illegible]

t	SKEWNESS						t
	6	7	.8	9	10	11	
.00	.395962	394892	393663	.392276	390734	.389040	00
.01	394758	.393496	392075	390499	388770	386893	01
02	.393522	392069	.390461	.388699	386788	384731	02
03	.392255	.390615	388821	386878	.384788	382555	03
.04	.390957	.389132	.387157	385035	382770	380366	04
05	389628	387621	.385468	.383171	380735	.378164	.05
.06	388270	.386084	.383756	381288	378684	375950	06
.07	.386882	384521	.382021	379385	.376618	373724	.07
08	385466	.382932	.380263	.377463	374536	371487	08
.09	.384021	.381318	.378483	375523	372440	369241	.09
10	382549	379679	.376683	373565	.370331	.366984	10
11	.381050	378016	.374862	.371591	368208	.364719	.11
12	379525	.376330	373021	369600	366073	362445	12
.13	377973	374621	371160	367593	363926	360163	.13
14	376396	.372891	369281	.365572	361768	357874	14
15	374795	371138	367384	.363536	359598	.355577	15
.16	.373169	.369364	365469	.361485	357419	353275	16
17	.371519	.367571	363537	359422	.355230	.350967	17
.18	.369846	365757	.361588	357345	353032	348653	18
19	.368151	363923	.359624	355257	350826	.346335	.19
.20	366433	362072	357645	353156	.348611	344013	20
.21	364694	.360201	.355651	.351045	.346389	341687	.21
.22	.362934	.358314	.353642	348923	344160	339357	22
.23	361153	356409	.351620	346791	341924	337025	.23
.24	359353	.354488	349585	.344649	339683	.334690	24
.25	.357533	.352550	.347538	.342498	.337436	332353	.25
26	.355694	350598	.345478	.340339	.335184	330015	.26
.27	353838	.348630	343407	.338172	332927	.327676	27
28	.351963	.346648	.341325	335997	.330667	.325336	28
.29	350071	344652	.339233	.333816	.328403	.322996	29
30	.348163	342643	.337131	.331627	326135	320657	30
31	346239	340621	335019	329433	323865	.318317	31
.32	344299	338587	332898	327233	321593	.315979	32
33	.342343	.336541	.330769	325028	319319	.313642	33
34	340374	.334484	328632	322818	317043	311307	.34
35	.338390	.332416	.326488	320604	.314766	308974	35
.36	.336393	330338	.324337	.318387	.312489	306644	36
37	334383	.328251	.322179	.316166	310212	.304316	37
.38	.332360	.326154	.320015	313942	.307934	.301991	38
39	.330325	.324048	317845	.311716	.305657	.299670	.39
.40	.328279	.321934	.315671	.309487	.303381	297352	.40
41	.326222	319812	.313491	307257	.301107	295039	.41
.42	324154	.317683	.311308	.305026	298834	.292730	42
.43	.322077	.315547	.309120	.302793	296563	.290426	.43
.44	.319989	313405	.306930	.300561	294294	288127	44
45	317893	.311256	.304736	.298328	.292028	.285833	45
.46	.315788	309102	302540	296095	289765	.283544	.46
.47	.313675	.306943	.300341	.293863	.287505	.281262	47
.48	.311555	.304780	298141	.291632	.285248	.278985	.48
.49	.309427	.302612	.295939	.289402	.282996	.276715	.49

	SKEWNESS						
t	0	1	2	3	.4	5	t
50	352065	344110	336369	.328830	321478	.314302	.50
.51	.350292	.342252	.334438	.326833	319426	312202	51
52	348493	.340372	332487	.324822	317362	.310093	52
53	346668	338470	330519	322796	315286	307976	.53
54	344818	336547	328533	.320756	313200	.305851	.54
55	342944	334604	.326529	318701	311103	.303719	.55
.56	.341046	332640	.324509	316634	308796	301579	56
.57	.339124	.330656	.322473	314554	.306879	299433	.57
58	.337180	328653	.320421	312461	304754	297281	58
.59	.335213	326632	.318354	.310357	302620	.295124	59
60	333225	.324592	.316273	308242	300477	.292961	60
61	331215	322535	.314178	.306115	298327	.290793	.61
62	329184	320461	312068	303979	296170	.288621	62
63	.327133	.318370	309946	.301833	294006	286445	.63
64	.325062	316264	307811	299677	291836	.284266	.64
.65	322972	314142	305565	297513	289660	282083	.65
.66	.320864	312004	303507	.295340	287478	.279898	.66
.67	.318737	.309853	301337	293159	285292	.277711	67
68	316593	307687	299157	290971	283101	275521	68
69	314432	305508	296968	288776	280906	.273331	.69
70	312254	303317	.294768	286575	.278707	271139	70
71	310060	301113	292560	284368	276505	.268946	.71
.72	.307851	298897	.290343	.282155	274301	266754	.72
73	.305627	296670	.288118	279937	272094	264561	73
74	303389	.294432	285886	277714	.269885	262369	.74
75	301137	.292184	283647	275488	.267674	.260177	.75
.76	.298872	289926	.281401	273257	265462	.257987	.76
77	296595	.287659	279148	271023	.263250	255798	77
78	294305	285384	.276891	268787	261037	253611	78
.79	292004	.283100	.274628	266548	258824	251426	79
.80	.289692	.280809	.272360	264307	.256612	249244	.80
.81	287369	.278510	.270089	262064	254400	247065	.81
82	285036	276205	267813	.259820	252189	244888	82
.83	.282694	273893	265535	257576	.249980	.242716	.83
84	.280344	271576	.263253	255331	247773	.240547	84
85	277985	.269254	.260969	.253086	245568	.238382	.85
86	275618	266928	258683	250842	243365	236222	.86
.87	.273244	264597	256396	.248598	241166	.234066	.87
88	.270864	262262	.254107	.246356	.238970	231915	.88
89	268477	259924	.251818	.244115	236777	229770	.89
.90	266085	.257584	249529	.241876	234588	227630	.90
.91	.263688	255241	.247240	239640	232403	.225496	.91
92	261286	.252897	244952	237406	230223	223368	.92
.93	258881	250551	242664	.235176	228048	.221247	.93
94	.256471	.248204	240378	.232949	225878	.219132	.94
95	.254059	245857	238094	.230725	223713	.217024	.95
96	251644	243510	.235812	228506	.221554	.214923	.96
.97	.249228	.241164	233533	.226291	.219401	.212829	.97
.98	.246809	.238818	.231256	224081	.217254	.210744	.98
.99	.244390	.236474	.228984	221876	.215114	.208665	.99

t	SKEWNESS						t
	.6	7	8	.9	1.0	1 1	
50	.307293	.300440	.293736	287174	280748	.274451	.50
.51	305152	298265	291533	.284948	.278504	.272195	51
.52	.303005	.296087	289330	.282725	.276265	269945	.52
53	300853	293907	.287126	.280504	.274031	267703	.53
54	298696	291724	.284923	.278286	271803	265468	.54
55	296535	289539	.282721	.276071	269580	.263241	.55
.56	294369	.287354	.280520	.273860	267363	.261022	.56
.57	.292200	285167	.278321	.271653	265152	258811	.57
.58	290028	282979	276124	.269449	.262947	256608	.58
59	287852	.280792	.273928	267251	.260749	254414	.59
60	285675	278604	271736	265057	.258558	.252228	.60
.61	283495	276417	269546	262868	.256373	.250052	61
62	.281313	274231	267359	260685	.254196	.247885	.62
.63	279131	.272046	265175	258506	252027	245727	63
64	276947	.269862	262996	256334	249865	.243578	.64
65	.274763	.267680	260820	254168	247711	.241439	.65
66	.272578	.265501	258649	252008	245565	.239310	.66
.67	.270394	263324	256482	249855	243428	.237190	67
.68	268211	.261150	.254320	.247708	.241299	235081	.68
.69	266028	.258978	.252164	245568	239178	.232981	.69
70	263847	256811	250012	243436	237066	.230892	.70
71	.261667	.254647	.247867	241310	234964	228813	.71
72	259489	.252487	245727	239193	.232870	226745	.72
73	.257314	.250331	243593	237083	.230786	.224688	73
74	255141	.248180	241466	.234981	228711	.222641	.74
75	252971	.246034	.239345	.232888	226646	.220605	.75
76	250805	.243893	.237231	230802	224590	.218581	.76
.77	248642	.241757	.235125	.228726	222544	216567	.77
78	.246482	.239627	.233025	.226658	.220509	214564	.78
79	.244327	.237503	230933	.224599	.218483	.212573	.79
80	.242177	235385	.228849	222548	.216468	.210593	.80
81	240031	.233274	226772	220508	214463	.208624	81
.82	237890	.231169	.224704	218476	212469	.206667	.82
.83	235754	.229071	222644	.216454	.210485	204722	.83
.84	.233624	.226980	.220592	.214442	.208512	.202788	.84
85	231500	224896	218549	212439	.206550	.200866	.85
.86	229381	.222820	.216515	210447	204598	.198955	.86
.87	227269	220751	214489	.208464	202658	197057	.87
88	225164	.218691	.212473	206492	200729	.195170	88
89	223066	.216638	.210466	.204529	198811	.193295	.89
.90	220974	.214594	.208469	.202578	.196904	191433	.90
91	218890	212559	206481	200637	.195009	189582	.91
.92	.216813	.210532	.204503	198706	.193125	.187743	.92
.93	214744	.208514	.202534	196786	.191252	.185917	.93
.94	.212683	.206505	.200576	.194877	.189391	184103	.94
.95	.210630	.204505	.198628	.192979	187541	182300	.95
.96	208585	.202515	196690	.191092	.185704	.180510	.96
97	206549	.200534	194762	189216	.183878	.178733	.97
98	.204521	.198562	.192845	.187351	.182063	176967	.98
.99	.202503	.196601	.190939	.185498	.180261	175214	.99

t	SKEWNESS						t
	0	.1	2	.3	.4	.5	
1 00	.241971	234132	.226714	219676	212981	.206595	1.00
1 01	239551	231792	.224449	217482	210854	204534	1.01
1 02	237132	229454	222189	.215295	.208736	.202480	1.02
1 03	.234714	.227120	.219933	.213113	.206624	200436	1.03
1.04	.232297	224789	.217683	.210938	.204521	198400	1 04
1 05	.229882	222462	215438	.208770	.202426	.196373	1.05
1.06	227470	220139	213199	206610	.200339	194356	1 06
1.07	225060	.217821	.210966	.204456	.198260	.192348	1.07
1.08	222653	215508	.208739	202311	.196191	190350	1.08
1 09	.220251	213200	.206520	200174	.194130	.188361	1.09
1 10	.217852	.210898	.204307	.198044	.192079	.186383	1.10
1.11	.215458	.208602	.202102	195924	.190037	184414	1 11
1.12	.213069	.206313	199905	193812	188004	.182456	1 12
1 13	210686	204031	.197716	.191709	.185982	180509	1.13
1 14	.208308	201755	.195535	189616	183969	.178572	1.14
1.15	.205936	199488	.193363	.187531	.181967	176645	1 15
1 16	203571	197228	.191200	.185457	179975	174730	1.16
1.17	201214	194977	.189046	.183393	.177993	.172826	1.17
1 18	198863	192734	.186901	.181338	.176023	.170933	1.18
1.19	196520	190500	.184766	179295	174063	.169051	1.19
1.20	194186	188275	182641	.177261	172114	.167180	1 20
1 21	191860	.186059	180526	.175239	.170176	.165321	1.21
1.22	189543	.183854	.178422	173227	168250	.163474	1.22
1 23	187235	181658	176328	.171227	166335	.161639	1.23
1 24	.184937	179473	.174245	169237	164432	.159815	1 24
1.25	.182649	.177298	172174	.167260	.162541	158003	1.25
1.26	.180371	.175134	170113	165294	.160661	.156203	1.26
1.27	178104	.172982	168064	.163339	158794	.154416	1.27
1.28	175847	.170840	.166027	161397	156938	152640	1.28
1.29	.173602	.168711	164002	.159467	.155095	.150877	1 29
1.30	171369	.166593	.161989	157549	.153264	.149127	1.30
1.31	.169147	.164487	159988	155644	151446	.147388	1 31
1.32	.166937	.162394	.158000	.153751	.149640	.145663	1.32
1.33	164740	.160313	156024	151871	147847	.143949	1.33
1.34	162555	158245	154062	.150003	146067	.142249	1.34
1.35	.160383	.156190	.152112	.148149	144299	.140561	1.35
1.36	.158225	154149	.150175	.146307	.142545	138886	1 36
1 37	.156080	152120	.148252	.144479	.140803	137224	1 37
1.38	153948	.150105	.146342	.142664	139074	.135574	1.38
1.39	151831	148104	144445	.140862	.137359	.133938	1 39
1.40	149727	146117	142563	.139074	135656	.132314	1.40
1 41	.147639	.144144	.140694	.137299	.133967	.130703	1.41
1 42	145564	142185	.138839	135538	.132292	129106	1 42
1 43	.143505	.140241	.136998	.133790	.130629	127521	1.43
1 44	.141460	.138311	.135171	.132057	.128980	.125949	1.44
1.45	.139431	.136396	.133359	130337	.127345	.124391	1 45
1.46	137417	.134496	.131561	128631	125723	.122845	1.46
1.47	135418	.132611	.129777	126939	.124114	.121313	1 47
1 48	133435	130741	128008	.125261	122515	119794	1.48
1 49	.131468	128886	.126253	.123598	.120938	.118287	1.49

t	SKEWNESS						t
	6	7	.8	9	1 0	1 1	
1 00	200493	194649	.189043	183655	178470	173473	1 00
1.01	.198493	192708	187158	.181824	176691	171745	1.01
1 02	196502	190777	.185284	.180005	.174925	.170028	1 02
1.03	194521	188856	.183421	178197	.173170	168324	1 03
1 04	192549	.186945	.181569	.176401	171427	166633	1 04
1 05	190588	.185046	179728	.174616	169696	.164954	1.05
1.06	.188636	.183157	177898	172843	167977	163287	1.06
1 07	186695	181278	.176080	171082	.166271	161632	1.07
1 08	184764	.179411	.174273	169333	164576	159990	1.08
1 09	.182843	177555	172478	167595	162894	158360	1 09
1 10	180933	175710	170694	165870	.161223	156743	1 10
1 11	.179034	173876	.168922	164156	.159565	155137	1 11
1 12	.177146	172053	.167161	162454	.157919	.153545	1 12
1.13	.175268	170242	165412	.160764	156285	151964	1.13
1 14	.173402	.168442	.163675	159086	.154664	.150396	1 14
1.15	.171547	166654	161950	.157421	153054	148840	1 15
1 16	169703	164877	160236	.155767	151457	147296	1.16
1 17	.167871	163112	158535	154125	149872	145764	1 17
1.18	166050	161359	.156845	152496	148299	144245	1 18
1.19	164241	159618	155168	.150878	146738	142737	1 19
1 20	.162444	.157889	153502	149273	145189	141242	1 20
1.21	160658	156171	151849	.147679	143653	.139759	1 21
1 22	.158884	154466	.150207	.146098	142128	138289	1 22
1.23	.157122	.152772	.148578	.144529	140616	136830	1.23
1 24	155372	.151091	146961	142972	139116	135383	1 24
1.25	.153634	149422	145356	.141427	137627	133948	1 25
1.26	.151908	.147765	.143763	139895	136151	132526	1 26
1.27	150195	.146120	.142183	.138374	.134687	131115	1 27
1.28	148493	144487	140614	136866	133235	.129716	1 28
1 29	.146804	.142867	.139058	135369	.131795	.128328	1 29
1 30	145127	.141259	.137514	133885	130367	126953	1 30
1 31	.143463	.139663	135982	.132413	128951	125590	1 31
1.32	.141811	.138080	.134462	.130953	127546	124238	1 32
1.33	.140172	136509	132955	.129505	126154	122897	1 33
1.34	.138545	.134950	131459	.128069	124773	.121569	1.34
1.35	.136931	.133404	.129976	126645	.123404	.120252	1.35
1 36	135329	131870	.128505	125232	.122047	118946	1 36
1 37	.133740	130348	.127047	123832	120702	117652	1.37
1.38	.132163	128839	125600	122444	.119368	116370	1 38
1 39	.130599	.127342	.124166	121068	.118046	.115099	1 39
1.40	129048	.125858	122743	.119703	.116736	.113839	1 40
1 41	127509	.124386	.121333	118350	115437	112590	1 41
1.42	.125983	122926	.119935	117009	114149	111353	1.42
1 43	.124470	.121479	.118549	115680	112873	.110127	1.43
1 44	122969	.120044	.117175	.114363	111609	.108912	1.44
1 45	.121482	118621	.115813	.113057	.110356	.107708	1.45
1.46	.120007	.117211	.114463	.111763	.109114	106515	1 46
1.47	.118544	.115813	.113125	.110481	.107883	.105333	1.47
1.48	117094	.114428	.111798	109210	.106664	.104162	1.48
1.49	.115657	.113054	.110484	.107950	.105456	.103002	1.49

t	SKEWNESS						t
	.0	1	2	.3	.4	.5	
1.50	129518	.127046	.124514	.121948	.119370	116794	1.50
1 51	.127583	.125222	.122789	.120312	.117816	115314	1 51
1 52	.125665	.123414	121079	.118691	.116275	.113848	1.52
1 53	123763	.121621	.119383	.117084	.114748	112394	1.53
1.54	121878	.119844	.117703	.115492	.113235	.110953	1.54
1 55	120009	.118082	.116038	.113913	.111736	.109526	1.55
1 56	.118157	.116337	.114388	.112349	.110250	.108112	1 56
1 57	116323	.114607	.112753	110799	.108778	106710	1.57
1 58	114505	.112894	.111133	109264	.107319	.105322	1.58
1 59	.112704	111196	.109528	.107743	.105874	.103947	1 59
1 60	.110921	.109515	.107939	.106237	.104443	.102585	1.60
1 61	.109155	107850	.106365	.104745	.103026	101236	1.61
1.62	107406	.106201	.104806	.103267	.101622	099900	1.62
1.63	.105675	104569	.103262	.101804	.100232	098577	1.63
1 64	.103961	.102953	.101733	.100355	.098856	.097266	1.64
1.65	102265	.101353	.100220	.098920	097493	095969	1.65
1 66	.100586	.099769	.098723	.097500	096143	094685	1 66
1 67	.098925	.098202	.097240	.096094	.094808	.093413	1.67
1.68	.097282	.096652	.095773	.094703	.093485	.092154	1.68
1.69	.095657	.095117	.094321	.093326	.092177	090908	1 69
1 70	094049	.093599	.092884	.091963	.090881	.089674	1 70
1 71	092459	.092098	.091463	.090615	089599	.088454	1.71
1.72	090887	090613	.090057	.089280	.088331	.087245	1 72
1 73	089333	.089144	.088666	.087960	.087076	.086050	1 73
1 74	.087796	087692	087290	.086654	085834	.084867	1.74
1.75	.086277	.086256	.085930	.085363	.084605	.083696	1 75
1 76	084776	084837	.084584	084085	.083390	.082538	1 76
1.77	083293	.083433	.083254	.082822	.082187	.081392	1.77
1.78	.081828	.082046	.081939	.081572	.080998	.080258	1.78
1 79	080380	080676	080639	.080336	.079822	.079137	1 79
1 80	078950	.079321	.079354	079115	.078659	.078028	1 80
1 81	077538	.077983	078083	.077907	077508	.076930	1 81
1 82	.076143	076661	076828	.076713	.076371	075845	1.82
1 83	074766	.075355	.075587	.075533	.075246	.074772	1.83
1.84	073407	074065	.074362	.074366	.074134	.073711	1.84
1 85	072065	.072791	073150	.073213	.073035	.072662	1.85
1 86	070740	.071533	071954	.072074	.071949	.071625	1.86
1.87	.069433	.070291	070772	.070948	.070875	.070599	1.87
1 88	068144	069065	.069605	.069835	069813	.069585	1.88
1 89	.066871	.067855	068452	.068736	068764	.068582	1.89
1 90	.065616	066660	.067314	067650	067727	067591	1.90
1 91	.064378	065480	.066189	066577	.066703	.066612	1.91
1 92	063157	.064317	065079	.065518	.065690	.065644	1.92
1.93	.061952	063168	063984	.064471	.064690	.064687	1.93
1 94	060765	.062035	062902	063438	.063702	.063741	1.94
1 95	.059595	060917	061834	.062417	062726	.062807	1.95
1 96	058441	.059815	060780	.061410	.061762	.061883	1.96
1 97	.057304	.058727	059740	060415	060809	.060971	1 97
1 98	056183	.057655	.058714	.059432	.059868	.060069	1.98
1 99	.055079	056598	057701	.058463	.058939	.059179	1.99

	SKEWNESS						
t	.6	7	8	9	1.0	1.1	t
1.50	114233	111693	.109182	.106702	104259	.101852	1.50
1.51	.112821	110344	.107891	105466	.103073	100714	1.51
1.52	111422	109008	.106612	104241	.101898	.099586	1.52
1.53	.110036	.107683	.105345	103027	.100733	098468	1.53
1.54	108662	106371	.104090	.101824	099580	097361	1.54
1.55	.107300	.105071	.102846	100633	.098438	096265	1.55
1.56	105952	103782	.101614	.099453	097307	095179	1.56
1.57	104616	102506	.100393	.098284	096186	.094103	1.57
1.58	.103292	101242	099184	097126	095076	.093037	1.58
1.59	.101981	.099990	097986	095979	.093976	091982	1.59
1.60	100682	.098750	096800	094844	092887	090937	1.60
1.61	099396	.097521	095625	.093719	091809	089902	1.61
1.62	098122	.096305	094462	.092605	090741	.088877	1.62
1.63	.096860	.095100	.093310	.091501	089683	087862	1.63
1.64	.095611	.093907	092169	090409	088636	086857	1.64
1.65	094374	092725	091039	.089327	087599	085862	1.65
1.66	093149	091555	089920	088256	086572	084877	1.66
1.67	.091936	090397	088813	.087195	.085555	083901	1.67
1.68	.090735	089251	087716	.086145	084548	082935	1.68
1.69	.089547	088115	086630	.085105	.083552	081979	1.69
1.70	088371	086992	085555	.084076	082565	081032	1.70
1.71	.087206	.085879	084491	.083057	081588	080094	1.71
1.72	.086053	084778	083438	.082048	.080621	079166	1.72
1.73	084913	083688	082395	.081050	.079663	078247	1.73
1.74	083784	.082610	081363	.080061	.078716	.077338	1.74
1.75	.082667	.081542	080342	079083	077777	076437	1.75
1.76	081561	.080486	079331	.078114	.076849	075546	1.76
1.77	080468	079440	.078331	.077156	075930	074664	1.77
1.78	079385	078406	077341	076208	.075020	073790	1.78
1.79	078315	077382	.076361	075269	074120	072926	1.79
1.80	077256	076370	.075392	.074340	073229	072071	1.80
1.81	076208	075368	.074432	073421	072347	071224	1.81
1.82	.075171	074376	.073483	.072511	.071474	.070386	1.82
1.83	074146	073396	.072544	071611	.070611	.069557	1.83
1.84	.073132	.072426	071615	.070720	.069756	.068736	1.84
1.85	.072129	071466	070696	.069839	068910	067924	1.85
1.86	.071138	070517	.069787	068967	068073	.067120	1.86
1.87	070157	069578	068887	068104	067245	.066324	1.87
1.88	.069187	068650	067998	067251	066426	065537	1.88
1.89	.068228	067732	067118	.066407	.065616	064758	1.89
1.90	067280	.066823	066247	.065571	.064814	063988	1.90
1.91	.066342	.065926	.065386	064745	064020	.063225	1.91
1.92	065416	065038	064535	.063928	063235	.062471	1.92
1.93	.064499	064160	.063692	063120	.062459	061724	1.93
1.94	.063594	063291	.062860	062320	.061690	.060986	1.94
1.95	.062699	062433	.062036	.061529	.060930	.060255	1.95
1.96	061814	061584	.061222	.060747	.060179	.059532	1.96
1.97	.060939	.060745	.060416	.059973	.059435	058817	1.97
1.98	060075	059916	.059620	.059208	058699	.058109	1.98
1.99	.059221	.059096	.058833	.058452	.057972	057409	1.99

	SKEWNESS						
t	0	1	2	.3	4	.5	t
2.00	053991	.055555	.056702	057505	058022	058299	2.00
2.01	052919	054527	.055717	.056560	.057116	057430	2.01
2.02	051864	.053513	.054745	.055628	056221	.056571	2.02
2.03	050824	.052515	.053786	.054708	.055338	.055723	2.03
2.04	.049800	.051530	.052840	.053799	.054466	.054886	2.04
2.05	.048792	.050560	.051907	052903	053605	.054059	2.05
2.06	.047800	.049604	050988	.052019	052755	.053242	2.06
2.07	046823	.048662	.050081	.051147	051916	.052435	2.07
2.08	045861	047734	049187	.050286	051088	.051638	2.08
2.09	.044915	.046820	048306	049437	.050271	.050852	2.09
2.10	043984	045920	.047437	048600	.049464	.050075	2.10
2.11	.043067	.045033	.046581	047774	048668	.049308	2.11
2.12	042166	044160	045737	046960	.047883	.048552	2.12
2.13	041280	.043300	044905	046157	047108	.047804	2.13
2.14	.040408	.042454	.044086	045365	046343	.047067	2.14
2.15	.039550	041620	043278	044584	.045589	046339	2.15
2.16	.038707	.040800	.042483	043814	.044845	.045620	2.16
2.17	037878	039993	041699	043055	044111	.044911	2.17
2.18	037063	039198	.040927	042307	.043387	.044211	2.18
2.19	.036262	.038416	.040167	.041569	.042673	043520	2.19
2.20	.035475	037647	039418	040842	041968	042838	2.20
2.21	.034701	036890	.038680	.040126	.041274	042166	2.21
2.22	033941	036145	037954	039420	.040589	041502	2.22
2.23	.033194	035413	.037239	038724	039913	040847	2.23
2.24	032460	.034693	.036535	.038038	039247	.040201	2.24
2.25	.031740	.033984	.035842	.037363	038591	039564	2.25
2.26	031032	.033287	035160	.036698	.037943	038935	2.26
2.27	030337	.032602	.034489	.036042	.037305	.038315	2.27
2.28	.029655	.031929	.033828	035396	036676	.037703	2.28
2.29	028985	.031267	033177	.034760	.036055	037100	2.29
2.30	028327	030616	032537	.034134	.035444	.036505	2.30
2.31	027682	029976	031908	033517	.034842	.035918	2.31
2.32	.027048	029347	.031288	032909	.034248	.035339	2.32
2.33	.026426	028730	.030678	.032310	033663	034768	2.33
2.34	.025817	028123	.030079	.031721	.033086	.034205	2.34
2.35	025218	.027526	029489	031141	.032518	.033650	2.35
2.36	.024631	026940	028909	030570	031958	033103	2.36
2.37	.024056	026365	.028338	.030008	031406	.032563	2.37
2.38	.023491	.025799	027777	029454	030862	.032031	2.38
2.39	022937	025244	.027225	.028909	030327	.031507	2.39
2.40	.022395	.024699	026683	.028373	.029800	030990	2.40
2.41	021862	024163	026149	027845	029280	030480	2.41
2.42	021341	023638	.025625	027326	.028768	029978	2.42
2.43	.020829	.023122	.025109	.026815	.028264	.029483	2.43
2.44	020328	.022615	.024603	.026312	.027768	.028995	2.44
2.45	.019837	.022118	.024105	.025817	.027279	.028514	2.45
2.46	019356	.021630	.023615	.025330	.026797	.028040	2.46
2.47	.018885	.021151	.023134	.024851	.026323	.027573	2.47
2.48	018423	020681	.022662	.024380	025856	.027113	2.48
2.49	.017971	.020220	.022197	.023916	.025397	.026659	2.49

t	SKEWNESS						t
	6	.7	8	9	1.0	1.1	
2 00	.058376	.058286	058054	057703	057252	056717	2 00
2.01	.057542	.057485	.057284	056963	056541	.056032	2 01
2.02	056718	056693	056523	056232	.055837	.055354	2 02
2 03	055903	055910	.055771	055508	.055140	.054684	2 03
2.04	055099	055137	.055028	054793	.054452	.054021	2.04
2 05	054304	054373	.054293	054086	.053771	053365	2 05
2 06	053518	053617	053566	.053386	.053098	.052717	2.06
2.07	.052743	.052871	.052848	.052695	052432	.052075	2 07
2.08	051976	052134	.052138	052011	.051773	051441	2.08
2.09	051219	.051405	.051436	.051335	051122	.050813	2 09
2.10	.050471	050685	.050743	.050667	.050478	.050192	2.10
2.11	.049733	049973	050057	.050007	049842	.049579	2 11
2 12	.049003	.049271	049380	049354	049212	048971	2 12
2 13	048283	048576	048710	048709	048590	048371	2 13
2.14	.047572	047890	048049	048071	047975	.047777	2.14
2 15	.046869	.047212	047395	047440	047366	047190	2.15
2.16	046176	046543	046749	.046817	.046765	046610	2 16
2.17	045491	.045882	046111	046201	.046170	.046036	2 17
2 18	.044814	.045229	045480	045592	045582	045468	2 18
2.19	.044147	044584	.044857	044990	045001	.044907	2.19
2 20	.043488	043947	044242	044395	044427	.044352	2 20
2.21	.042837	.043317	043634	.043808	.043859	.043803	2 21
2.22	.042194	042696	043033	.043227	.043297	.043260	2 22
2.23	.041560	042082	.042439	.042653	042742	.042724	2 23
2.24	.040934	041277	041853	.042086	042194	.042194	2.24
2.25	040317	.040878	041274	.041525	.041652	.041669	2 25
2 26	039707	.040287	.040702	.040972	041116	041151	2 26
2 27	039105	.039704	040137	.040424	.040586	040638	2.27
2.28	038511	039128	.039578	039884	.040063	040131	2.28
2 29	037925	.038559	.039027	039349	.039545	039630	2.29
2.30	.037347	.037998	038482	038822	.039034	039135	2 30
2 31	036776	.037443	037945	.038300	.038529	.038645	2 31
2.32	.036213	.036896	037413	.037785	.038029	038161	2 32
2.33	.035657	036356	.036889	.037276	.037536	.037683	2.33
2.34	.035109	035823	.036371	036773	037048	037210	2.34
2 35	.034568	.035297	035859	.036277	.036566	.036743	2 35
2 36	.034034	.034777	035354	.035786	.036089	.036280	2 36
2.37	.033508	.034264	034856	.035301	035619	035824	2.37
2.38	032988	.033758	.034363	.034822	.035154	.035372	2.38
2.39	032476	.033259	033877	.034350	.034694	.034926	2 39
2 40	.031971	.032766	.033397	.033882	.034240	.034485	2 40
2.41	.031472	.032279	.032923	.033421	.033791	034049	2 41
2.42	.030981	.031799	.032454	.032965	.033348	033618	2.42
2.43	.030496	.031325	.031992	032515	.032910	.033192	2.43
2.44	.030018	.030858	.031536	.032071	.032478	.032771	2.44
2.45	.029546	.030397	031086	.031632	.032050	.032355	2.45
2.46	.029081	.029942	.030641	031198	.031628	031944	2 46
2.47	.028622	.029492	.030202	.030770	031210	.031538	2 47
2.48	.028170	.029049	.029769	.030347	030798	031137	2 48
2.49	.027724	.028612	.029342	.029929	.030391	030740	2.49

t	SKEWNESS .0	.1	.2	.3	.4	.5	t
2.50	.017528	.019768	.021741	.023461	.024944	.026212	2.50
2.51	.017095	.019324	.021293	.023012	.024499	.025772	2.51
2.52	.016670	.018888	.020852	.022571	.024060	.025338	2.52
2.53	.016254	.018461	.020420	.022137	.023629	.024911	2.53
2.54	.015848	.018043	.019995	.021711	.023204	.024490	2.54
2.55	.015449	.017632	.019578	.021292	.022785	.024075	2.55
2.56	.015060	.017230	.019168	.020879	.022374	.023666	2.56
2.57	.014678	.016835	.018766	.020474	.021968	.023264	2.57
2.58	.014305	.016448	.018371	.020075	.021570	.022867	2.58
2.59	.013940	.016068	.017983	.019684	.021177	.022477	2.59
2.60	.013583	.015697	.017603	.019298	.020791	.022093	2.60
2.61	.013234	.015332	.017229	.018920	.020411	.021714	2.61
2.62	.012892	.014975	.016862	.018548	.020038	.021341	2.62
2.63	.012558	.014625	.016502	.018182	.019670	.020974	2.63
2.64	.012232	.014282	.016149	.017823	.019308	.020612	2.64
2.65	.011912	.013946	.015802	.017470	.018952	.020256	2.65
2.66	.011600	.013617	.015462	.017123	.018602	.019905	2.66
2.67	.011295	.013295	.015128	.016782	.018257	.019560	2.67
2.68	.010997	.012979	.014800	.016447	.017919	.019220	2.68
2.69	.010706	.012670	.014479	.016118	.017585	.018886	2.69
2.70	.010421	.012367	.014163	.015795	.017258	.018556	2.70
2.71	.010143	.012071	.013854	.015477	.016935	.018232	2.71
2.72	.009871	.011780	.013551	.015165	.016619	.017913	2.72
2.73	.009606	.011496	.013253	.014859	.016307	.017599	2.73
2.74	.009347	.011218	.012961	.014558	.016000	.017289	2.74
2.75	.009094	.010945	.012675	.014262	.015699	.016985	2.75
2.76	.008847	.010679	.012395	.013972	.015403	.016685	2.76
2.77	.008605	.010418	.012120	.013687	.015112	.016391	2.77
2.78	.008370	.010163	.011850	.013408	.014825	.016100	2.78
2.79	.008140	.009913	.011586	.013133	.014544	.015815	2.79
2.80	.007915	.009669	.011326	.012863	.014267	.015534	2.80
2.81	.007697	.009430	.011072	.012598	.013995	.015258	2.81
2.82	.007483	.009196	.010823	.012338	.013728	.014986	2.82
2.83	.007274	.008967	.010579	.012083	.013465	.014718	2.83
2.84	.007071	.008743	.010340	.011833	.013206	.014455	2.84
2.85	.006873	.008524	.010106	.011587	.012953	.014195	2.85
2.86	.006679	.008310	.009876	.011346	.012703	.013941	2.86
2.87	.006491	.008101	.009651	.011109	.012458	.013690	2.87
2.88	.006307	.007897	.009431	.010877	.012217	.013443	2.88
2.89	.006127	.007697	.009215	.010649	.011981	.013201	2.89
2.90	.005953	.007501	.009003	.010425	.011748	.012962	2.90
2.91	.005782	.007310	.008796	.010206	.011519	.012727	2.91
2.92	.005616	.007123	.008593	.009990	.011295	.012496	2.92
2.93	.005454	.006941	.008394	.009779	.011075	.012269	2.93
2.94	.005296	.006763	.008199	.009572	.010858	.012046	2.94
2.95	.005143	.006588	.008009	.009368	.010645	.011826	2.95
2.96	.004993	.006418	.007822	.009169	.010436	.011610	2.96
2.97	.004847	.006252	.007639	.008974	.010231	.011398	2.97
2.98	.004705	.006089	.007460	.008782	.010029	.011189	2.98
2.99	.004567	.005930	.007285	.008594	.009831	.010984	2.99

t	SKEWNESS						t
	.6	.7	8	9	1.0	1 1	
2 50	.027285	.028181	028920	.029517	029989	.030348	2 50
2.51	026851	.027756	.028503	.029110	.029591	.029960	2 51
2.52	026424	027336	.028092	028708	.029199	.029578	2 52
2.53	026002	026922	.027686	028311	.028811	.029199	2 53
2.54	025587	.026513	.027285	.027919	.028428	028825	2.54
2.55	025178	.026110	.026890	027532	028049	.028456	2 55
2.56	.024774	.025713	026500	.027149	027676	.028091	2 56
2.57	.024376	.025321	026115	.026772	027306	.027730	2.57
2.58	.023984	.024935	.025735	.026400	.026942	.027374	2.58
2.59	.023597	.024553	.025360	.026032	.026582	027022	2 59
2 60	023216	.024177	.024990	.025669	.026226	026674	2.60
2.61	022841	023806	.024625	.025310	.025875	026331	2.61
2.62	022471	023441	024265	024956	025528	025991	2.62
2.63	.022106	.023080	.023909	024607	.025185	.025655	2 63
2.64	021747	022724	023559	024262	.024847	025324	2 64
2 65	.021392	022374	023213	.023922	.024513	024996	2.65
2.66	021043	.022028	.022871	023586	.024183	.024673	2 66
2 67	020700	021687	022535	023254	023857	024353	2.67
2 68	020361	.021351	.022202	.022927	023535	024038	2.68
2 69	.020027	021019	.021874	.022603	.023217	023726	2.69
2.70	019698	.020693	.021551	.022284	022903	023418	2 70
2.71	.019374	020370	.021232	.021970	.022593	023113	2 71
2.72	.019055	020053	.020917	.021659	.022287	.022812	2 72
2 73	.018740	019740	.020607	.021352	.021985	.022515	2 73
2.74	.018430	.019431	020301	.021049	.021687	.022222	2 74
2 75	018125	.019127	.019999	020751	.021392	.021932	2 75
2.76	.017824	018827	019701	.020456	.021101	.021646	2 76
2.77	.017528	.018531	.019407	.020165	.020814	.021363	2.77
2 78	017237	.018240	019117	019878	.020530	.021084	2.78
2.79	.016949	.017952	018831	.019594	.020250	.020808	2 79
2.80	.016666	.017669	.018549	.019315	.019974	.020535	2.80
2 81	.016388	.017390	018271	.019039	.019701	020266	2.81
2 82	.016113	017115	017997	018767	.019432	.020000	2.82
2 83	015843	.016844	.017727	.018498	.019166	.019738	2 83
2.84	.015577	.016577	.017460	.018233	.018903	.019478	2.84
2.85	015315	016314	.017197	.017971	.018644	.019222	2.85
2.86	015057	.016054	.016937	.017713	.018388	.018969	2.86
2 87	.014803	.015798	.016682	.017459	.018135	018719	2 87
2 88	.014552	.015546	016430	017207	.017886	018473	2.88
2 89	.014306	.015298	.016181	.016959	.017640	.018229	2.89
2 90	.014064	015054	015936	.016715	.017397	017988	2 90
2 91	.013825	.014813	.015694	016474	017157	.017751	2.91
2 92	.013590	.014575	.015456	.016235	.016920	.017516	2 92
2 93	.013358	.014341	.015221	.016001	016687	.017284	2.93
2.94	.013130	.014111	.014989	.015769	.016456	.017055	2.94
2.95	012906	.013884	.014760	.015540	.016228	.016829	2 95
2.96	.012685	.013660	.014535	.015315	.016003	.016606	2.96
2.97	.012468	.013439	.014313	.015092	.015782	016386	2.97
2.98	.012254	.013222	.014094	.014873	.015563	.016168	2.98
2.99	.012044	013008	.013879	.014657	.015347	.015955	2.99

t	SKEWNESS						t
	.0	1	.2	.3	.4	.5	
3.00	004432	005775	007114	008409	.009637	.010782	3.00
3 01	004301	005624	006946	008228	009446	010583	3.01
3.02	.004173	005476	.006782	008051	.009258	010388	3 02
3 03	.004049	005332	.006621	.007877	.009074	010196	3.03
3.04	003928	005191	006464	007706	008893	010007	3.04
3.05	003810	005053	006310	.007539	008716	.009822	3.05
3.06	003695	004919	006159	.007375	008541	009640	3.06
3.07	.003584	.004788	006012	007215	008370	009460	3.07
3.08	003475	004660	005867	007057	.008202	.009284	3 08
3 09	.003370	004535	005726	.006903	008037	009111	3.09
3.10	.003267	.004413	005588	.006751	007875	.008941	3 10
3.11	.003167	004295	005453	.006603	007716	.008773	3.11
3.12	003070	004178	.005321	.006458	.007560	.008609	3 12
3.13	002975	004065	.005192	.006315	007407	008447	3.13
3 14	002884	003955	005066	006176	.007257	008289	3.14
3.15	002794	003847	.004942	.006039	.007109	.008133	3 15
3 16	002707	003742	.004821	.005905	006965	007979	3 16
3 17	002623	003640	.004703	.005774	006823	007829	3.17
3.18	002541	.003540	004588	005645	006683	.007680	3 18
3 19	.002462	003442	004475	.005519	006546	007535	3.19
3.20	.002384	003347	004365	.005396	.006412	.007392	3.20
3.21	002309	003255	004257	.005275	006280	007252	3.21
3.22	002236	003165	004151	005157	006151	007114	3 22
3 23	.002165	.003077	004048	.005041	.006025	.006978	3 23
3 24	.002096	.002991	003947	.004927	005900	.006845	3.24
3 25	002029	.002907	.003849	.004816	005778	.006714	3 25
3 26	001964	.002826	.003753	004707	005659	.006586	3 26
3 27	001901	002746	.003659	.004600	005541	006459	3.27
3.28	.001840	002669	.003567	.004496	005426	.006335	3.28
3.29	.001780	.002594	.003477	004394	005313	.006214	3.29
3.30	001723	002520	003390	004293	005202	006094	3.30
3.31	001667	002449	.003304	004195	005094	.005977	3.31
3 32	001612	002379	.003220	004099	004987	005861	3 32
3 33	.001560	002311	003139	.004005	.004883	.005748	3 33
3.34	001508	002245	.003059	003914	004780	.005637	3 34
3.35	001459	002181	.002981	003823	.004680	.005527	3.35
3.36	001411	002118	.002905	.003735	.004581	.005420	3.36
3 37	001364	002057	002830	003649	.004485	.005315	3.37
3 38	.001319	001998	.002758	003565	004390	005211	3.38
3 39	.001275	001940	002687	.003482	004297	.005110	3.39
3 40	001232	001883	.002618	003401	004206	.005010	3.40
3.41	001191	.001829	.002550	.003322	.004117	.004912	3 41
3.42	.001151	.001775	.002484	003245	.004029	004816	3.42
3.43	.001112	.001723	.002420	003169	.003944	.004721	3.43
3.44	001075	.001673	.002357	.003095	.003860	.004629	3.44
3.45	.001038	001624	002296	.003022	.003777	.004538	3 45
3.46	.001003	.001576	002236	002952	003696	.004448	3.46
3 47	000969	.001529	002177	.002882	.003617	.004361	3.47
3.48	000936	.001484	.002120	.002814	.003540	.004275	3.48
3 49	000904	.001440	002065	.002748	003464	004190	3.49

t	SKEWNESS						t
	6	7	8	9	1.0	1 1	
3 00	011836	.012798	013666	014443	015133	015741	3 00
3 01	011632	012590	013456	014233	014923	015531	3 01
3 02	011432	012386	013249	014025	014715	.015321	3 02
3 03	011234	012184	013046	.013820	014510	015120	3.03
3 04	011040	011986	012845	.013618	014308	014918	3 04
3 05	010848	011791	012647	013418	014108	014719	3 05
3 06	010660	011598	012452	013222	013911	.014523	3 06
3.07	010475	011409	012259	013028	013717	.014328	3 07
3 08	010293	011222	012070	.012837	013525	014137	3 08
3 09	010113	011038	.011883	.012648	013335	013947	3 09
3 10	009937	.010857	.011699	012462	013148	013761	3 10
3 11	009763	.010679	011517	012279	012964	013576	3 11
3 12	009592	.010503	011339	012098	012782	013394	3 12
3 13	009424	.010330	011162	.011919	012603	013214	3 13
3 14	.009259	.010160	010989	011743	012425	.013037	3 14
3 15	.009096	009992	010817	.011570	.012251	012861	3 15
3.16	008936	009827	010649	011399	012078	012688	3 16
3 17	.008779	009665	010483	.011230	011908	012517	3 17
3 18	008624	009505	.010319	011064	011740	.012349	3 18
3.19	008472	009347	.010157	010900	011574	012182	3 19
3 20	008322	009192	009998	.010738	.011411	.012018	3.20
3 21	008174	009039	.009842	010578	011250	011856	3 21
3 22	.008029	008889	009687	010421	011090	.011696	3.22
3 23	007887	008741	009535	010266	010933	.011538	3 23
3 24	007747	008595	009385	010113	010778	011381	3 24
3.25	007609	.008452	009237	009962	010626	011227	3 25
3 26	.007473	008311	009092	009814	.010475	011075	3 26
3 27	007340	.008172	008949	.009667	010326	.010925	3.27
3.28	.007208	.008035	.008807	.009523	010179	.010777	3 28
3 29	007080	007900	008668	.009380	.010035	010631	3.29
3 30	006953	007767	008531	.009240	009892	010486	3.30
3 31	006828	007637	008396	009101	009751	010344	3 31
3.32	.006705	.007508	008263	008965	009612	010203	3 32
3 33	.006585	.007382	008132	.008830	009475	010065	3.33
3 34	006466	007257	008003	.008698	009340	009927	3 34
3 35	006350	.007135	007876	008567	009206	.009792	3 35
3 36	006235	007014	.007750	.008438	.009075	009659	3.36
3 37	006122	006896	007627	008311	.008945	009527	3 37
3 38	006011	006779	007506	.008186	008817	009397	3 38
3.39	005903	.006664	007386	008062	.008690	009269	3.39
3 40	005796	006551	007268	007941	.008566	009142	3 40
3 41	005690	006440	007152	.007821	008443	.009017	3 41
3 42	005587	006330	007037	.007702	.008322	008894	3 42
3.43	.005485	.006222	006925	007586	008202	008772	3.43
3.44	.005385	006116	006814	.007471	.008084	008652	3 44
3 45	.005287	006012	.006704	.007358	007968	.008533	3.45
3 46	005190	.005909	.006597	.007246	007853	.008416	3 46
3.47	005095	.005808	006491	.007136	007740	008301	3 47
3 48	005002	005709	006386	.007028	.007629	008187	3.48
3.49	.004910	.005611	006284	.006921	.007519	.008074	3.49

	SKEWNESS						
t	0	.1	.2	3	.4	5	t
3.50	000873	.001397	.002010	002683	003389	004107	3.50
3.51	000843	001356	001957	002619	003316	004026	3 51
3 52	.000814	.001315	.001906	002557	003245	.003946	3.52
3 53	.000785	001276	001855	.002497	003174	003868	3 53
3.54	.000758	001238	.001806	002437	003106	.003791	3 54
3.55	.000732	001200	001758	002379	003038	003715	3 55
3.56	.000706	.001164	001711	002322	002972	.003641	3 56
3.57	000681	.001129	.001666	002267	002908	.003568	3.57
3.58	.000657	.001095	.001621	.002212	002845	003497	3 58
3 59	000634	001062	001578	.002159	002782	.003427	3 59
3 60	000612	001029	.001536	002107	002722	003358	3 60
3.61	000590	.000998	.001494	002057	002662	003291	3 61
3.62	000569	.000967	.001454	.002007	002604	.003225	3 62
3.63	.000549	.000938	.001415	001959	002547	003160	3.63
3 64	000529	.000909	.001377	001911	002491	003096	3 64
3.65	000510	000881	001339	001865	002436	003034	3.65
3.66	000492	.000854	001303	001820	002382	.002972	3.66
3.67	000474	.000827	001268	.001775	002330	.002912	3 67
3 68	.000457	.000802	001233	.001732	002278	002853	3 68
3 69	.000441	.000777	001200	001690	002228	002795	3 69
3.70	000425	000753	001167	001649	002179	002739	3 70
3.71	.000409	000729	001135	.001608	.002130	002683	3 71
3 72	.000394	.000706	001104	001569	002083	002628	3 72
3.73	.000380	000684	.001073	.001530	.002037	002575	3 73
3 74	.000366	000663	.001044	001493	001991	.002522	3 74
3.75	.000353	000642	001015	001456	.001947	002471	3.75
3.76	000340	000622	000987	001420	.001903	002420	3 76
3.77	.000327	.000602	000960	001385	.001861	.002371	3 77
3.78	000315	000583	000933	001351	.001819	002322	3.78
3.79	000303	.000565	000907	.001317	.001778	002275	3.79
3.80	000292	.000547	.000882	001284	001738	002228	3.80
3.81	.000281	000529	.000857	001253	.001699	.002182	3.81
3.82	.000271	.000512	.000833	001221	.001661	002137	3.82
3.83	.000260	.000496	000810	.001191	.001624	.002093	3 83
3.84	000251	.000480	000787	001161	.001587	002050	3.84
3.85	.000241	000465	.000765	001132	.001551	002008	3 85
3 86	.000232	000450	000744	001104	.001516	001966	3 86
3.87	.000223	000435	.000723	001076	001482	.001925	3.87
3.88	000215	000421	000702	001049	001448	001885	3 88
3.89	.000207	000407	000682	.001023	.001415	.001846	3 89
3 90	.000199	000394	.000663	.000997	.001383	001808	3 90
3.91	.000191	.000381	.000644	.000972	001351	.001770	3 91
3 92	.000184	.000369	000626	000947	001321	.001733	3 92
3.93	.000177	.000357	000608	000923	.001290	.001697	3.93
3.94	.000170	.000345	000591	000900	.001261	.001662	3.94
3 95	.000163	.000334	000574	000877	001232	001627	3.95
3 96	.000157	.000323	000557	000855	.001204	.001593	3 96
3 97	.000151	.000312	000541	.000833	001176	.001560	3 97
3.98	.000144	000302	000526	.000811	001149	001527	3 98
3 99	.000139	000292	.000511	.000791	.001123	001495	3.99

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
3.50	.004820	.005515	.006182	.006815	.007410	.007963	3.50
3.51	.004732	.005420	.006083	.006712	.007303	.007854	3.51
3.52	.004645	.005327	.005985	.006609	.007197	.007746	3.52
3.53	.004559	.005236	.005888	.006509	.007093	.007639	3.53
3.54	.004475	.005146	.005793	.006409	.006991	.007534	3.54
3.55	.004392	.005057	.005699	.006311	.006889	.007430	3.55
3.56	.004311	.004970	.005607	.006215	.006790	.007327	3.56
3.57	.004231	.004884	.005516	.006120	.006691	.007226	3.57
3.58	.004153	.004799	.005426	.006026	.006594	.007126	3.58
3.59	.004076	.004716	.005338	.005934	.006498	.007028	3.59
3.60	.004000	.004635	.005251	.005843	.006404	.006931	3.60
3.61	.003926	.004554	.005166	.005753	.006311	.006835	3.61
3.62	.003853	.004475	.005082	.005665	.006219	.006740	3.62
3.63	.003781	.004398	.004999	.005578	.006128	.006647	3.63
3.64	.003711	.004321	.004917	.005492	.006039	.006555	3.64
3.65	.003641	.004246	.004837	.005407	.005951	.006464	3.65
3.66	.003573	.004172	.004758	.005324	.005864	.006374	3.66
3.67	.003506	.004099	.004680	.005242	.005779	.006286	3.67
3.68	.003441	.004028	.004604	.005161	.005694	.006199	3.68
3.69	.003376	.003957	.004528	.005082	.005611	.006112	3.69
3.70	.003313	.003888	.004454	.005003	.005529	.006027	3.70
3.71	.003251	.003820	.004381	.004926	.005448	.005944	3.71
3.72	.003190	.003753	.004309	.004850	.005368	.005861	3.72
3.73	.003129	.003687	.004238	.004774	.005290	.005779	3.73
3.74	.003070	.003623	.004169	.004701	.005212	.005699	3.74
3.75	.003013	.003559	.004100	.004628	.005136	.005619	3.75
3.76	.002956	.003497	.004033	.004556	.005060	.005541	3.76
3.77	.002900	.003435	.003966	.004485	.004986	.005464	3.77
3.78	.002845	.003374	.003901	.004415	.004913	.005387	3.78
3.79	.002791	.003315	.003836	.004347	.004840	.005312	3.79
3.80	.002738	.003256	.003773	.004279	.004769	.005238	3.80
3.81	.002686	.003199	.003711	.004213	.004699	.005165	3.81
3.82	.002635	.003142	.003649	.004147	.004630	.005092	3.82
3.83	.002585	.003087	.003589	.004082	.004561	.005021	3.83
3.84	.002536	.003032	.003529	.004019	.004494	.004951	3.84
3.85	.002487	.002978	.003471	.003956	.004428	.004881	3.85
3.86	.002440	.002926	.003413	.003894	.004362	.004813	3.86
3.87	.002393	.002874	.003356	.003833	.004298	.004745	3.87
3.88	.002347	.002823	.003301	.003773	.004234	.004679	3.88
3.89	.002303	.002772	.003246	.003714	.004172	.004613	3.89
3.90	.002258	.002723	.003192	.003656	.004110	.004548	3.90
3.91	.002215	.002674	.003138	.003599	.004049	.004484	3.91
3.92	.002173	.002627	.003086	.003542	.003989	.004421	3.92
3.93	.002131	.002580	.003035	.003487	.003930	.004359	3.93
3.94	.002090	.002534	.002984	.003432	.003872	.004298	3.94
3.95	.002050	.002489	.002934	.003378	.003814	.004237	3.95
3.96	.002010	.002444	.002885	.003325	.003758	.004178	3.96
3.97	.001971	.002400	.002837	.003273	.003702	.004119	3.97
3.98	.001933	.002357	.002789	.003221	.003647	.004061	3.98
3.99	.001896	.002315	.002742	.003170	.003593	.004003	3.99

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
4.00	.000134	.000282	.000496	.000771	.001097	.001463	4.00
4.01	.000129	.000273	.000482	.000751	.001071	.001433	4.01
4.02	.000124	.000264	.000468	.000732	.001047	.001403	4.02
4.03	.000119	.000255	.000454	.000713	.001022	.001373	4.03
4.04	.000114	.000246	.000441	.000694	.000999	.001344	4.04
4.05	.000109	.000238	.000428	.000676	.000976	.001316	4.05
4.06	.000105	.000230	.000416	.000659	.000953	.001288	4.06
4.07	.000101	.000222	.000404	.000642	.000931	.001260	4.07
4.08	.000097	.000215	.000392	.000625	.000909	.001234	4.08
4.09	.000093	.000208	.000380	.000609	.000888	.001208	4.09
4.10	.000089	.000201	.000369	.000593	.000867	.001182	4.10
4.11	.000086	.000194	.000358	.000578	.000847	.001157	4.11
4.12	.000082	.000187	.000348	.000563	.000827	.001132	4.12
4.13	.000079	.000181	.000337	.000548	.000808	.001108	4.13
4.14	.000076	.000175	.000328	.000534	.000789	.001084	4.14
4.15	.000073	.000169	.000318	.000520	.000770	.001061	4.15
4.16	.000070	.000163	.000308	.000506	.000752	.001039	4.16
4.17	.000067	.000157	.000299	.000493	.000735	.001016	4.17
4.18	.000064	.000152	.000290	.000480	.000717	.000995	4.18
4.19	.000062	.000147	.000282	.000468	.000700	.000973	4.19
4.20	.000059	.000142	.000273	.000455	.000684	.000952	4.20
4.21	.000057	.000137	.000265	.000443	.000668	.000932	4.21
4.22	.000054	.000132	.000257	.000432	.000652	.000912	4.22
4.23	.000052	.000128	.000250	.000420	.000636	.000892	4.23
4.24	.000050	.000123	.000242	.000409	.000621	.000873	4.24
4.25	.000048	.000119	.000235	.000398	.000606	.000854	4.25
4.26	.000046	.000115	.000228	.000388	.000592	.000836	4.26
4.27	.000044	.000111	.000221	.000377	.000578	.000818	4.27
4.28	.000042	.000107	.000214	.000367	.000564	.000800	4.28
4.29	.000040	.000103	.000208	.000357	.000551	.000783	4.29
4.30	.000039	.000099	.000201	.000348	.000538	.000766	4.30
4.31	.000037	.000096	.000195	.000339	.000525	.000749	4.31
4.32	.000035	.000093	.000189	.000330	.000512	.000733	4.32
4.33	.000034	.000089	.000184	.000321	.000500	.000717	4.33
4.34	.000032	.000086	.000178	.000312	.000488	.000701	4.34
4.35	.000031	.000083	.000173	.000304	.000476	.000686	4.35
4.36	.000030	.000080	.000167	.000296	.000465	.000671	4.36
4.37	.000028	.000077	.000162	.000288	.000454	.000656	4.37
4.38	.000027	.000074	.000157	.000280	.000443	.000642	4.38
4.39	.000026	.000072	.000152	.000272	.000432	.000628	4.39
4.40	.000025	.000069	.000148	.000265	.000421	.000614	4.40
4.41	.000024	.000067	.000143	.000258	.000411	.000601	4.41
4.42	.000023	.000064	.000139	.000251	.000401	.000588	4.42
4.43	.000022	.000062	.000134	.000244	.000392	.000575	4.43
4.44	.000021	.000060	.000130	.000237	.000382	.000562	4.44
4.45	.000020	.000058	.000126	.000231	.000373	.000550	4.45
4.46	.000019	.000056	.000122	.000225	.000364	.000538	4.46
4.47	.000018	.000054	.000119	.000219	.000355	.000526	4.47
4.48	.000018	.000052	.000115	.000213	.000346	.000514	4.48
4.49	.000017	.000050	.000111	.000207	.000338	.000503	4.49

t	SKEWNESS						t
	6	7	8	9	1 0	1 1	
4.00	.001859	.002273	.002696	.003121	.03539	.003947	4.00
4.01	.001823	.002233	.002651	.003071	.003486	.003891	4.01
4.02	.001788	.002192	.002607	.003023	.003434	.003836	4.02
4.03	.001753	.002153	.002563	.002975	.003383	.003782	4.03
4.04	.001719	.002114	.002520	.002928	.003333	.003728	4.04
4.05	.001686	.002076	.002477	.002882	.003283	.003676	4.05
4.06	.001653	.002039	.002436	.002836	.003234	.003624	4.06
4.07	.001621	.002002	.002394	.002791	.003186	.003572	4.07
4.08	.001589	.001966	.002354	.002747	.003138	.003522	4.08
4.09	.001558	.001930	.002314	.002704	.003091	.003472	4.09
4.10	.001528	.001895	.002275	.002661	.003045	.003423	4.10
4.11	.001498	.001861	.002237	.002618	.002999	.003374	4.11
4.12	.001469	.001827	.002199	.002577	.002954	.003326	4.12
4.13	.001440	.001794	.002162	.002536	.002910	.003279	4.13
4.14	.001412	.001761	.002125	.002496	.002866	.003232	4.14
4.15	.001384	.001729	.002089	.002456	.002823	.003186	4.15
4.16	.001357	.001698	.002054	.002417	.002781	.003141	4.16
4.17	.001330	.001667	.002019	.002378	.002739	.003096	4.17
4.18	.001304	.001637	.001984	.002340	.002698	.003052	4.18
4.19	.001278	.001607	.001951	.002303	.002658	.003009	4.19
4.20	.001253	.001577	.001918	.002266	.002618	.002966	4.20
4.21	.001228	.001549	.001885	.002230	.002578	.002924	4.21
4.22	.001204	.001520	.001853	.002194	.002539	.002882	4.22
4.23	.001180	.001492	.001821	.002159	.002501	.002841	4.23
4.24	.001157	.001465	.001790	.002125	.002463	.002800	4.24
4.25	.001134	.001438	.001760	.002091	.002426	.002760	4.25
4.26	.001112	.001412	.001730	.002057	.002390	.002721	4.26
4.27	.001090	.001386	.001700	.002024	.002354	.002682	4.27
4.28	.001068	.001361	.001671	.001992	.002318	.002644	4.28
4.29	.001047	.001336	.001642	.001960	.002283	.002606	4.29
4.30	.001026	.001311	.001614	.001929	.002248	.002569	4.30
4.31	.001005	.001287	.001586	.001898	.002214	.002532	4.31
4.32	.000985	.001263	.001559	.001867	.002181	.002496	4.32
4.33	.000966	.001240	.001533	.001837	.002148	.002460	4.33
4.34	.000947	.001217	.001506	.001808	.002115	.002425	4.34
4.35	.000928	.001195	.001480	.001778	.002083	.002390	4.35
4.36	.000909	.001173	.001455	.001750	.002052	.002356	4.36
4.37	.000891	.001151	.001430	.001722	.002021	.002322	4.37
4.38	.000873	.001130	.001405	.001694	.001990	.002289	4.38
4.39	.000856	.001109	.001381	.001667	.001960	.002256	4.39
4.40	.000838	.001088	.001357	.001640	.001930	.002223	4.40
4.41	.000822	.001068	.001334	.001613	.001901	.002191	4.41
4.42	.000805	.001048	.001311	.001587	.001872	.002160	4.42
4.43	.000789	.001029	.001288	.001561	.001843	.002129	4.43
4.44	.000773	.001010	.001266	.001536	.001815	.002098	4.44
4.45	.000758	.000991	.001244	.001511	.001788	.002068	4.45
4.46	.000742	.000972	.001223	.001487	.001760	.002038	4.46
4.47	.000727	.000954	.001201	.001463	.001733	.002009	4.47
4.48	.000713	.000937	.001181	.001439	.001707	.001980	4.48
4.49	.000698	.000919	.001160	.001416	.001681	.001951	4.49

t	SKEWNESS						t
	0	.1	.2	.3	4	5	
4.50	000016	000048	000108	.000201	.000330	000492	4.50
4 51	000015	000046	000104	.000196	000321	000481	4.51
4 52	000015	000044	000101	.000190	000314	000470	4 52
4 53	000014	000043	.000098	000185	000306	000460	4 53
4 54	000013	000041	000095	000180	000298	.000450	4.54
4 55	000013	.000040	000092	000175	000291	000440	4 55
4 56	.000012	.000038	.000089	000170	000284	.000430	4.56
4.57	.000012	000037	000086	000165	.000277	.000420	4 57
4 58	000011	.000036	000084	000161	000270	000411	4.58
4.59	000011	000034	.000081	000156	.000263	.000402	4.59
4 60	000010	.000033	000078	000152	000257	.000393	4 60
4 61	000010	000032	000076	.000148	.000251	.000384	4.61
4.62	.000009	.000031	000073	000144	000244	.000375	4.62
4 63	000009	.000029	000071	.000140	000238	.000367	4.63
4 64	000008	.000028	000069	000136	.000232	000359	4 64
4 65	000008	000027	000067	000132	000227	000351	4 65
4 66	000008	000026	000065	.000128	.000221	000343	4 66
4.67	000007	000025	.000062	000125	.000216	000335	4.67
4 68	.000007	000024	000060	000121	000210	000328	4 68
4 69	000007	.000023	.000059	.000118	000205	000320	4.69
4 70	.000006	000023	000057	000115	.000200	000313	4 70
4.71	000006	000022	000055	000111	.000195	.000306	4.71
4 72	.000006	000021	000053	000108	000190	000299	4 72
4 73	.000006	000020	.000051	000105	.000185	000292	4.73
4 74	000005	000019	000050	000102	000181	.000286	4.74
4.75	000005	000019	000048	.000099	000176	000279	4.75
4 76	000005	000018	000047	.000097	000172	.000273	4 76
4 77	.000005	.000017	000045	000094	000167	000267	4.77
4 78	000004	000017	000044	000091	000163	.000261	.478
4 79	.000004	000016	000042	000089	000159	000255	4.79
4 80	000004	000015	000041	000086	000155	.000249	4 80
4 81	.000004	000015	000039	000084	000151	.000244	4.81
4 82	.000004	000014	.000038	000081	000147	.000238	4 82
4 83	.000003	000014	000037	.000079	000144	.000233	4.83
4 84	.000003	000013	.000036	000077	000140	000227	4 84
4 85	.000003	000013	000035	000075	000136	000222	4 85
4.86	.000003	000012	000033	000072	000133	000217	4 86
4.87	.000003	000012	000032	.000070	000130	.000212	4 87
4.88	000003	.000011	.000031	.000068	000126	000207	4 88
4.89	000003	000011	000030	000066	000123	000203	4 89
4 90	.000002	000010	000029	000064	000120	.000198	4 90
4 91	000002	.000010	.000028	.000063	000117	.000193	4.91
4 92	.000002	.000010	000027	000061	000114	.000189	4.92
4.93	000002	.000009	.000026	000059	.000111	000185	4 93
4.94	.000002	.000009	.000026	.000057	000108	000180	4 94
4 95	.000002	000008	000025	.000056	000105	.000176	4.95
4 96	.000002	.000008	.000024	000054	.000103	.000172	4.96
4 97	.000002	.000008	.000023	000053	.000100	000168	4.97
4.98	.000002	000007	000022	.000051	.000098	000164	4 98
4.99	000002	000007	000022	.000050	.000095	000161	4.99

t	SKEWNESS						t
	.6	.7	8	.9	1.0	1.1	
4 50	000684	000902	001140	.001393	001655	.001923	4.50
4.51	000670	.000885	001120	.001370	.001630	.001895	4 51
4 52	000657	.000869	.001101	.001348	.001605	.001868	4 52
4.53	000643	000852	.001082	.001326	.001581	001841	4.53
4.54	000630	000836	.001063	.001304	001556	.001814	4.54
4 55	.000618	.000821	001044	.001283	.001533	.001788	4.55
4.56	000605	000805	.001026	.001262	.001509	.001762	4.56
4 57	000593	.000790	001008	.001242	.001486	.001737	4.57
4.58	000581	000775	000991	001222	.001463	.001712	4.58
4.59	000569	.000761	.000973	001202	.001441	.001687	4.59
4.60	000557	.000747	000956	.001182	.001419	.001663	4.60
4.61	.000546	.000733	.000940	.001163	.001397	.001638	4.61
4 62	.000535	000719	000923	.001144	.001376	001615	4.62
4 63	.000524	.000705	000907	.001125	001354	.001591	4.63
4 64	000513	.000692	000891	.001107	.001334	.001568	4.64
4.65	000503	.000679	000876	.001089	001313	.001545	4.65
4 66	000492	.000666	.000860	.001071	.001293	001523	4 66
4.67	.000482	000654	.000845	.001053	.001273	.001501	4.67
4 68	000472	000641	000831	.001036	.001253	.001479	4.68
4 69	.000463	.000629	000816	.001019	.001234	.001458	4.69
4:70	.000453	000617	000802	.001002	.001215	.001436	4 70
4 71	.000444	000606	000788	.000986	.001196	.001416	4.71
4 72	000435	000594	.000774	.000970	001178	.001395	4.72
4.73	000426	000583	.000760	.000954	.001160	.001375	4.73
4.74	000417	000572	000747	.000938	.001142	.001355	4.74
4.75	000408	000561	000734	.000923	.001124	.001335	4.75
4.76	000400	.000550	000721	.000908	.001107	.001315	4.76
4 77	.000392	000540	.000708	.000893	.001090	.001296	4 77
4 78	.000384	.000530	.000696	.000878	.001073	.001277	4 78
4 79	000376	000520	000683	000863	.001056	.001259	4.79
4 80	000368	.000510	000671	.000849	.001040	.001240	4 80
4 81	.000360	000500	.000659	.000835	.001024	001222	4.81
4 82	000353	.000490	.000648	.000821	.001008	.001204	4.82
4 83	000346	000481	000636	.000808	000993	.001187	4.83
4.84	.000338	000472	.000625	.000795	.000977	.001170	4 84
4.85	000331	000463	.000614	000781	.000962	.001152	4 85
4.86	.000325	000454	000603	000769	.000947	.001136	4.86
4 87	.000318	.000445	.000592	000756	.000932	001119	4 87
4 88	000311	000437	000582	000743	.000918	001103	4.88
4 89	000305	.000429	000572	000731	.000904	.001086	4.89
4 90	000298	000420	.000561	000719	000890	.001071	4.90
4 91	000292	000412	.000551	000707	.000876	.001055	4.91
4.92	000286	.000404	.000542	000695	.000862	001039	4.92
4.93	.000280	000397	.000532	.000684	.000849	.001024	4 93
4.94	.000274	.000389	.000523	.000672	.000836	.001009	4.94
4 95	.000269	000382	.000513	.000661	000823	.000994	4.95
4 96	.000263	.000374	.000504	.000650	000810	.000980	4.96
4 97	000257	.000367	.000495	.000640	.000797	.000965	4 97
4.98	000252	.000360	.000486	.000629	.000785	.000951	4.98
4.99	000247	000353	.000478	.000618	.000773	.000937	4.99

	SKEWNESS						
t	0	1	2	.3	.4	.5	t
5.00	.000001	000007	000021	000048	000093	.000157	5.00
5.01	000001	000007	000020	000047	000090	000153	5.01
5.02	.000001	000006	.000019	000045	000088	000150	5.02
5.03	000001	000006	000019	000044	000086	000146	5.03
5.04	000001	000006	000018	000043	.000083	.000143	5.04
5.05	000001	000006	000018	000042	000081	000140	5.05
5.06	000001	000005	.000017	000040	000079	000136	5.06
5.07	000001	000005	000016	000039	000077	000133	5.07
5.08	000001	000005	000016	000038	000075	000130	5.08
5.09	000001	000005	000015	000037	000073	.000127	5.09
5.10	.000001	000005	.000015	.000036	000071	.000124	5.10
5.11	.000001	.000004	000014	000035	.000070	000121	5.11
5.12	.000001	.000004	.000014	000034	000068	000118	5.12
5.13	000001	000004	.000013	000033	000066	000116	5.13
5.14	000001	000004	000013	000032	.000064	000113	5.14
5.15	.000001	000004	000012	.000031	000063	.000110	5.15
5.16	000001	000004	.000012	000030	000061	000108	5.16
5.17	000001	000003	000012	000029	000059	000105	5.17
5.18	000001	000003	000011	000028	.000058	000103	5.18
5.19	000001	.000003	000011	000027	000056	000100	5.19
5.20	000001	.000003	000010	000027	000055	000098	5.20
5.21	000001	000003	000010	000026	000053	.000096	5.21
5.22		000003	000010	.000025	000052	000093	5.22
5.23		000003	.000009	000024	.000051	000091	5.23
5.24		000003	000009	000024	000049	000089	5.24
5.25		.000002	.000009	000023	.000048	.000087	5.25
5.26		.000002	000008	000022	000047	000085	5.26
5.27		.000002	000008	000021	000046	000083	5.27
5.28		000002	000008	000021	000044	000081	5.28
5.29		000002	000008	000020	.000043	.000079	5.29
5.30		000002	000007	.000020	000042	000077	5.30
5.31		000002	000007	000019	000041	.000075	5.31
5.32		000002	000007	000018	000040	.000074	5.32
5.33		000002	000007	000018	000039	000072	5.33
5.34		000002	000006	000017	000038	.000070	5.34
5.35		000002	000006	.000017	000037	000069	5.35
5.36		000002	.000006	.000016	000036	000067	5.36
5.37		000001	000006	000016	000035	000065	5.37
5.38		.000001	000006	000015	000034	.000064	5.38
5.39		000001	.000005	000015	.000033	.000062	5.39
5.40		000001	.000005	.000014	.000032	.000061	5.40
5.41		000001	000005	000014	000031	.000059	5.41
5.42		000001	000005	.000014	.000030	.000058	5.42
5.43		.000001	.000005	.000013	000030	000056	5.43
5.44		000001	.000004	.000013	.000029	.000055	5.44
5.45		000001	000004	000012	000028	000054	5.45
5.46		000001	.000004	000012	000027	.000053	5.46
5.47		000001	.000004	.000012	.000027	.000051	5.47
5.48		000001	000004	.000011	000026	000050	5.48
5.49		000001	.000004	.000011	.000025	000049	5.49

	SKEWNESS						
t	6	7	8	.9	1.0	1 1	t
5 00	000242	.000346	000469	.000608	000761	000924	5.00
5 01	000236	000340	.000461	.000598	000749	000910	5.01
5 02	000232	000333	000453	000588	000737	000897	5.02
5 03	000227	000327	000444	000578	000726	000883	5.03
5.04	000222	.000320	.000437	.000569	.000714	000870	5 04
5.05	.000217	000314	000429	000559	.000703	000857	5.05
5 06	000213	000308	000421	000550	.000692	.000845	5 06
5 07	.000208	000302	.000414	000541	.000681	000833	5 07
5 08	000204	000296	000406	000532	000671	000820	5.08
5 09	000199	.000290	000399	000523	.000660	000808	5.09
5 10	000195	000285	000392	.000514	.000650	000796	5.10
5 11	000191	000279	000385	.000506	.000640	.000785	5 11
5.12	000187	000274	000378	000497	000630	.000773	5 12
5.13	000183	000269	.000371	.000489	000620	000762	5.13
5 14	000179	.000263	.000364	000481	.000610	000750	5.14
5 15	000175	000258	.000358	.000473	000600	000739	5.15
5 16	000172	.000253	000351	000465	000591	.000728	5.16
5 17	000168	000248	000345	.000457	.000582	000718	5 17
5 18	000165	000243	.000339	.000449	000573	.000707	5 18
5 19	000161	000239	.000333	000442	.000564	000697	5.19
5 20	.000158	000234	000327	.000434	000555	000686	5.20
5 21	.000154	.000230	.000321	000427	000546	000676	5 21
5 22	.000151	000225	000315	.000420	.000537	000666	5.22
5.23	000148	000221	000309	000413	000529	000656	5 23
5 24	000145	000216	.000304	000406	000521	000647	5 24
5.25	000142	.000212	.000298	.000399	.000513	000637	5 25
5 26	000139	.000208	.000293	.000392	000504	000628	5.26
5 27	000136	000204	000288	.000386	000497	.000618	5.27
5 28	000133	.000200	000282	000379	.000489	000609	5 28
5 29	000130	000196	000277	.000373	.000481	000600	5.29
5 30	000127	000192	000272	.000367	.000473	000591	5 30
5.31	.000124	000188	000267	.000360	.000466	000582	5 31
5 32	.000122	.000185	000263	.000354	.000459	.000574	5 32
5 33	000119	000181	000258	000348	.000451	.000565	5.33
5 34	000117	000178	.000253	.000342	000444	000557	5.34
5 35	000114	.000174	000248	000337	.000437	.000549	5.35
5 36	.000112	.000171	.000244	000331	.000430	000540	5.36
5 37	000109	000167	000240	000325	.000424	000532	5 37
5 38	000107	.000164	000235	000320	000417	000524	5 38
5.39	000105	.000161	000231	000314	000410	.000517	5.39
5 40	000102	000158	000227	.000309	.000404	000509	5 40
5.41	000100	.000154	.000223	000304	000397	.000501	5.41
5 42	.000098	000151	000219	.000299	.000391	000494	5.42
5 43	.000096	.000148	000215	.000294	.000385	000487	5.43
5.44	000094	.000146	.000211	.000289	.000379	000479	5.44
5.45	.000092	.000143	000207	000284	000373	000472	5.45
5.46	000090	.000140	.000203	.000279	.000367	.000465	5 46
5 47	.000088	000137	000199	000274	000361	.000458	5 47
5 48	.000086	000134	.000196	.000270	000355	.000451	5.48
5 49	.000084	.000132	·000192	.000265	000350	.000445	5.49

t	SKEWNESS						t
	0	1	.2	.3	.4	5	
5.50		000001	000004	.000011	000025	.000048	5.50
5 51		000001	.000003	.000010	000024	000047	5 51
5 52		000001	000003	.000010	000023	.000046	5.52
5.53		000001	000003	000010	000023	000044	5 53
5 54		.000001	.000003	000009	.000022	.000043	5.54
5.55		000001	.000003	.000009	000021	.000042	5.55
5.56		000001	000003	.000009	.000021	.000041	5.56
5.57		000001	.000003	.000009	000020	.000040	5.57
5.58		000001	000003	.000008	.000020	.000039	5.58
5.59		.000001	000003	000008	000019	.000038	5.59
5 60		.000001	000003	000008	000019	000037	5.60
5.61		000001	000002	000008	.000018	000037	5.61
5 62			000002	000007	000018	000036	5 62
5 63			.000002	000007	.000017	000035	5 63
5 64			.000002	.000007	.000017	000034	5 64
5 65			000002	000007	.000016	000033	5 65
5 66			000002	.000006	000016	000032	5.66
5 67			.000002	000006	000015	000032	5 67
5 68			000002	000006	.000015	000031	5 68
5.69			000002	000006	.000015	.000030	5 69
5.70			000002	.000006	000014	000029	5 70
5.71			000002	.000006	000014	.000029	5 71
5.72			.000002	000005	.000013	000028	5.72
5.73			000002	000005	000013	000027	5 73
5.74			000001	.000005	.000013	000027	5 74
5 75			.000001	000005	000012	000026	5.75
5 76			000001	000005	.000012	000025	5 76
5.77			000001	000005	000012	.000025	5 77
5 78			000001	000004	.000011	.000024	5 78
5 79			000001	000004	000011	000024	5.79
5.80			000001	.000004	000011	.000023	5.80
5.81			.000001	.000004	000010	.000022	5 81
5.82			.000001	000004	.000010	.000022	5 82
5 83			.000001	000004	000010	000021	5.83
5 84			000001	000004	000010	000021	5 84
5 85			.000001	.000004	.000009	000020	5 85
5 86			000001	000003	000009	000020	5 86
5 87			.000001	000003	000009	.000019	5 87
5.88			.000001	.000003	000009	.000019	5 88
5.89			.000001	000003	.000008	.000018	5 89
5 90			000001	000003	.000008	.000018	5 90
5.91			000001	.000003	000008	000017	5.91
5.92			000001	.000003	000008	000017	5.92
5.93			.000001	.000003	000007	.000017	5 93
5.94			.000001	000003	000007	.000016	5.94
5.95			000001	000003	.000007	.000016	5 95
5.96			000001	000002	.000007	.000015	5 96
5 97			.000001	.000002	000007	000015	5 97
5 98			000001	000002	000007	000015	5.98
5.99			.000001	000002	.000006	000014	5.99

t	SKEWNESS						t
	6	.7	8	.9	1 0	1.1	
5 50	000082	000129	000189	.000261	.000344	000438	5 50
5 51	000080	000127	000185	000256	.000339	000432	5 51
5 52	.000079	.000124	000182	000252	.000333	000425	5 52
5 53	.000077	000122	000178	.000248	.000328	000419	5.53
5 54	000075	000119	.000175	.000243	000323	000412	5 54
5 55	.000074	.000117	.000172	000239	.000318	000406	5 55
5.56	.000072	.000114	000169	000235	.000313	000400	5.56
5.57	.000071	000112	000166	000231	000308	.000394	5.57
5.58	.000069	000110	000163	000227	000303	000388	5 58
5.59	000067	000108	000160	000223	.000298	000383	5 59
5 60	.000066	000106	000157	000220	.000293	.000377	5 60
5.61	.000065	000104	000154	000216	000289	.000371	5 61
5.62	.000063	.000101	.000151	000212	.000284	000366	5 62
5.63	000062	000099	.000148	.000208	000279	000360	5 63
5 64	000060	.000097	000146	000205	000275	000355	5 64
5 65	.000059	000096	000143	000201	000271	000349	5 65
5.66	000058	000094	000140	000198	000266	.000344	5.66
5 67	.000057	000092	000138	.000195	.000262	000339	5 67
5 68	000055	000090	000135	000191	.000258	.000334	5.68
5 69	.000054	.000088	.000133	.000188	000254	000329	5.69
5 70	000053	000086	000130	.000185	000250	000324	5 70
5 71	000052	000084	000128	000182	.000246	000319	5 71
5 72	.000051	000083	000125	000179	000242	.000314	5 72
5.73	000050	000081	000123	.000175	.000238	.000310	5 73
5.74	.000048	000080	000121	000172	000234	.000305	5.74
5 75	.000047	.000078	.000119	000170	000230	000300	5.75
5.76	000046	.000076	000116	000167	000227	000296	5 76
5.77	000045	.000075	.000114	.000164	.000223	.000291	5.77
5 78	000044	000073	.000112	000161	000219	000287	5 78
5.79	000043	.000072	.000110	000158	000216	000283	5 79
5 80	.000042	.000071	000108	.000155	.000212	.000279	5.80
5.81	000041	.000069	.000106	.000153	.000209	000274	5 81
5.82	000041	000068	.000104	.000150	000206	000270	5 82
5.83	.000040	.000066	000102	.000148	000202	.000266	5 83
5 84	000039	.000065	.000100	000145	.000199	000262	5.84
5 85	000038	000064	.000098	000143	.000196	000258	5 85
5 86	000037	000062	.000097	.000140	000193	.000254	5.86
5 87	000036	.000061	.000095	000138	.000190	.000250	5 87
5.88	.000035	.000060	.000093	000135	000187	000247	5 88
5.89	000035	000059	.000091	.000133	.000184	.000243	5 89
5.90	.000034	000058	.000090	000131	.000181	.000239	5 90
5.91	.000033	.000056	000088	000128	000178	000236	5.91
5.92	000032	000055	000086	000126	000175	000232	5 92
5.93	.000032	000054	000085	.000124	.000172	000229	5 93
5 94	000031	000053	.000083	.000122	000169	.000225	5.94
5.95	.000030	000052	.000082	000120	000167	000222	5 95
5 96	.000030	000051	.000080	.000118	.000164	000218	5.96
5.97	.000029	.000050	.000079	000116	.000161	.000215	5 97
5.98	000028	000049	000077	.000114	000159	.000212	5.98
5.99	.000028	.000048	.000076	.000112	.000156	.000209	5.99

t	SKEWNESS						t
	.0	.1	2	.3	4	5	
6.00			000001	.000002	.000006	000014	6 00
6.01			000001	.000002	.000006	000014	6 01
6 02			.000001	.000002	.000006	000013	6 02
6 03			.000001	000002	.000006	000013	6.03
6.04				000002	.000005	000013	6 04
6.05				.000002	.000005	.000012	6 05
6.06				000002	000005	.000012	6 06
6 07				000002	.000005	.000012	6 07
6 08				.000002	.000005	000011	6 08
6.09				000002	.000005	.000011	6.09
6.10				000002	000005	.000011	6 10
6 11				.000002	.000005	000011	6 11
6.12				.000001	000004	000010	6.12
6.13				000001	000004	.000010	6.13
6.14				000001	000004	000010	6 14
6 15				000001	.000004	000010	6 15
6.16				.000001	.000004	000009	6.16
6.17				000001	.000004	000009	6.17
6.18				.000001	.000004	.000009	6 18
6 19				000001	000004	000009	6.19
6 20				000001	.000003	000008	6.20
6.21				.000001	000003	000008	6.21
6.22				.000001	.000003	000008	6 22
6.23				000001	000003	.000008	6 23
6.24				000001	.000003	.000008	6 24
6.25				.000001	.000003	.000007	6 25
6 26				.000001	000003	.000007	6 26
6 27				.000001	.000003	.000007	6 27
6 28				000001	000003	.000007	6 28
6.29				000001	.000003	.000007	6 29
6 30				000001	000003	.000007	6 30
6 31				.000001	.000003	.000006	6 31
6.32				000001	.000002	.000006	6.32
6.33				000001	000002	000006	6 33
6.34				000001	.000002	000006	6 34
6 35				000001	000002	.000006	6.35
6.36				000001	000002	.000006	6.36
6.37				.000001	000002	.000005	6.37
6.38				000001	.000002	.000005	6 38
6 39				.000001	.000002	.000005	6 39
6.40				000001	.000002	.000005	6.40
6.41				.000001	.000002	.000005	6.41
6 42				000001	.000002	.000005	6.42
6 43				.000001	000002	.000005	6.43
6 44				.000001	000002	.000005	6 44
6.45					000002	.000004	6.45
6 46					.000002	000004	6 46
6 47					.000002	.000004	6.47
6.48					.000002	.000004	6.48
6 49					.000002	.000004	6.49

t	SKEWNESS						t
	6	.7	.8	9	1 0	1 1	
6 00	000027	000047	000074	000110	000154	000206	6 00
6 01	.000027	000046	000073	000108	000151	000202	6 01
6 02	.000026	000045	000072	000106	000149	.000199	6 02
6 03	000025	000044	.000070	000104	000146	000196	6 03
6 04	.000025	000043	000069	000102	000144	000193	6 04
6 05	000024	000042	000068	000101	000142	000190	6 05
6 06	000024	000041	000066	000099	000139	000188	6.06
6 07	000023	000041	000065	000097	000137	000185	6 07
6 08	000023	000040	000064	000096	000135	000182	6 08
6.09	000022	000039	000063	000094	000133	000179	6.09
6.10	000022	000038	000062	000092	000131	000176	6 10
6 11	.000021	000037	000060	.000091	000128	000174	6 11
6.12	.000021	.000037	000059	000089	000126	000171	6 12
6 13	000020	000036	000058	000088	000124	000169	6 13
6 14	000020	000035	000057	000086	000122	000166	6 14
6 15	000019	000034	000056	000085	000120	000163	6 15
6 16	000019	000034	000055	000083	000118	000161	6 16
6 17	000018	000033	000054	000082	.000116	000159	6 17
6 18	000018	000032	000053	000080	000115	000156	6 18
6 19	.000018	000032	000052	000079	000113	000154	6.19
6 20	000017	000031	000051	000077	000111	000151	6 20
6.21	000017	000030	000050	000076	000109	000149	6 21
6 22	000016	000030	.000049	000075	000107	000147	6 22
6 23	.000016	000029	000048	000073	.000106	000145	6 23
6 24	000016	.000029	000047	.000072	.000104	.000142	6 24
6 25	.000015	000028	.000046	.000071	000102	.000140	6 25
6 26	000015	000027	000045	000070	.000101	000138	6.26
6 27	000015	000027	000045	000068	000099	000136	6 27
6 28	.000014	.000026	000044	.000067	.000097	000134	6 28
6 29	000014	.000026	000043	.000066	000096	.000132	6.29
6 30	.000014	000025	000042	000065	000094	000130	6.30
6 31	.000013	000025	.000041	000064	.000093	000128	6 31
6 32	000013	.000024	000041	000063	.000091	.000126	6 32
6.33	000013	000024	000040	000062	000090	000124	6 33
6 34	000013	.000023	000039	000061	000088	.000122	6 34
6 35	.000012	000023	000038	000059	000087	000120	6 35
6 36	000012	.000022	000038	000058	000085	000118	6 36
6 37	000012	000022	000037	000057	.000084	.000117	6 37
6.38	000011	000021	000036	000056	000083	000115	6.38
6 39	000011	000021	.000035	000055	.000081	000113	6.39
6.40	.000011	.000021	.000035	.000054	.000080	000111	6 40
6.41	.000011	000020	000034	000053	000079	000110	6 41
6 42	.000010	.000020	.000033	000053	000077	000108	6 42
6.43	000010	000019	.000033	000052	000076	000106	6.43
6 44	.000010	.000019	000032	000051	000075	000105	6 44
6 45	.000010	.000018	.000032	000050	000074	.000103	6 45
6.46	.000010	000018	000031	.000049	000072	000102	6.46
6.47	000009	000018	.000030	000048	000071	000100	6 47
6.48	000009	.000017	.000030	000047	000070	000099	6 48
6 49	000009	000017	000029	000046	000069	000097	6.49

t	SKEWNESS						t
	0	.1	2	3	4	.5	
6.50					.000001	000004	6 50
6.51					.000001	.000004	6.51
6 52					.000001	000004	6 52
6 53					000001	000004	6 53
6 54					000001	.000004	6.54
6.55					000001	.000003	6 55
6 56					.000001	000003	6 56
6.57					000001	000003	6 57
6 58					000001	000003	6 58
6.59					.000001	.000003	6 59
6 60					000001	.000003	6 60
6 61					.000001	.000003	6 61
6 62					.000001	.000003	6 62
6 63					.000001	000003	6 63
6.64					000001	.000003	6.64
6.65					000001	000003	6.65
6.66					000001	000003	6.66
6 67					.000001	000003	6 67
6.68					.000001	.000002	6.68
6 69					.000001	.000002	6 69
6 70					000001	.000002	6 70
6 71					000001	000002	6 71
6 72					.000001	.000002	6 72
6.73					.000001	.000002	6.73
6 74					.000001	.000002	6.74
6 75					.000001	.000002	6 75
6.76					000001	000002	6 76
6 77					000001	.000002	6.77
6 78					.000001	.000002	6 78
6.79					.000001	.000002	6.79
6 80					.000001	.000002	6.80
6 81					.000001	.000002	6.81
6 82					.000001	.000002	6.82
6 83					.000001	.000002	6.83
6.84					000001	.000002	6.84
6 85					000001	.000002	6.85
6.86					.000001	.000002	6.86
6 87						.000002	6 87
6 88						.000001	6.88
6.89						.000001	6.89
6.90						.000001	6.90
6.91						.000001	6.91
6 92						.000001	6.92
6.93						.000001	6.93
6.94						.000001	6 94
6.95						.000001	6.95
6.96						.000001	6.96
6 97						.000001	6.97
6.98						.000001	6.98
6.99						.000001	6.99

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
6.50	.000009	.000017	.000029	.000046	.000068	.000096	6.50
6.51	.000008	.000016	.000028	.000045	.000067	.000094	6.51
6.52	.000008	.000016	.000028	.000044	.000066	.000093	6.52
6.53	.000008	.000016	.000027	.000043	.000065	.000091	6.53
6.54	.000008	.000015	.000027	.000042	.000063	.000090	6.54
6.55	.000008	.000015	.000026	.000042	.000062	.000088	6.55
6.56	.000008	.000015	.000026	.000041	.000061	.000087	6.56
6.57	.000007	.000014	.000025	.000040	.000060	.000086	6.57
6.58	.000007	.000014	.000025	.000040	.000059	.000084	6.58
6.59	.000007	.000014	.000024	.000039	.000058	.000083	6.59
6.60	.000007	.000014	.000024	.000038	.000057	.000082	6.60
6.61	.000007	.000013	.000023	.000038	.000057	.000081	6.61
6.62	.000007	.000013	.000023	.000037	.000056	.000079	6.62
6.63	.000006	.000013	.000022	.000036	.000055	.000078	6.63
6.64	.000006	.000012	.000022	.000036	.000054	.000077	6.64
6.65	.000006	.000012	.000022	.000035	.000053	.000076	6.65
6.66	.000006	.000012	.000021	.000034	.000052	.000075	6.66
6.67	.000006	.000012	.000021	.000034	.000051	.000073	6.67
6.68	.000006	.000011	.000020	.000033	.000050	.000072	6.68
6.69	.000006	.000011	.000020	.000033	.000050	.000071	6.69
6.70	.000005	.000011	.000020	.000032	.000049	.000070	6.70
6.71	.000005	.000011	.000019	.000031	.000048	.000069	6.71
6.72	.000005	.000010	.000019	.000031	.000047	.000068	6.72
6.73	.000005	.000010	.000018	.000030	.000046	.000067	6.73
6.74	.000005	.000010	.000018	.000030	.000046	.000066	6.74
6.75	.000005	.000010	.000018	.000029	.000045	.000065	6.75
6.76	.000005	.000010	.000017	.000029	.000044	.000064	6.76
6.77	.000005	.000009	.000017	.000028	.000043	.000063	6.77
6.78	.000005	.000009	.000017	.000028	.000043	.000062	6.78
6.79	.000004	.000009	.000016	.000027	.000042	.000061	6.79
6.80	.000004	.000009	.000016	.000027	.000041	.000060	6.80
6.81	.000004	.000009	.000016	.000026	.000041	.000059	6.81
6.82	.000004	.000008	.000016	.000026	.000040	.000058	6.82
6.83	.000004	.000008	.000015	.000025	.000039	.000057	6.83
6.84	.000004	.000008	.000015	.000025	.000039	.000056	6.84
6.85	.000004	.000008	.000015	.000024	.000038	.000056	6.85
6.86	.000004	.000008	.000014	.000024	.000037	.000055	6.86
6.87	.000004	.000008	.000014	.000024	.000037	.000054	6.87
6.88	.000004	.000007	.000014	.000023	.000036	.000053	6.88
6.89	.000004	.000007	.000014	.000023	.000036	.000052	6.89
6.90	.000003	.000007	.000013	.000022	.000035	.000051	6.90
6.91	.000003	.000007	.000013	.000022	.000034	.000051	6.91
6.92	.000003	.000007	.000013	.000022	.000034	.000050	6.92
6.93	.000003	.000007	.000013	.000021	.000033	.000049	6.93
6.94	.000003	.000007	.000012	.000021	.000033	.000048	6.94
6.95	.000003	.000006	.000012	.000020	.000032	.000048	6.95
6.96	.000003	.000006	.000012	.000020	.000032	.000047	6.96
6.97	.000003	.000006	.000012	.000020	.000031	.000046	6.97
6.98	.000003	.000006	.000011	.000019	.000031	.000045	6.98
6.99	.000003	.000006	.000011	.000019	.000030	.000045	6.99

	SKEWNESS						
t	.0	.1	.2	.3	.4	.5	t
7.00						.000001	7.00
7.01						.000001	7.01
7.02						.000001	7.02
7.03						.000001	7.03
7.04						.000001	7.04
7.05						.000001	7.05
7.06						.000001	7.06
7.07						.000001	7.07
7.08						.000001	7.08
7.09						.000001	7.09
7.10						.000001	7.10
7.11						.000001	7.11
7.12						.000001	7.12
7.13						.000001	7.13
7.14						.000001	7.14
7.15						.000001	7.15
7.16						.000001	7.16
7.17						.000001	7.17
7.18						.000001	7.18
7.19						.000001	7.19
7.20						.000001	7.20
7.21						.000001	7.21
7.22						.000001	7.22
7.23						.000001	7.23
7.24						.000001	7.24
7.25						.000001	7.25
7.26						.000001	7.26
7.27						.000001	7.27
7.28						.000001	7.28
7.29							7.29
7.30							7.30
7.31							7.31
7.32							7.32
7.33							7.33
7.34							7.34
7.35							7.35
7.36							7.36
7.37							7.37
7.38							7.38
7.39							7.39
7.40							7.40
7.41							7.41
7.42							7.42
7.43							7.43
7.44							7.44
7.45							7.45
7.46							7.46
7.47							7.47
7.48							7.48
7.49							7.49

t	SKEWNESS						t
	6	7	8	9	10	11	
7 00	000003	000006	000011	000019	000030	000044	7 00
7 01	000003	000006	000011	000018	000029	000043	7 01
7 02	000003	000006	000011	000018	000029	000043	7.02
7 03	000003	000005	.000010	000018	000028	000042	7 03
7 04	000002	000005	.000010	000017	000028	.000041	7 04
7 05	.000002	000005	000010	000017	000027	000041	7 05
7 06	000002	000005	000010	000017	000027	.000040	7 06
7 07	000002	.000005	.000010	000016	000026	000039	7 07
7 08	000002	000005	000009	000016	000026	.000039	7 08
7 09	.000002	000005	000009	000016	000025	000038	7 09
7 10	.000002	000005	.000009	.000016	000025	000038	7 10
7.11	.000002	000005	.000009	000015	000025	.000037	7 11
7 12	000002	000004	000009	000015	000024	.000037	7 12
7 13	.000002	000004	000008	.000015	000024	000036	7 13
7 14	.000002	.000004	.000008	000015	000023	000035	7 14
7 15	.000002	000004	000008	.000014	000023	000035	7 15
7 16	.000002	000004	000008	000014	000023	000034	7.16
7 17	.000002	000004	.000008	000014	000022	000034	7 17
7.18	000002	000004	000008	000014	000022	000033	7 18
7 19	.000002	.000004	000008	000013	000022	000033	7 19
7 20	000002	000004	000007	000013	000021	000032	7 20
7.21	000002	000004	000007	000013	000021	000032	7.21
7 22	000002	000004	000007	000013	.000021	000031	7 22
7 23	000002	000004	000007	.000012	.000020	.000031	7.23
7 24	000002	.000003	000007	000012	.000020	000030	7.24
7.25	.000001	000003	.000007	.000012	000019	.000030	7.25
7.26	000001	000003	000007	000012	.000019	000029	7 26
7 27	000001	000003	000006	000012	000019	000029	7 27
7 28	000001	000003	000006	000011	000019	000028	7 28
7 29	000001	000003	000006	000011	000018	000028	7.29
7 30	000001	000003	000006	000011	000018	.000028	7.30
7 31	000001	000003	.000006	000011	.000018	.000027	7.31
7 32	000001	000003	000006	000011	000017	000027	7 32
7 33	000001	000003	000006	000010	.000017	000026	7 33
7 34	.000001	000003	000006	000010	000017	000026	7 34
7 35	000001	000003	000006	000010	000016	000025	7.35
7 36	000001	.000003	000005	.000010	000016	000025	7 36
7 37	000001	000003	000005	.000010	000016	.000025	7.37
7 38	000001	.000003	000005	000009	000016	000024	7 38
7 39	000001	000003	.000005	000009	000015	000024	7 39
7.40	000001	000002	000005	000009	000015	000024	7 40
7 41	000001	000002	000005	000009	000015	000023	7 41
7.42	000001	.000002	000005	000009	.000015	000023	7 42
7 43	000001	.000002	000005	000009	.000014	.000022	7.43
7.44	.000001	000002	000005	000008	000014	.000022	7.44
7 45	.000001	000002	.000005	000008	.000014	000022	7 45
7 46	000001	.000002	000004	000008	.000014	000021	7 46
7 47	000001	000002	.000004	000008	000013	000021	7 47
7 48	000001	.000002	000004	000008	000013	000021	7.48
7 49	.000001	000002	.000004	000008	000013	000020	7.49

t	SKEWNESS						t
	0	.1	2	3	4	5	
7.50							7.50
7.51							7.51
7.52							7.52
7.53							7.53
7.54							7.54
7.55							7.55
7.56							7.56
7.57							7.57
7.58							7.58
7.59							7.59
7.60							7.60
7.61							7.61
7.62							7.62
7.63							7.63
7.64							7.64
7.65							7.65
7.66							7.66
7.67							7.67
7.68							7.68
7.69							7.69
7.70							7.70
7.71							7.71
7.72							7.72
7.73							7.73
7.74							7.74
7.75							7.75
7.76							7.76
7.77							7.77
7.78							7.78
7.79							7.79
7.80							7.80
7.81							7.81
7.82							7.82
7.83							7.83
7.84							7.84
7.85							7.85
7.86							7.86
7.87							7.87
7.88							7.88
7.89							7.89
7.90							7.90
7.91							7.91
7.92							7.92
7.93							7.93
7.94							7.94
7.95							7.95
7.96							7.96
7.97							7.97
7.98							7.98
7.99							7.99

t	SKEWNESS						t
	6	7	8	9	1.0	1.1	
7 50	000001	.000002	.000004	.000008	.000013	.000020	7 50
7 51	000001	000002	000004	000007	.000013	.000020	7.51
7 52	000001	000002	.000004	.000007	.000012	.000020	7.52
7 53	.000001	000002	.000004	000007	000012	.000019	7.53
7.54	000001	.000002	000004	000007	.000012	000019	7.54
7.55	000001	000002	.000004	000007	000012	.000019	7.55
7 56	000001	000002	000004	.000007	.000012	000018	7.56
7 57	000001	000002	.000004	.000007	000011	.000018	7 57
7 58	.000001	000002	.000004	000007	000011	000018	7.58
7.59	000001	000002	000003	000006	000011	000017	7.59
7.60	000001	000002	.000003	.000006	.000011	.000017	7.60
7 61	.000001	000002	.000003	.000006	.000011	.000017	7.61
7 62	.000001	.000002	000003	.000006	.000010	.000017	7.62
7 63	000001	000001	.000003	000006	.000010	.000016	7.63
7.64	000001	.000001	.000003	000006	000010	000016	7 64
7 65	000001	.000001	.000003	000006	000010	000016	7.65
7 66	000001	000001	000003	000006	000010	000016	7.66
7 67	.000001	.000001	.000003	000006	000010	000015	7.67
7 68	.000001	000001	000003	000005	000009	.000015	7.68
7 69	.000001	.000001	.000003	000005	000009	000015	7.69
7 70	000001	.000001	000003	.000005	.000009	000015	7.70
7.71		.000001	000003	000005	.000009	.000014	7.71
7 72		.000001	.000003	000005	.000009	.000014	7.72
7 73		.000001	.000003	.000005	.000009	.000014	7 73
7.74		000001	000003	.000005	.000009	.000014	7.74
7.75		.000001	.000002	.000005	000008	.000014	7.75
7.76		000001	000002	.000005	.000008	.000013	7.76
7.77		.000001	.000002	.000005	.000008	000013	7.77
7 78		000001	.000002	000005	.000008	000013	7.78
7 79		000001	000002	.000004	.000008	.000013	7.79
7 80		000001	.000002	.000004	.000008	.000013	7 80
7.81		.000001	000002	.000004	.000008	.000012	7.81
7 82		.000001	.000002	000004	.000007	.000012	7.82
7.83		000001	.000002	.000004	.000007	000012	7 83
7 84		.000001	.000002	.000004	000007	000012	7.84
7 85		.000001	.000002	.000004	.000007	000012	7.85
7.86		.000001	.000002	000004	.000007	000011	7 86
7.87		000001	000002	000004	.000007	.000011	7.87
7 88		000001	.000002	.000004	.000007	.000011	7.88
7 89		000001	.000002	.000004	.000007	.000011	7.89
7 90		000001	000002	000004	.000007	.000011	7.90
7.91		000001	000002	.000004	.000006	.000011	7.91
7 92		000001	000002	.000004	000006	.000010	7.92
7 93		.000001	.000002	.000003	000006	.000010	7.93
7 94		.000001	000002	.000003	.000006	.000010	7 94
7 95		.000001	.000002	.000003	.000006	000010	7.95
7.96		000001	000002	.000003	000006	.000010	7.96
7.97		000001	.000002	.000003	000006	.000010	7.97
7.98		.000001	000002	.000003	.000006	.000009	7.98
7.99		.000001	.000002	.000003	.000006	.000009	7.99

	SKEWNESS						
t	0	.1	.2	.3	4	.5	t
8.00							8.00
8.01							8.01
8.02							8.02
8.03							8.03
8.04							8.04
8.05							8.05
8.06							8.06
8.07							8.07
8.08							8.08
8.09							8.09
8.10							8.10
8.11							8.11
8.12							8.12
8.13							8.13
8.14							8.14
8.15							8.15
8.16							8.16
8.17							8.17
8.18							8.18
8.19							8.19
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8.23							8.23
8.24							8.24
8.25							8.25
8.26							8.26
8.27							8.27
8.28							8.28
8.29							8.29
8.30							8.30
8.31							8.31
8.32							8.32
8.33							8.33
8.34							8.34
8.35							8.35
8.36							8.36
8.37							8.37
8.38							8.38
8.39							8.39
8.40							8.40
8.41							8.41
8.42							8.42
8.43							8.43
8.44							8.44
8.45							8.45
8.46							8.46
8.47							8.47
8.48							8.48
8.49							8.49

t	SKEWNESS						t
	.6	7	8	9	1 0	1 1	
8.00		000001	000002	.000003	000005	000009	8 00
8 01		000001	000001	.000003	000005	000009	8 01
8.02		.000001	000001	000003	000005	.000009	8 02
8.03		.000001	.000001	000003	000005	000009	8.03
8 04		000001	000001	000003	.000005	000009	8.04
8 05		.000001	.000001	.000053	.000005	.000008	8 05
8 06		000001	000001	000003	.000005	000008	8 06
8 07		000001	000001	000003	000005	000008	8.07
8.08		.000001	000001	.000003	.000005	000008	8.08
8 09		.000001	000001	000003	000005	000008	8 09
8.10		000001	.000001	000003	000005	000008	8 10
8 11		000001	000001	.000002	000005	000008	8 11
8 12		000001	000001	.000002	000004	000008	8 12
8 13			.000001	.000002	.000004	000007	8 13
8 14			000001	000002	000004	000007	8 14
8.15			000001	.000002	000004	000007	8 15
8 16			.000001	.000002	000004	000007	8.16
8 17			000001	.000002	000004	000007	8 17
8 18			.000001	.000002	000004	000007	8 18
8 19			000001	000002	000004	.000007	8 19
8 20			.000001	000002	000004	.000007	8 20
8.21			000001	.000002	000004	.000007	8 21
8 22			.000001	.000002	000004	000006	8 22
8.23			.000001	000002	.000004	000006	8 23
8 24			000001	000002	000004	000006	8.24
8.25			000001	.000002	.000004	000006	8 25
8 26			.000001	000002	.000004	.000006	8 26
8 27			000001	000002	.000003	000006	8 27
8 28			000001	000002	.000003	000006	8 28
8 29			.000001	000002	.000003	000006	8 29
8 30			000001	000002	.000003	.000006	8 30
8 31			000001	000002	.000003	000006	8 31
8 32			000001	000002	.000003	.000006	8.32
8 33			.000001	.000002	.000003	000005	8.33
8 34			.000001	000002	000003	000005	8 34
8 35			.000001	.000002	.000003	000005	8 35
8 36			000001	.000002	000003	000005	8 36
8.37			.000001	000002	000003	000005	8 37
8 38			.000001	000002	.000003	000005	8 38
8 39			000001	000001	000003	000005	8 39
8.40			000001	000001	000003	000005	8.40
8 41			.000001	.000001	000003	000005	8 41
8 42			000001	.000001	.000003	000005	8 42
8 43			000001	000001	.000003	000005	8 43
8 44			000001	000001	000003	000005	8.44
8.45			.000001	000001	.000003	000004	8 45
8 46			000001	000001	.000003	.000004	8.46
8 47			000001	.000001	000002	.000004	8.47
8.48			.000001	.000001	000002	000004	8 48
8.49			.000001	.000001	000002	000004	8.49

	SKEWNESS						
t	0	1	2	3	4	.5	t
8.50							8.50
8.51							8.51
8.52							8.52
8.53							8.53
8.54							8.54
8.55							8.55
8.56							8.56
8.57							8.57
8.58							8.58
8.59							8.59
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8.71							8.71
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8.90							8.90
8.91							8.91
8.92							8.92
8.93							8.93
8.94							8.94
8.95							8.95
8.96							8.96
8.97							8.97
8.98							8.98
8.99							8.99

	SKEWNESS						
t	.6	.7	.8	.9	1.0	1.1	t
8.50			.000001	.000001	.000002	.000004	8.50
8.51			.000001	.000001	.000002	.000004	8.51
8.52			.000001	.000001	.000002	.000004	8.52
8.53			.000001	.000001	.000002	.000004	8.53
8.54			.000001	.000001	.000002	.000004	8.54
8.55			.000001	.000001	.000002	.000004	8.55
8.56				.000001	.000002	.000004	8.56
8.57				.000001	.000002	.000004	8.57
8.58				.000001	.000002	.000004	8.58
8.59				.000001	.000002	.000004	8.59
8.60				.000001	.000002	.000004	8.60
8.61				.000001	.000002	.000003	8.61
8.62				.000001	.000002	.000003	8.62
8.63				.000001	.000002	.000003	8.63
8.64				.000001	.000002	.000003	8.64
8.65				.000001	.000002	.000003	8.65
8.66				.000001	.000002	.000003	8.66
8.67				.000001	.000002	.000003	8.67
8.68				.000001	.000002	.000003	8.68
8.69				.000001	.000002	.000003	8.69
8.70				.000001	.000002	.000003	8.70
8.71				.000001	.000002	.000003	8.71
8.72				.000001	.000002	.000003	8.72
8.73				.000001	.000002	.000003	8.73
8.74				.000001	.000002	.000003	8.74
8.75				.000001	.000002	.000003	8.75
8.76				.000001	.000001	.000003	8.76
8.77				.000001	.000001	.000003	8.77
8.78				.000001	.000001	.000003	8.78
8.79				.000001	.000001	.000003	8.79
8.80				.000001	.000001	.000003	8.80
8.81				.000001	.000001	.000003	8.81
8.82				.000001	.000001	.000002	8.82
8.83				.000001	.000001	.000002	8.83
8.84				.000001	.000001	.000002	8.84
8.85				.000001	.000001	.000002	8.85
8.86				.000001	.000001	.000002	8.86
8.87				.000001	.000001	.000002	8.87
8.88				.000001	.000001	.000002	8.88
8.89				.000001	.000001	.000002	8.89
8.90				.000001	.000001	.000002	8.90
8.91				.000001	.000001	.000002	8.91
8.92				.000001	.000001	.000002	8.92
8.93				.000001	.000001	.000002	8.93
8.94				.000001	.000001	.000002	8.94
8.95				.000001	.000001	.000002	8.95
8.96				.000001	.000001	.000002	8.96
8.97				.000001	.000001	.000002	8.97
8.98				.000001	.000001	.000002	8.98
8.99					.000001	.000002	8.99

	SKEWNESS						
t	0	1	2	.3	4	.5	t
9.00							9.00
9.01							9.01
9.02							9.02
9.03							9.03
9.04							9.04
9.05							9.05
9.06							9.06
9.07							9.07
9.08							9.08
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9.42							9.42
9.43							9.43
9.44							9.44
9.45							9.45
9.46							9.46
9.47							9.47
9.48							9.48
9.49							9.49

t	SKEWNESS						t
	6	7	8	9	1.0	1.1	
9.00					.000001	.000002	9.00
9.01					.000001	.000002	9.01
9.02					.000001	.000002	9.02
9.03					.000001	.000002	9.03
9.04					.000001	.000002	9.04
9.05					.000001	.000002	9.05
9.06					.000001	.000002	9.06
9.07					.000001	.000002	9.07
9.08					.000001	.000002	9.08
9.09					.000001	.000002	9.09
9.10					.000001	.000002	9.10
9.11					.000001	.000002	9.11
9.12					.000001	.000002	9.12
9.13					.000001	.000002	9.13
9.14					.000001	.000001	9.14
9.15					.000001	.000001	9.15
9.16					.000001	.000001	9.16
9.17					.000001	.000001	9.17
9.18					.000001	.000001	9.18
9.19					.000001	.000001	9.19
9.20					.000001	.000001	9.20
9.21					.000001	.000001	9.21
9.22					.000001	.000001	9.22
9.23					.000001	.000001	9.23
9.24					.000001	.000001	9.24
9.25					.000001	.000001	9.25
9.26					.000001	.000001	9.26
9.27					.000001	.000001	9.27
9.28					.000001	.000001	9.28
9.29					.000001	.000001	9.29
9.30					.000001	.000001	9.30
9.31					.000001	.000001	9.31
9.32					.000001	.000001	9.32
9.33					.000001	.000001	9.33
9.34					.000001	.000001	9.34
9.35					.000001	.000001	9.35
9.36					.000001	.000001	9.36
9.37					.000001	.000001	9.37
9.38					.000001	.000001	9.38
9.39					.000001	.000001	9.39
9.40						.000001	9.40
9.41						.000001	9.41
9.42						.000001	9.42
9.43						.000001	9.43
9.44						.000001	9.44
9.45						.000001	9.45
9.46						.000001	9.46
9.47						.000001	9.47
9.48						.000001	9.48
9.49						.000001	9.49

	SKEWNESS						
t	.0	1	2	3	4	.5	t
9.50							9.50
9.51							9.51
9.52							9.52
9.53							9.53
9.54							9.54
9.55							9.55
9.56							9.56
9.57							9.57
9.58							9.58
9.59							9.59
9.60							9.60
9.61							9.61
9.62							9.62
9.63							9.63
9.64							9.64
9.65							9.65
9.66							9.66
9.67							9.67
9.68							9.68
9.69							9.69
9.70							9.70
9.71							9.71
9.72							9.72
9.73							9.73
9.74							9.74
9.75							9.75
9.76							9.76
9.77							9.77
9.78							9.78
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9.80							9.80
9.81							9.81
9.82							9.82
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9.85							9.85
9.86							9.86
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9.88							9.88
9.89							9.89
9.90							9.90
9.91							9.91
9.92							9.92
9.93							9.93
9.94							9.94
9.95							9.95
9.96							9.96
9.97							9.97
9.98							9.98
9.99							9.99

t	SKEWNESS						t
	.6	.7	8	9	1 0	1 1	
9 50						000001	9.50
9.51						000001	9 51
9 52						000001	9 52
9 53						.000001	9 53
9.54						.000001	9.54
9.55						.000001	9 55
9.56						.000001	9.56
9.57						.000001	9 57
9.58						000001	9 58
9.59						.000001	9 59
9 60						.000001	9 60
9.61						.000001	9 61
9.62						.000001	9 62
9.63						.000001	9 63
9 64						.000001	9 64
9.65						.000001	9 65
9 66						.000001	9.66
9.67						.000001	9 67
9 68						.000001	9 68
9.69						.000001	9.69
9.70						.000001	9 70
9 71						.000001	9.71
9 72						.000001	9.72
9.73						.000001	9 73
9.74						.000001	9 74
9 75						.000001	9.75
9 76						.000001	9.76
9.77						.000001	9 77
9.78						.000001	9 78
9.79						.000001	9.79
9 80						.000001	9 80
9 81						.000001	9 81
9.82							9.82
9 83							9 83
9.84							9.84
9.85							9 85
9.86							9.86
9.87							9.87
9.88							9.88
9.89							9 89
9.90							9.90
9 91							9.91
9.92							9.92
9.93							9.93
9.94							9 94
9.95							9 95
9.96							9.96
9.97							9.97
9.98							9 98
9.99							9.99

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THE USE OF LINEAR FUNCTIONS TO DETECT HIDDEN PERIODS IN DATA SEPARATED INTO SMALL SETS

By

EDWARD L. DODD

I.—INTRODUCTION

Readers who have access to the Handbook of Mathematical Statistics[1] will find in chapter XI a synopsis of a periodogram analysis by W. L Crum, with references to some of the important papers on period testing

My own interest in this subject was aroused several years ago by Dr J. A Udden,[2] Director of the Bureau of Economic Geology at the University of Texas, who had in his possession measurements of the thicknesses of successive layers of anhydrite (CaSO) taken from a Texas oil well. The material, Dr. Udden noted, was "suggestive of cycles" (p. 351), but one difficulty was mentioned: "Probably 2 per cent of the layers are indistinct." It was not always possible to tell whether the number recorded as the thickness of a layer represents a single deposit or two or more deposits insufficiently separated by the usual bituminous demarcation. The analogous difficulty in distinguishing consecutive rings of big trees[3] was met by comparison of the rings of trees from the same forest. But such companion records were not available for the rock lamina.

A little reflection will show that the usual method of testing

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1 Rietz, Houghton Mifflin Co., 1924.

2 "Laminated Anhydrite in Texas." *Bulletin of the Geological Society of America,* Vol. 35 (1924), pp 347-354

3 A E Douglass, "Climatic Cycles and Tree Growth," Carnegie Institution of Washington, Publication No 289 (1919)

for cycles, from data arranged in columns becomes vitiated if in several instances merging of layers has taken place—not so much because of the exaggerated size of certain items, but because the items get into the wrong columns. When a step is lost, all subsequent items are misplaced.

My purpose is then to explain how tests for periods can be made by first using the data in small sets—thus minimizing the vicious effects of a merger—and then by suitably combining the results obtained from these small sets.

We might as well admit at the start that a demonstration of a periodicity is in general impossible. Perhaps the revolution of the earth on its axis represents a demonstrated periodicity. But for the most part, announced periodicities are merely improbable or probable. There is no absolute proof that they exist. We know that what we call "pure chance," typified by the throws of a coin, will sometimes yield irregularities of oscillation between two states, the minimum and the maximum, which to all appearances is a "periodicity." The question arises: About how often will pure chance thus deceive us? All we can do is to compute certain relative frequencies or probabilities. If the probability found is very, very small, that the apparent periodicity had its origin in pure chance, we assert with some assurance that a real periodicity exists In this mode of approach, this paper will follow rather closely Arthur Schuster,[1] whose work is fundamental.

Although Schuster's main interest was in the quadratic function, "intensity"—at first, in the square root of intensity, Terrestrial Magnetism, loc cit.—he pointed out (p. 27) how certain conclusions could be reached through integrals—substantially linear functions, if integration is regarded as summation. It is this approach to period testing through linear functions that I am setting forth in his paper. Some special attention must be given to phase in the application of this methdd.

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[1] "On the investigation of hidden periodicities with application to a supposed 26-day period of meteorological phenomena." *Terrestrial Magnetism*, Vol 3 (1898), pp 13-41. In my paper, "The probability law for the intensity of a trial period, with data subject to the Gaussian law," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 681-684, I referred to Schuster's paper in the *Proceedings of the Royal Society of London* Reference should have been made also to the above paper in *Terrestrial Magnetism*, where the probability law is given for the square root of intensity (p 21), which can easily be thrown into the form given in my paper. Schuster, however, postulated (p 20) that "$2\pi \quad p$ is a submultiple of a right angle"—a condition which would not always be satisfied—also (p. 21) that the vectors be distributed according to the law of errors centered at the *origin*, an inconvenient restriction, and his method did not bring out the different law of distribution needed for the case when the period is equal to two

Most of the methods for detecting periodicities make use of the trigonometric functions, with their well known properties, in particular, use is made of the Sines and cosines of an angle and its multiples, as in harmonic analysis and Fourier series. With the aid of these harmonic multipliers, linear functions are first formed; and from these, by squaring and adding, a quadratic function, which plays the central role, as "intensity." In the method set forth in this paper, however, the linear functions themselves are the most important, not merely for graphical representation, but for determining probabilities.

Suppose, then, that a set of numbers is furnished us—perhaps from an unknown source—for example a set of ten numbers consisting of 5's and 1's alternating:

$$5,\ 1,\ 5,\ 1,\ 5,\ 1,\ 5,\ 1,\ 5,\ 1$$

Has this set of numbers the period *two*? If this question means: Is there a function of period *two* which takes on these ten values, the answer is: Yes, namely—

$$3+2\cos\pi r \qquad\qquad r=0,\ 1,\ 2,\ \ldots .\ 9$$

Here, as usual, π means 180°, obtained from a complete revolution of 360° by dividing by *two*. If, in place of an integer r, we take a continuous variable x, and plot

$$y=3+2\ \cos\pi x$$

from $x=0$ to $x=10$, a wave curve is formed with each upper crest at 5, and each depression at 1.

But usually in period testing, something is desired beyond the mere possibility of making a mathematical curve fit the data Perhaps a farmer on each 10 acres of his farm has raised 5 bales of cotton, 1 bale of cotton, 5 bales of cotton, etc., alternately for 10 years, under apparently the same conditions as to labor, fertilizer, etc. He would like to know whether this is due to mere chance or to some recurrence at two-year intervals of droughts, pests, or adverse conditions. Stranger events do, indeed, occur by pure chance than the foregoing hypothetical yield of cotton. But the regularity postulated above would strike almost anyone as exceptional, and it would be prudent for our farmer to believe that there was some non-fortuitous cause of the regularity, and to try to discover it.

Let us, indeed, set up a chance situation to correspond to the foregoing yield of cotton. If the two faces of a coin are marked 5 and 1, and are recorded as such, the probability for ten throws starting with 5 and alternating between 1 and 5 is only 1/1024. A bet of $1,023 against $1 would measure the unusualness of the specified succession of 5's and 1's.

That this occurrence is unusual may be signalized by another test and method of approach Let X_r denote the result of the r th trial of an independent chance variable, which with equal likelihood ($p_1 = 1/2 = p_2$) takes on the values 5 or 1, and can take on no other value. The "mean value" of X_r is then, by definition,

$$p_1(5) + p_2(1) = \tfrac{1}{2}(5) + \tfrac{1}{2}(1) = 3$$

This would, indeed, be also the average value of the five 5's and five 1's in the illustration. The "mean error" ε of X_r would be found from

$$\varepsilon^2 = \frac{1}{2}(5-3)^2 + \frac{1}{2}(1-3)^2 = 4 \,, \qquad \varepsilon = 2$$

This would be also the standard deviation σ of the numbers in the illustration—that is

$$\sigma^2 = \frac{1}{10}\left[(5-3)^2 + (1-3)^2 + (5-3)^2 + \cdots + (1-3)^2\right] = 4 \,, \qquad \sigma = 2$$

Now let

$$X = X_1 - X_2 + X_3 - \cdots - X_{10}$$

Since the signs alternate, the mean value of X is zero; since there are ten terms, the mean error of X is $\varepsilon\sqrt{10} = 2(3.16) = 6.32$. If now X_r should take on alternately the values of 5 and 1, then X would become 20. It would thus exceed its mean value zero by more than three times its mean error, or more than four and one-half times its "probable error" This is commonly regarded as "significant."

To see a little more clearly into the mechanism of the above result, let us pass from the numbers X_r to their deviations from their mean value 3.

Let

$$x_1 = X_1 - 3\,, \quad x_2 = X_2 - 3\,, \quad \cdots\,, \quad x_r = X_r - 3\,, \quad \cdots$$

Then the mean value of x_r is zero, and its mean error is 2 Now let

$$x = x_1 - x_2 + x_3 - \cdots - x_{10}$$

Then the mean value of x is zero and its mean error is $\varepsilon\sqrt{10}$; in both respects it resembles X Furthermore it takes on the same value 20 that X takes on when the 5 and 1 alternate; since $x_1 - x_2 = (X_1 - 3) - (X_2 - 3) = X_1 - X_2$, etc. And here again 20 is a remarkable value for x since it represents an excess of more than three times its mean error But let us now find x directly from the values taken on by x_r, when X_r alternates between 5 and 1.

$$x = 1\,(2) - 1\,(-2) + 1\,(2) - \cdots - 1\,(-2) = 20$$

The feature to be noted is that the successive values of x_r and of $\cos \pi(r - 1)$ match in sign, for $r = 1, 2, 3 \ldots 10$.

$$x_r = 2,\ -2,\ 2,\ -2,\ \cdots,\ -2$$

$$\cos \pi(r-1) = 1,\ -1,\ 1,\ -1,\ \cdots,\ -1$$

Each product $x_r \cos \pi\,(r - 1)$ is then positive, and this accounts for the large value of x. This matching in sign of the deviations of the data with the successive terms of a test function $\cos 2\pi(r - 1)/k$ or perhaps $\cos 2\pi r/k$ when k is given a particular value—here $k = 2$—is, indeed, fundamental Also the similarity between the properties of X and x will be found to be maintained in more general cases.

The foregoing illustrates the method of period testing to be set forth in this paper A general assumption is at first made, that the data contain no periodic constituent, but on the contrary represent mere chance fluctuations. Certain linear functions of the data are found with coefficients which are the cosines or sines of multiples of the angle associated with a given period For these functions, the fluctuations usually to be expected are to be computed—assuming that the measurements represent chance data If the actual values which these functions take on are greatly in excess of what is expected of them, the initial assumption that the data are due to chance is called into question It may be more reasonable to suppose that to some extent the data conform to the period associated with the cosine multipliers involved in the test These "harmonic" multipliers, indeed, pass through a succession of positive and negative values in a regular way If the positive and negative fluctuations of the measurements from their average value are well "timed" with those of the harmonic multipliers, we get a sum of products nearly all positive, thus a much larger result than if positive numbers were not matched with positive, negative with negative numbers.

As preliminary to all tests the data may be divided into fairly large groups of consecutive measurements—say with 120 measurements in a group; for 120 is a multiple of 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 40, 60, numbers quite suitable for trial periods The arithmetic mean and standard deviation of each such group may be computed. These may usually be accepted as close approximations to the mean value and mean error of the measurements of the group

To illustrate further the nature of the tests to be applied, let us imagine that the 120 measurements of a group are recorded on slips of papers, these slips put into a bag, drawn at random, and recorded as drawn. This set of numbers would have the same arithmetic mean and standard deviation, noted above, no matter in what order they are drawn and recorded But periodicities depend upon the order of the measurements. A chance order of measurements x_r, such as established by drawing from a bag, would very seldom match sufficiently well a periodic function like

$\cos 2\pi r/2$ with period 2, or $\cos 2\pi r/3$ with period 3, etc., to make a test function $C(k)=\Sigma X_r \cos 2\pi r/k$ noticeably large. Thus, if for some particuler k, the function $C(k)$, computed from the data in their actual given order, turns out to be significantly large, the indication is that the data contain a constituent with period k.

We mean here that each measurement of the set may be thought of as the sum of certain constituents, one of which is periodic with period k. Another constituent may perhaps have a different period k' Still another constituent may be a chance variable with no regularity which can properly be called periodic.

II Trigonometric Formulas.

Of considerable use are the simple formulas:

$$(1) \qquad \sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$$

$$(2) \qquad \cos a \cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$$

Indeed, by using (1) in summing the product $\sin(r\theta+\alpha)\sin\theta/2$ from $r=0$ to $(n-1)$ there is obtained,[1] in case θ is not a multiple of 360°,

$$(3) \qquad \sum_{r=0}^{n-1} \sin(r\theta+\alpha) = \sin\left(\frac{(n-1)\theta}{2}+\alpha\right)\frac{\sin n\theta/2}{\sin\theta/2}$$

Likewise, for $\theta \not\equiv 0 \pmod{360^\circ}$; i e., θ not a multiple of 360°,

$$(4) \qquad \sum_{r=0}^{n-1} \cos(r\theta+\alpha) = \cos\left(\frac{(n-1)\theta}{2}+\alpha\right)\frac{\sin n\theta/2}{\sin\theta/2}$$

As important special cases, we have when $n\theta$ is a multiple of 360°,

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[1] For formulas suitable for period testing and for a historical review of this subject with references, the reader is referred to the article of H. Burkhardt in *Encyklopädie der Mathematischen Wissenschaften,* II A 9a, pp. 642-694.

$$(5)\qquad \sum_{r=0}^{n-1} \sin(r\theta+\alpha) = 0 = \sum_{r=0}^{n-1} \cos(r\theta+\alpha)\,, \quad \begin{array}{l}\theta \not\equiv 0 \pmod{360^\circ}\\ n\theta \equiv 0 \pmod{360^\circ}\end{array}$$

As an application of (5), let $X_1, X_2, \ldots X_r, \ldots X_n$ be any set of n numbers, let C be any constant. Then

$$(6)\qquad \sum_{r=0}^{n-1} (X_r - C)\cos(r\theta+\alpha) = \sum_{r=0}^{n-1} X_r \cos(r\theta+\alpha)\,, \quad \begin{array}{l}\theta \not\equiv 0 \pmod{360^\circ}\\ n\theta \equiv 0 \pmod{360^\circ}\end{array}$$

Likewise for $\sin(r\theta+\alpha)$

The above signifies that if the X_r represent data to be subjected to tests with harmonic multipliers, where an integral number of complete cycles is taken, it is immaterial where the origin for the data is taken. In the theory, the C will be usually taken as the arithmetic mean of the data, in computation, the C may be some simple number which will reduce the number of significant figures in the data.

III. Chance Data in Distributions with close contact at extremities

Chance data distributed normally will be considered first. Given n numbers or variates $X_1, X_2, \ldots, X_n$, the arithmetic mean M and standard deviation σ are determined by

$$(7)\qquad M = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)\,, \quad \sigma^2 = \frac{1}{n}\left[(X_1 - M)^2 + \cdots (X_n - M)^2\right].$$

Let

$$(8)\qquad \phi(t) = \frac{2}{\sqrt{2\pi}}\int_0^t e^{-\frac{t^2}{2}}\,dt$$

The data will be said to be normally distributed if the number of variates lying between $M+\lambda_1$ and $M+\lambda_2$ is approximately

$\frac{n}{2}\left[\phi(\lambda_2/\sigma) - \phi(\lambda_1/\sigma)\right]$ for all values of $\lambda_1 < \lambda_2$. Here n is supposed to be at least moderately large. To express this in the language of probability, suppose the n numbers X_1, X_2, are recorded on slips and put into a bag, and suppose a slip is drawn out Then, for the X_r thus drawn—

(9) Probability that $\lambda < X_r < \lambda + d\lambda$ is $\frac{d\lambda}{\sigma\sqrt{2\pi}} e^{-(\lambda-M)^2/2\sigma^2}$

where, if $d\lambda$ is taken rather small, the n is to be thought of as rather large.

The important theorem needed here—substantially explained, if not proven, in most books on probability—is that if sets of k of these variates are drawn at random, and linear functions with fixed coefficients, such as

(10) $$F(k) = a_1 X_1 + a_2 X_2 + \cdots + a_k X_k$$

are formed, these functions $F(k)$ as determined in sets of drawings will be normally distributed with standard deviation σ_k, where

(11) $$\sigma_k^2 = \sigma^2(a_1^2 + a_2^2 + \qquad + a_k^2) .$$

If, in particular $a_r = \cos(r\theta + \alpha)$ making $2a_r^2 = 1 + \cos(2r\theta + 2\alpha)$ and if further $k\theta = 360°$ with $k > 2$, it follows from (5) that

(12) $$\sigma_k^2 = k\sigma^2/2$$

Let us now in (6) set $C = M$; or rather, what amounts to the same thing, change the origin for the data so as to make $M = 0$. Then the "mean value" or "expected value" of $F(k)$ in (10) is zero. Then, with the use of (8) and (12), it follows that

(13) Probability that $\left|F(k)\right| > 3\sigma\sqrt{k/2}$ is $1 - \phi(3)$
=0.0027

This small probability by no means implies impossibility However, if the computed $\left|F(k)\right|$ exceeds $3\sigma\sqrt{k/2}$, there is some ground for doubting the original hypothesis that the data under consideration exhibit a chance arrangement. Sometimes such evidence gathered from different sections of the data can be made cumulative. A comparatively large value for $F(k)$ in (10) is likely to result when the signs of the X_r match the signs of the a_r, taken as in (11) and (12) as $\cos(r\theta + \alpha)$, giving a cycle or period of k items.

To what extent evidence is thus afforded for the specific period of k needs further consideration. But, until we have found an adequate number of instances in which some inequality like (13) is satisfied we have obtained little evidence of any periodicity at all.

Thus far we have considered normally distributed data, conforming to the well-known symmetric bell-shaped probability curve. But this is more restrictive than necessary. Results substantially the same can usually be obtained for distributions—even those not symmetric and not mesokurtic—which at both ends taper off in slender tails. Although the particular numerical value of the probability given in (13) is no longer applicable to these curves, the probability nevertheless is usually very small, as presented geometrically as slices of the two tails.

Moreover, the equations (11) and (12) arise from the general theory of expected values. Suppose that p_r is the probability that the chance variable X will take on the value ξ_i, where $p_1 + p_2 + \cdots + p_s = 1$. Then the expected value of X is, by definition

$$(14) \qquad E(X) = p_1 \xi_1 + p_2 \xi_2 + \cdots + p_s \xi_s = E_1 ,$$

and its mean error $\epsilon(X)$ is defined by

$$(15) \qquad \epsilon^2(X) = p_1 (\xi_1 - E_1)^2 + \cdots + p_s (\xi_s - E_1)^2 = E(X - E_1)^2$$

It is common to identify expected value and mean error with arithmetic mean and standard deviation as approximations. In applying (6), the supposition was made that the origin be taken so as to make $M = 0$. With this adjustment, we may take $E(X) = 0 = E$, in (14) and (15). As the X_j are regarded as independent, the theory of expected values applied to (10) leads first from $E(X) = 0$ to $E[F(k)] = 0$, as mentioned before, and then to (11)—noting that when $i \neq j$, $E(X_i X_j) = 0$, in the expression for $E\left[F(k)-0\right]^2$.

IV. Data with Periodic Constituents.

We now consider data of the form

$$(16) \qquad W_r = X_r + Y_r + Z_r ,$$

where X_r is, as before, a chance variable; but

$$(17) \qquad Y_r = b\cos(r\theta+\beta)\ ; \quad Z_r = c\cos(r\theta'+\gamma) ,$$

$$(18) \qquad k\theta = 2\pi = 360^\circ = k'\theta'$$

Here Y_r and Z_r are periodic with periods k and k', not necessarily integral, amplitudes b an c, phases β and γ, respectively. Dealing first with Y_r, let m and n be whole numbers such that $n = mk$. Then, in analogy with (10), but applied to n of the $Y's$ take

$$(19) \qquad F(n) = \sum_{r=0}^{n-1} Y_r \cos(r\theta+\alpha) = \frac{nb}{2}\cos(\alpha-\beta)\ ; \qquad k>2$$

as may be shown from (2) and (5). The magnitude of $F(n)$ depends materially upon the phase difference $(\alpha-\beta)$ But

$$(20) \qquad \left|\cos(\alpha-\beta)\right| > 0.92 , \quad \text{if } \left|\alpha-\beta\right| \leq 22\tfrac{1}{2}^\circ$$

Thus if the phase α of the test function $\cos(r\theta+\alpha)$ differs from the phase β of the data, taken now as Y_r in (17), by not more than 22½°, the absolue value of $F(n)$ in (19) will fall below its maximum, $nb/2$, by less than 8%. The phase β of the Y constituent of data would in general be unknown; but if for α we take *eight* consecutive multiples of 45°, one of these would fall within 22½° of any designated angle β, (mod 360°) Moreover, if in (19), α is increased by 180°, $F(n)$ merely changes its sign, and thus gives no essentially new information. Hence, instead of *eight* multiples of

45°, the four multiples—90°, 0°, −45°, 45°—will be adequate. These, taken in the above order, give

$$(21)\qquad S=\sum_{r=0}^{n-1} Y_r \sin r\theta \qquad ; \quad C=\sum_{r=0}^{n-1} Y_r \cos r\theta$$

$$(22)\qquad S'=\sum_{r=0}^{n-1} Y_r \sin(r\theta+45^\circ)\,; \quad C'=\sum_{r=0}^{n-1} Y_r \cos(r\theta+45^\circ)\,.$$

Furthermore, it is not necessary to compute S' and C' in (22) directly from the data, since

$$(23)\qquad S' \; \frac{\sqrt{2}}{2}(C+S) \quad ; \qquad C' \; \frac{\sqrt{2}}{2}(C-S) \quad ;$$

but a direct computation of S' or C' would serve well as a check upon (21) Thus, if in (19) we assign to α the four values mentioned above, we get S, C, S', C', in (21), (22) such that for one of these quantities (20) is satisfied, which makes $F(n)$ in (19) take a value almost equal to $nb/2$. This increases as n itself—not merely as the square root of n, an increase typical for $\sum X_r \cos(r\theta+\alpha)$, see (12), (16), with k replaced by n

Let us now consider the function.

$$(24)\quad G(n)=\sum_{r=0}^{n-1} Z_r \cos(r\theta+\alpha)=C\sum_{r=0}^{n-1}\cos(r\theta+\alpha)\cdot\cos(r\theta'+\gamma).$$

By (2), the terms above have the form

$$(25)\quad \frac{C}{2}\cos\left[r(\theta+\theta')+\alpha+\gamma\right]+\frac{C}{2}\cos\left[r(\theta-\theta')+\alpha-\gamma\right].$$

In order to use (5), we postulate that neither $\theta+\theta'$ nor $\theta-\theta'$ is zero or any other multiple of 360°; in particular $\theta \neq \theta'$. With $n = mk$, as before, (18) gives $n\theta = mk\theta = m(2\pi)$. Hence, it follows that

$$\sin n(\theta+\theta')/2 = \pm\sin n\theta'/2 = \pm\sin mk\pi/k'\,.$$

Likewise, $\sin n\,(\theta-\theta')/2 = \mp \sin mk\pi/k'$.

Hence, from (4), (25) it follows that $G(n)$ in (24) contains the factor sin $mk\pi/k$. Thus $G(n) = 0$, if

$$(26) \qquad k' = mk,\ \frac{mk}{2},\ \frac{mk}{3},\ \cdots\ \frac{mk}{m-1},\ \frac{mk}{m+1},\ \frac{mk}{m+2},\ \cdots$$

This may also be written

$$(27) \qquad qk' = mk, \quad q = \text{any whole number} \neq m.$$

Thus, if m cycles of a period k are used as multipliers in the form (24) upon a set of mk numbers Z, with period $k = mk/q$, where q is any whole number except m the result is zero It should be noted that in order to apply (4) to (24) (25), to get (26), it was necessary to require that $k' \neq k$, which would make $q \neq m$ in (27) To illustrate 3 cycles, each with period $k = 4$, will "annihilate" a set of 12 numbers if these are the successive terms of $C\cos(\gamma + 2\pi r/k')$ with period k' equal to 12, or 12/2, or 12/4, or 12/5, etc., but not 12/3

Indeed, $G(n)$, instead of vanishing when k is set equal to k in (24), making $\theta' = \theta$, takes on just about its maximum value $nc/2$ in this case when the phase α is properly chosen—see (19), (20) Inasmuch as $G(n)$ in (24) is a continuous function of θ', it follows that if k' is taken very close to k, $G(n)$ would be almost as large as for $k' = k$ But from (26) we learn that $G(n)$ goes down to zero if k' is allowed to be as small as $mk/(m+1)$ or as large as $mk\ (m-1)$

Thus, if significantly large results are obtained when using the test function cos $(r\theta + \alpha)$ with period $k = 2\pi/\theta$, the individual period k itself is not necessarily indicated. But rather, the test furnishes evidence that *some period close to* k *is present in the data*, this proximity being expressed by the inequality (see 26)

$$(28) \qquad \frac{m}{m+1}\,k < k' < \frac{m}{m-1}\,k\,.$$

The relations involved here can perhaps be set forth in greatest simplicity by using integration to effect summations—cf (34). In the test function cos $(r\theta + \alpha)$, set the phase $\alpha = 0$, and take $x = r\theta$, where $\theta = 2\pi/k$ Suppose k is rational, and take an even integer m such that $n = mk$ is an even integer Consider the test as covering the data, from $x = -m\pi$ to $x = m\pi$. Also, in (24), take $\gamma = 0$, $\theta' = t\theta$, $= 1$ This leads naturally to

$$(29) \qquad g(t, m) = \frac{1}{m\pi}\int_{-m\pi}^{m\pi} \cos x \cdot \cos tx \, dx$$

where the coefficient $1/m\pi$ is chosen to make $g(1, m) = 1$ With the aid of (2), it is easily seen that

$$(30) \qquad g(t, m) = \frac{2t \sin m\pi t}{m\pi(t^2 - 1)}, \qquad t \neq 1$$

for a given m, the plot of $g(t, m)$ as a function of t consists of a crest above the interval from $t = 1 - 1/m$ to $t = 1 + 1/m$, flanked on each side by depressions only about one-fourth or one-fifth as great in size or amplitude followed by waves of still smaller size—a "vibration" *strongly* "damped" on each side of $t = 1$ It has essentially the same characteristics as curves frequently occurring in periodogram analysis.[1] Only the interval from $t = 1 - 1/m$ to $1 + 1/m$ has in general much significance Sometimes the two adjacent waves[2] need a little attention But as $\theta' = t\theta$, the above interval is described by

$$(31) \qquad 1 - \frac{1}{m} < \frac{k}{k'} < 1 + \frac{1}{m},$$

which is another way of writing (28).

As an illustration, suppose that 4 cycles of 12 terms each of cos $(r\,30^\circ + \alpha)$ are used in a test with a significantly large result. Here $k = 12$, $m = 4$ Then (28) would recommend to our consideration periods between 9.6 and 16 Perhaps only those between 10 and 15 would deserve serious attention Since at points $t = 1 \pm 3/4m$, the curve (30) is less than half as high as at $t = 1$. Another interesting form[3] of (28) is

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[1] Rietz-Handbook Loc. cit p. 172, Figure 17

[2] Schuster, *Terrestrial Magnetism*, Vol. 3 (1898), p. 30

[3] Cf the Schuster criterion, Rietz Loc cit p. 173, Schuster, Loc. cit, p. 30

$$(32) \qquad \left| k' - k \right| < \frac{k'}{m}$$

Before leaving (30), it may be well to note that $g(t, m)$ does *not* take its maximum exactly at $t = 1$, but at

$$(33) \qquad t = 1 + \frac{3}{3 + 2m^2\pi^2}$$

as may be seen by setting $t = 1 + T$ in (30), expanding $\sin m\pi t =$ $\sin m\pi T$ in powers of T, and setting the first derivative equal to zero When $m = 1$, $t = 1.13$, when $m = 2$, $t = 1.04$; when m is moderately large, t is very close to 1. In all cases, however, the test function which yields the largest result, when applied to a cosine function with period k' is not that one which exactly fits, but one with period $k = k't$, where in the ideal case represented by (29) this value of t is given by (33). Inasmuch as $t > 1$, there is some *danger*, then, of *overestimating the size of the unknown period* k', if the attempt is made to get a close approximation to k' by using several test periods k in the immediate vicinity of k', and selecting the k giving the maximum result This is not due to the fact that $G(n)$ in (24) is a *linear* function of the Z_r's. For, if in (29), we should change $\cos x$ to $\sin x$, to get the mate of $g(t, m)$, this mate would be zero. Thus, the usual *quadratic* function would reduce to the square of $g(t, m)$, and would have its maximum at the same place given by (33) If the main purpose of an investigation is merely to locate with fair precision those periods whose existence have high probabilities, it may not be necessary to refer to (33).

Going back to the constituents of W_r in (16) we see that if n terms of $\Sigma W_r \cos(r\theta + \alpha)$ are taken, the Y contribution to this sum increases directly as n, the X contribution, being of chance origin, increases usually about as the square root of n; while the Z contribution oscillates about zero

V Convenient Forms for Test Functions

The main points of the theory needed for testing data for periods with the aid of linear functions have now been set forth. In the first place, it appears impossible to demonstrate a periodicity. At best, we can merely make certain suppositions appear more or less probable or improbable The method outlined here starts with the assumption that the *order* of the sizes exhibited in the data is a *chance* arrangement Certain functions are to be computed which under chance conditions would ordinarily keep generally within a certain range If these functions show no marked tendency to jump the bounds, then the tests yield no positive evidence of periodicity On the other hand, if these functions take on extremely high values, it appears reasonable to relinquish the supposition that the order of sizes is a chance arrangement and to suppose, rather, that such a periodicity exists as would naturally make the function large. If the data have as a constituent a cosine fluctuation and this is matched by a test cosine curve of the same period and phase, it is easy to see that the sum of products all positive obtained from similarly placed ordinates may be abnormally large Any k which gives these large results is to be regarded as approximating a probable period

That the tests may all be conducted in a systematic and uniform manner, some further properties and details may well be noted.

In the first place, only values of $k \geqq 2$ need be considered if the data are regarded as representing a sequence of discrete values, corresponding to values of the time (or other argument) spaced at unit intervals. For suppose that p/q, the period of $\cos\left[2\pi rq/p\right]$ is less than 2 Then for each integer r, $\cos\left[2\pi rq/p\right] = \cos\left[2\pi r(p-q)/p\right]$, the latter with period $p/(p-q) > 2$ This applies, indeed, to the case where the discrete values are integrated values. In fact, since

$$(34) \qquad \int_r^{r+1} \cos\left(\frac{2\pi t}{k}+\beta\right)dt = A\cos\left(\frac{2\pi r}{k}+\beta'\right),$$

where $A = (k/\pi)\sin \pi/k$, $\beta' = (\pi/k) + \beta$, it follows that if there is growth or deposit of $K + \cos\left[2\pi t/k + \beta\right]dt$ in time dt —thus, with period k —then the total deposits in time intervals 0 to 1, 1 to 2, 2 to 3, etc, form a sequence with the same period k.

In the second place, it should be noticed that the case of $k = 2$ is peculiar. In place of (12), we have

$$(35) \qquad \sigma_2^2 = 2\sigma^2\cos^2\alpha$$

as follows directly from the fact that when $k = 2$, $\theta = 180^\circ$, and $\cos(180^\circ + \alpha) = -\cos\alpha$. With the phase α small, we have approximately $\sigma_2 = \sigma/\sqrt{2}$.

Let us now suppose that the data in given order are divided into sets of convenient size—say sets of 120 measurements Let the arithmetic mean and standard deviation of each set be found. If these quantities—in particular, the standard deviation—show violent fluctuations as we pass from one set to the next set, it may be necessary to handle the material in different sets. But suppose these fluctuations appear to keep within reasonable bounds

In (6), the data were represented by X_r. Later, in order to emphasize the possibility of different constituents, W_r was used in (16). But, for simplicity, let us now return to X_r as a symbol for the rth element of the data In the first tests, let the period k be a whole number. Moreover, in place of the functions (21), (22) let us introduce the following, for $k > 2$.

$$(36) \qquad u = u_j(k) = \frac{1}{\sigma}\sqrt{\frac{2}{k}} \sum_{jk-k}^{jk-1} X_r \sin r\theta\,, \quad j = 1,\ 2,\ 3,$$

$$(37) \qquad v = v_j(k) = \frac{1}{\sigma}\sqrt{\frac{2}{k}} \sum X_r \cos r\theta\,, \quad k\theta = 360^\circ$$

$$(38) \qquad u' = u'_j(k) = \frac{1}{\sigma}\sqrt{\frac{2}{k}} \sum X_r \sin(r\theta + 45^\circ)$$

$$(39) \qquad v' = v'_j(k) = \frac{1}{\sigma}\sqrt{\frac{2}{k}} \sum X_r \cos(r\theta + 45^\circ)$$

In the case of $k = 2$, replace the radical by $1/\sqrt{2}$. If tests for fractional k are desirable, replace k in (36) to (39) by n, where $n = mk$, m and n whole numbers as in (21).

Here for each individual set—say of 120 measurements—it is assumed that each measurement has the same "expected value" or "Probable value," approximated by the arithmetic mean, and the same "mean error," approximated by the standard deviation σ. In this case, u, v, u', and v' all have the same expected value, zero, by (5), noting that the distributive law holds for expected values Moreover, when $k > 2$—see (11), (12), (14), (15)—the mean error of u, v, u', and v' is in each case unity. This is also true when $k = 2$, if the phase has been properly matched—see (35).

To make the tests, then, the functions u, v, u', and v' are computed for certain values of k —perhaps for the sub-multiples 2, 3, 4, 5, 6, 8, . . . of 120 In this way, for $k = 6$, twenty values would be found for each of the four functions. The information thus found may not be very significant But if not, we may combine results as follows. Let $s = q^2$, where q is a whole number Let

$$(40) \qquad U_1 = \frac{1}{q}\,(u_1 + u_2 + \cdots + u_s) \;; \quad U_2 = \frac{1}{q}\,(u_{s+1} + \cdots + u_{2s}),$$

etc., and form similar expressions for $V_1, V_2, \ldots U_1', U_2', \ldots V_1', V_2', \ldots$ Each of these functions has expected value *zero* and mean error *unity*. To illustrate—suppose that $u_1(6) = 1.8$, $u_2(6) = 2.1$; $u_3(6) = 1.7$; $u_4(6) = 1.8$ These results taken individually would not furnish strong evidence for a period of 6. Some statisticians regard a variation equal to three times the "probable error" or two times the standard deviation as "significant"—in which case (6) 2.1 would be significant. But such evidence is not overwhelming. But, by (40), $U_1 = 3.7$. Here U_1, with mean value zero, has jumped up to an absolute value 3.7 times its standard deviation, unity. On a pure chance basis, in normal distributions, this would happen only about twice in 10,000 trials, on the average Altho $U_1 = 3.7$ affords no demonstration of a period of 6, the result is at least highly significant. If such high values occur repeatedly in using $k = 6$, we would be justified in asserting that the data contain a constituent with period somewhere near 6.

Moreover, the process (40) is subject to iteration—as long as the data hold out If $s' = q'^2$, then $(U_1 + U_2 + \ldots + U_s)/q'$ is a function with mean value zero, and mean error unity

When the change in standard deviation is fairly gradual from set to set, the values of $u_1, u_2 \ldots$ can be computed without interruption, using proper adjustments for those values of u_j whose terms arise partly from two sets, such as u_8 (16).

Such a result as $u_2(6) = 2.1$ would furnish evidence only for the six measurements from which it was computed; and in the light of (28), with $m = 1$, the implication at most would be for some period greater than 3. But $U_1 = 3.7$ would furnish strong evidence that in the 24 measurements covered there was a constituent with period between 4.8 and 8—taking $m = 4$ in (28)

The technique of computation would present a few problems. In some cases (6) would be utilized Certain tables[1] of products with the harmonic factors as multiplicands may be of assistance. Or certain tables may be constructed for use with the aid of an adding machine—with complements listed to take the place of negative numbers Only u and v in (36), (37) need be computed directly; for u' and v' may be found at once—see (23). But it would seem advisable to compute u' or v' as a check. Graphs showing the progress of the functions u, v, etc, may be constructed

The interpretation of the results would often be difficult because different sections of the data would frequently give different indications Again, if two layers of rock are counted as one, an error would be introduced But this would affect the u_j, v_j, . . involved, not the preceding or following u_j, v_j. Indeed, if an actual period is present, as indicated by the u's, an error of merging may merely shift the "burden of proof" to one of the other functions v, u', or v'. Certain cyclic changes, bringing u, u', v, v' into prominence in rotation, may indicate that the test period k is close to an actual period but with a discrepancy large enough to produce a systematic advance of phase Many similar principles commonly employed in period testing could be used to advantage in the method here outlined

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[1] E. g. L. W. Pollak "Rechentafeln zur Harmonischen Analyse"

Edward L. Dodd

SYNOPSIS OF ELEMENTARY MATHEMATICAL STATISTICS[1]

by

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SECTION IV. THE GRAPHICAL REPRESENTATION OF FREQUENCY DISTRIBUTIONS

25. The investigation of a frequency distribution is greatly facilitated by presenting the data graphically by means of either a *Frequency Polygon* or a *Histogram*, depending upon the nature of the distribution.

For a distribution of discrete variates the frequencies are represented by ordinates whose lengths are proportional to the various frequencies and whose abscissae correspond to the variates of the distribution. The shape of the distribution is rendered more apparent by either connecting the tops of the ordinates by straight lines, thus forming a *Frequency Polygon*, or drawing a *Frequency Curve* that approximately passes through the vertices of the polygon. Figure I presents the Frequency Polygon derived from the data of Table XI. In addition a curve has been drawn to illustrate the general trend of the distribution.

If the frequency distribution under examination be one of grouped discrete or continuous variates it will be found that the *Histogram* is best suited for graphical representation. A Histogram is a series of rectangles erected on bases that are proportional to the class intervals and with altitudes proportional to the respective class frequencies. Thus, in this case, the frequencies are represented by areas. The shape of the distribution may be emphasized by constructing a continuous frequency curve such that the areas under the curve between the ordinates at the lower and upper boundaries of the various rectangles should equal approximately the areas of the corresponding rectangles. Two examples are presented, both the distributions are composed of continuous variates, one exhibiting positive skewness and the second negative. The numerical data and corresponding Histograms are presented in Tables XII and XIII and Figures II and III respectively

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[1] A continuation of an article bearing the same caption appearing in Vol. I, No. 1, of the ANNALS.

TABLE XI

Distribution of Frequency of glands in the right fore-leg of 2,000 female swine[1]

v	f_v	t	f_t
0	15	-2.083	.013
1	209	-1.488	.176
2	365	- .893	.307
3	482	- .298	.405
4	414	.297	.348
5	277	.892	.233
6	134	1.487	.113
7	72	2.082	.061
8	22	2.677	.018
9	8	3.272	.007
10	2	3.867	.002

$M_v = 3.501$ $N = 2000$

$\sigma_v = 1.68077$ $\frac{1}{\sigma} = .594965$

$\alpha_{3,v} = .508462$ $\frac{\sigma}{N} = .000840385$

26 It has previously been stated that the three fundamental statistical functions are the Mean, Standard Deviation, and Skewness. The Mean has been defined as a convenient average, and the Standard

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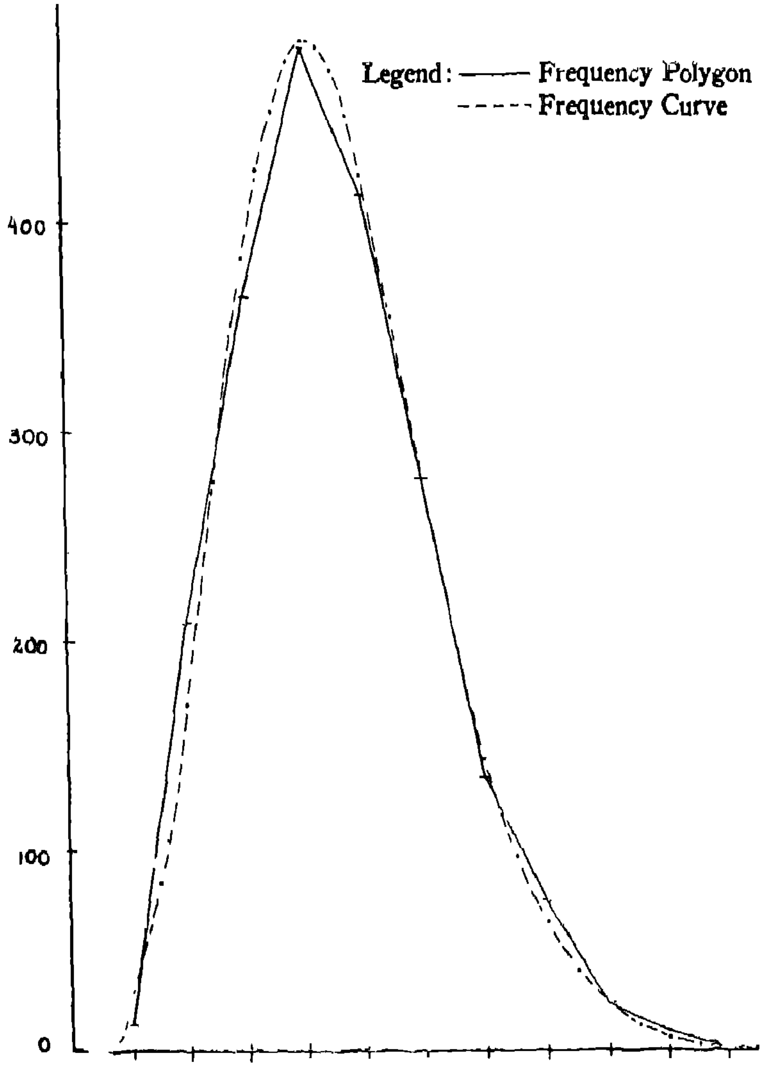
[1] Davenport, "Statistical Methods," page 35

FIGURE I

Frequency Distribution of glands in the right fore-leg of 2,000 female swine



Deviation measures the concentration of the variates about this average. Skewness has not, however, been so clearly explained. If the variates of a distribution be symmetrically arranged about their mean, then μ_{3v} or the third moment about the mean will be zero. Under these conditions α_{3v}, or the coefficient of skewness, must also be zero Thus α_{3v} measures the degree to which a frequency distribution is symmetrical. If α_{3v} is zero, then from the standpoint

TABLE XII

Weights of White Boys - 30 to 33 months
(Correct to nearest pound)

Class Mark	f	t	ft
21	3	-2.90	009
22	3	-2.50	009
23	11	-2.09	.032
24	27	-1 69	.079
25	65	-1.28	191
26	101	- .88	.297
27	135	- 47	.397
28	136	- .07	400
29	128	34	.376
30	105	.75	.309
31	59	1.15	.173
32	30	1.56	.088
33	15	1.96	044
34	7	2.37	.021
35	5	2.77	.015
36	8	3.18	024
37	1	3.58	.003
38	1	3.99	.003

$M_v = 28.16190$ $N = 840$

$\sigma_v = 2\ 46837$ $\frac{1}{\sigma} = .405126$

$\alpha_{3v} = 427969$ $\frac{\sigma}{N} = .00293854$

of the present synopsis the distribution may be considered normal, for if such a distribution be graphed in standard units it will follow the locus of the well known Normal Curve of Error. Accordingly it would seem logical to expect that for each value of α_3 there is *one* standard curve which is the locus toward which all distributions with that degree

TABLE XIII

Barometric Heights for Daily Observations During Thirteen Years at Llandudno, England[1]

(Original measurements to nearest millimeter)

Class Mark	t	f	ft
28 35	-4 38	1	001
28 55	-3 82	2	001
28 75	-3 26	8	005
28.95	-2.71	30	018
29.15	-2.15	74	045
29.35	-1 59	166	102
29.55	-1 04	368	.226
29 75	- 48	509	.313
29.95	08	656	403
30.15	.63	580	356
30.35	1 19	353	.217
30.55	1 75	140	086
30 75	2 31	30	018
30 95	2 86	5	003

$M_v = 29.9221$ $\quad N = 2922$

$\sigma_v = .359014$ $\quad \frac{1}{\sigma} = 2.78541$

$\alpha_{3v} = -.32919$ $\quad \frac{\sigma}{N} = 000614329$[2]

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[1] Karl Pearson and A Lee, "Philosophic Transactions," p 428 (1897).

[2] This formula assumes that the class interval is unity, the proper value of $\frac{\sigma}{N}$ is therefore 5 times the value as ordinarily computed

FIGURE II

Weights of White Boys (30 to 33 months)

Histogram and Frequency Curve

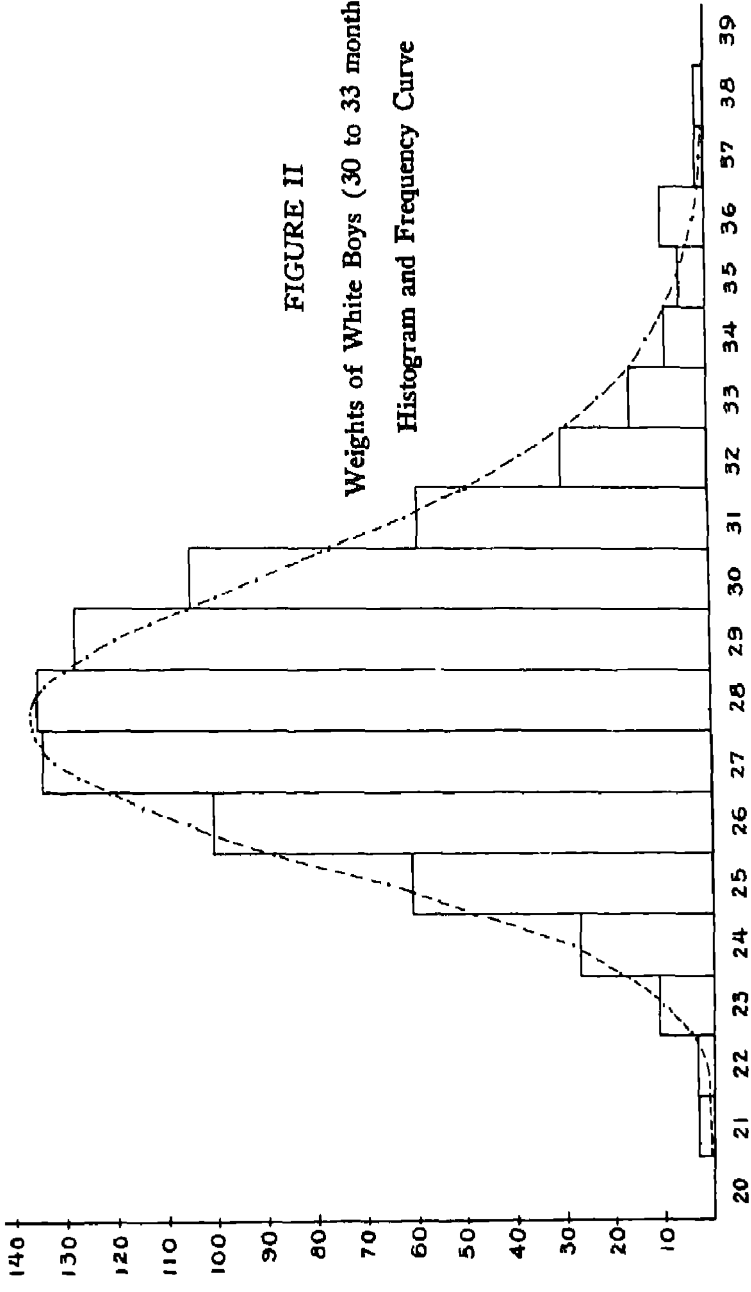
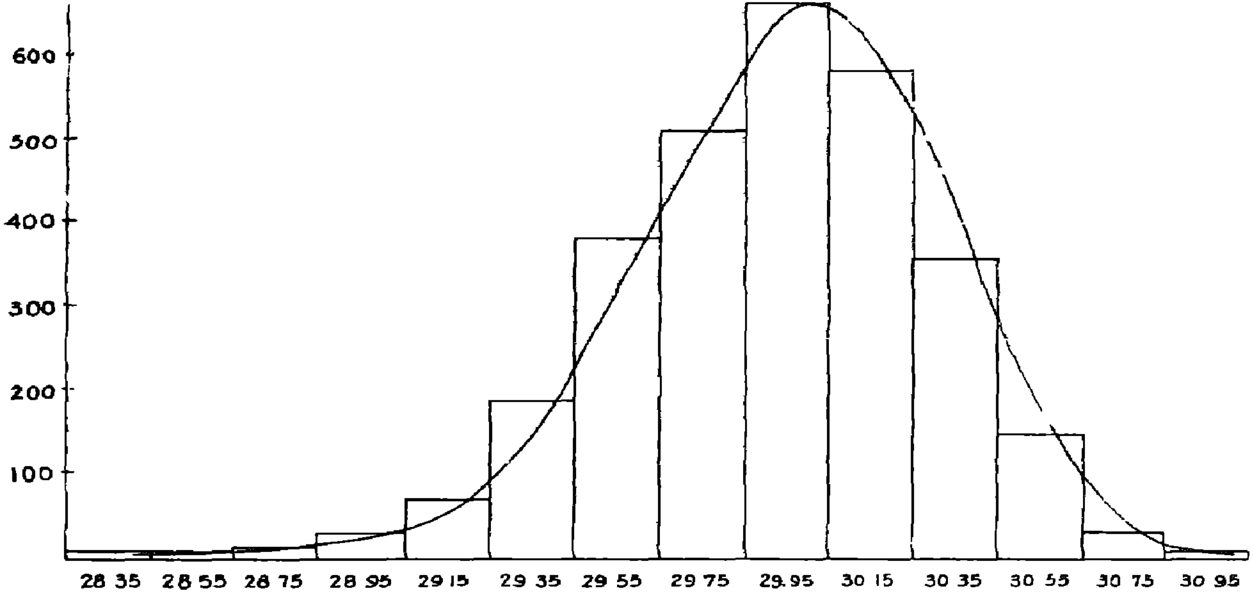


FIGURE III

Barometric Heights Recorded Daily at Llandudno, England



of skewness approach. The one essential is that the unit of measurement must be removed from the data, that is each distribution should be expressed in terms of the standard variates t and the corresponding frequencies f_t As before, the standard variate t_1 corresponding to v_1 is obtained from the following formula:

$$t_1 = \frac{v_1 - M_v}{\sigma_v} = \frac{\bar{V}_1}{\sigma_v}$$

Similarly, the frequencies for each of the standard variates is defined as follows.

$$f_t = \frac{\sigma_v}{N} f_v \tag{26}$$

These two formulae will enable one to analyze all distributions entirely independent of the unit involved In Figure IV the three distributions graphically presented in Figures I, II and III are shown contrasted with the Normal Curve The numerical values of t and f_t for each distribution are given in the corresponding table. The values may be obtained in each case by employing the continuous process described in Section I It will be noticed that the two distributions with positive skewness of .5 and .4 respectively reach their maximum in advance of the Normal Curve and approach the zero limit more gradually for positive values of the standard variates Accordingly, for the distribution exhibiting negative skewness, the positions are reversed and the more gradual approach to the zero limit occurs for the negative values of the standard variates In general, a distribution having skewness within the limits ± 3 will exhibit very little deviation from the normal curve when presented graphically in this manner.

Summary of Section IV—

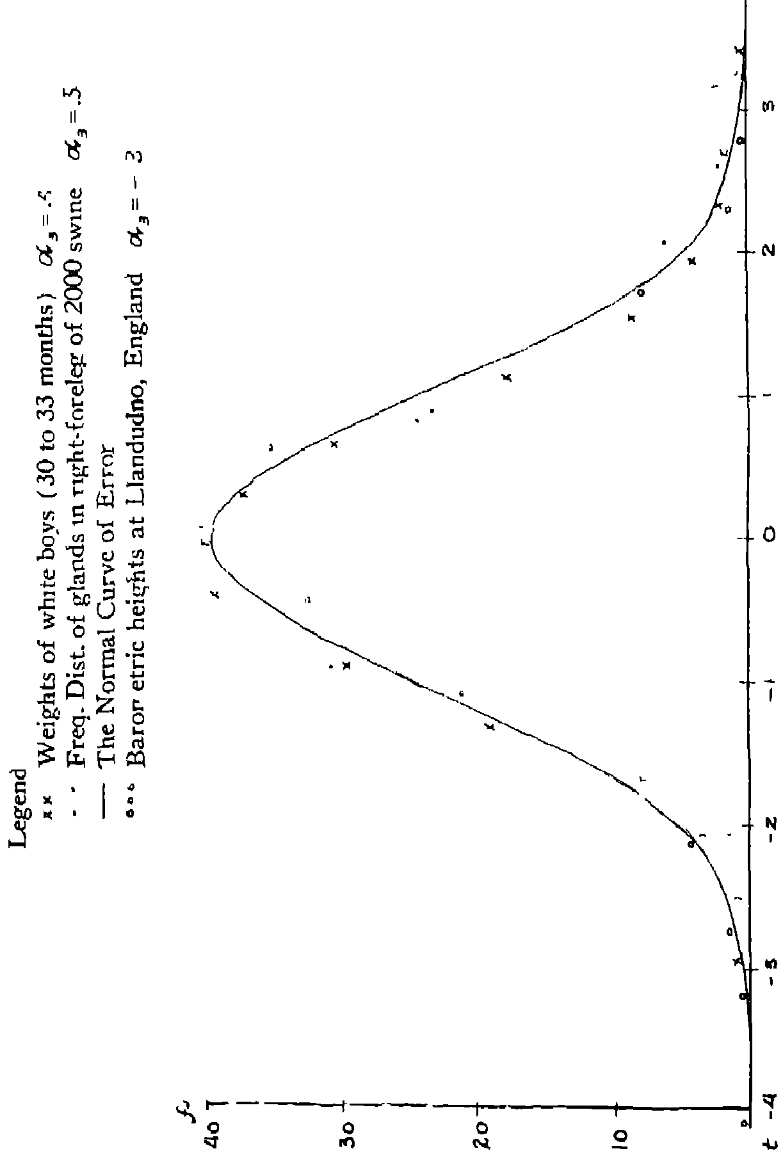
It is usually found very advantageous in the investigation of frequency distributions to present the data graphically A distribution of discrete variates should be represented by a Frequency Polygon and one of continuous variates by a Histogram. In either case a free hand curve may be drawn indicating the general trend of the distribution and is called the Frequency Curve. The *Standardized* Curve is obtained by plotting the variates and their corresponding frequencies in *standard* form by means of the following formulae:

$$t_1 = \frac{V_1 - M_v}{\sigma_v} = \frac{\bar{V}_1}{\sigma_v}$$

$$f_t = \frac{\sigma_v}{N} \cdot f_v \, .$$

FIGURE IV

The Graphical Comparison of Three Distributions with the Normal Curve



SECTION V. THE INVERSE PROBLEM

27. From the standpoint of Elementary Mathematical Statistics we may say that the Mean, Standard Deviation, and Skewness together with its total frequency completely characterize a distribution. If this statement were accurate it would be possible to reproduce any distribution if its three elementary functions and total frequency were known. A tabulation of Pearson's Type III Curves for various degrees of skewness affords, for the purposes of Elementary Statistics, the most satisfactory representation of frequency distributions from the point of view of both effectiveness and facility in using[1] In order to illustrate the method several numerical examples are included. In Table XIV the illustration is one of discrete variates

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[1] L. R. Salvosa, "Tables of Pearson's Type III Function," The Annals of Mathematical Statistics, May, 1930.

TABLE XIV

Frequency Distribution of Number of Glands in the Right Foreleg of 2,000 Female Swine

V (1)	t (2)	f_t (3)	Predicted Frequency (4)	Observed Frequency (5)
0	−2.08	.026952	32	15
1	−1.49	141661	169	209
2	−.89	.320068	381	365
3	−.30	409193	487	482
4	.30	353689	421	414
5	.89	229770	274[1]	277
6	1.49	118287	141	134
7	2.08	.051638	62[1]	72
8	2.68	.019220	23	22
9	3.27	.006459	8	8
10	3.87	001925	2	2
Total			2000	2000

$M = 3.501$ $N = 2000$

$\sigma = 1.68077$ $\frac{1}{\sigma} = 594965$

$\alpha_3 = 508462$ $\frac{N}{\sigma} = 1189.93$

Explanation In every case the value of α_3 is taken to the nearest tenth and the value of t to the nearest hundredth. In the examples included no interpolation has been made for any value

Columns (1) and (2) of Table XIV contain the variates and the corresponding values of t obtained by means of the continuous process. Column (3) is obtained directly from the Table of Ordinates of the Pearson Type III Function. All values may be found in the column with skewness = .5 and opposite the respective value of t . Since these are the Standard Frequencies f_t , the predicted frequencies for each variate may be obtained from the following formula.

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[1] In order to obtain $N = 2000$ it was necessary to increase these frequencies by 1, although the fractional value was slightly less than than the necessary 5.

$$\because f_t = \frac{\sigma}{N} \quad f_v$$

$$\therefore f_v = \frac{N}{\sigma} \cdot f_t \qquad (27)$$

The predicted frequencies in column (4), therefore, are obtained by multiplying column (3) by the value 1189.93 These values are the *graduated* frequencies. The actual observed frequencies are given in column (5).

TABLE XV

Distribution of Weights of White Boys - 30 to 33 Months

(Measurements correct to nearest pound)

Lower Limit of Class (1)	t at Lower Limit (2)	Accumulative Percent Frequency (3)	Percent Frequency (4)	Predicted Frequency (5)	Observed Frequency (6)
20 5	-3.10	.000021	000357	0	3
21 5	-2 70	.000378	002990	3	3
22.5	-2 29	.003368	.013174	11	11
23.5	-1 89	016542	.039440	33	27
24 5	-1.48	055982	079399	67	65
25 5	-1 08	135381	127331	107	101
26.5	- 67	262712	154908	130	135
27.5	- 27	417620	163707	138	136
28.5	14	.581327	.140357	118	128
29 5	54	.721684	.110157	93	105
30.5	95	831841	073061	61	59
31.5	1 35	904902	045897	39	30
32 5	1 76	950799	.025019	21	15
33 5	2 16	975818	.013225	11	7
34 5	2 57	.989043	006176	5	5
35 5	2 97	.995219	.002845	2	8
36.5	3 38	998064	001172	1	1
37.5	3.78	999236	000483	0	1
38.5	4 19	999719	000218	0	0
Total				840	840

$M = 28.16190$ $\qquad N = 840$

$\sigma = 2.46837$ $\qquad \frac{1}{\sigma} = 404126$

$\alpha_3 = .427969$

Explanation:

28. Since Table XV is a distribution of continuous variates, it is necessary to use the Table of Areas of the Pearson Type III Curve. The values in this table are the *accumulated percent* of the standard curve *below* a specified value of t. The method of prediction is therefore to estimate the per cent of the distribution lying *between* the consecutive lower limits of each class. In column (1) of Table XV are given the lower limit of each class and in Column (2) the value of t at this lower limit. Column (3) is taken directly from the Table of Areas of the Pearson Type III Function, $\alpha_3 = 4$, and represent the percent of the distribution lying *below* the particular value of t. In order to find the percentage of the distribution in each class, it is merely necessary, therefore, to difference column (3). For example, the first value, .000357, is found by subtracting .000021 from .000378. In order to find the predicted frequencies in column (5), N, or the total frequency, should be multiplied by each value in column (4). The observed frequencies are given in column (6).

TABLE XVI

Barometric Heights for Daily Observations During Thirteen Years at Llandudno, England

(Correct to the nearest millemeter)

Lower Limit of Class (1)	t (2)	Accumulative Percent Frequency (3)	Percent Frequency (4)	Predicted Frequency (5)	Observed Frequency (6)
28.25	−4.66	999956	000171	1	1
28.45	−4.10	999785	.000739	2	2
28.65	−3.54	999046	.002716	8	8
28.85	−2.99	996330	009081	27	30
29.05	−2.43	.987249	.025770	75	74
29.25	−1.87	.961479	.061380	179	166
29.45	−1.31	.900099	.117068	342	368
29.65	− .76	.783031	.185122	541	509
29.85	− .20	.597909	.222074	649	656
30.05	.36	.375835	.192952	564	580
30.25	.91	.182883	.121528	355	353
30.45	1.47	.061355	.048526	142	140
30.65	2.03	012829	.011285	33	30
30.85	2.58	001544	.001468	4	5
31.05	3.14	.000076	.000076	0	0
Total				2922	2922

$M = 29.92207$ $N = 2922$

$\sigma = .359014$ $\frac{1}{\sigma} = 2.78541$

$\alpha_3 = -.32919$

Explanation:

29. Although the data of Table XVI is also a distribution of continuous variates, it will be noticed that in this case the coefficient of

skewness is negative Since the Tables include only positive values of α_3, it seems desirable to explain the procedure for such a distribution If a frequency curve having pronounced positive skewness be graphed on rather fine paper and then held to the light or in front of a mirror, it will be seen that the distribution will seem to show negative skewness to the same degree in which it formerly displayed positive. This being true, it is possible to use the Tables for all cases of negative skewness by merely changing the sign of t, and if an area is desired it is necessary to reverse the order of differencing. Three examples are given in order to cover as many different cases.

Illustration 1, $\alpha_3 = -5$, required the percentage of the area of the standardized curve lying between $t = -2\ 43$ and $t = -1.98$. From the tables under the column for skewness = 5.

$t = +2.43$, percent of area = 983883

$t =\ 1.98$, percent of area = 964416

The percentage lying between these two values of t is therefore $.983883 - .964416 = .019467$

Illustration 2, if $\alpha_3 = -8$, required the percentage of the area lying between $t = -.02$ and $t = 25$ Using the Table of Areas in the column for skewness of .8,

If $t = +.02$, percent of area = .561064

$t = -25$, percent of area = .450687

To find the percent of the area merely subtract as before, $.561064 - 450687 = .110377$.

Illustration 3, if $\alpha_3 = -.2$, required the percentage of the area lying between $t = .52$ and $t = 1.63$. Again referring to the Tables of Areas, we find for $\alpha_3 = .2$

If $t = -\ .52$, percent of area = 310015

If $t = -1.63$, percent of area = .045108

Accordingly, the required percentage is $.310015 - .045108 = .264907$.

TABLE XVII

Expansion of $(5/6+1/6)^{180}$

v (1)	t (2)	f_t (3)	Pred. Freq. (4)	Obs. Freq. (5)
14	−3.21	.001468	0	0
15	−3.01	.003013	1	1
16	−2.81	.005919	1	1
17	−2.61	.010954	2	2
18	−2.41	.019227	4	4
19	−2.21	.032053	7	6
20	−2.01	.050807	10	10
21	−1.81	.076658	15	16
22	−1.60	.112095	23	23
23	−1.40	.153377	31	31
24	−1.20	.200401	40	41
25	−1.00	.250281	50	51
26	− .80	.299057	60	61
27	− .60	.342196	69	69
28	− .40	.375301	76	75
29	− .20	.394857	80	79
30	.01	.398640	80	80
31	.21	.386166	78	77
32	.41	.359746	72	72
33	.61	.322535	65	64
34	.81	.278510	56	56
35	1.01	.231792	47	46
36	1.21	.186059	38	37
37	1.41	.144144	29	29
38	1.62	.106201	21	22
39	1.82	.076661	15	16
40	2.02	.053513	11	11
41	2.22	.036145	7	8
42	2.42	.024163	5	5
43	2.62	.014975	3	3
44	2.82	.009196	2	2
45	3.02	.005476	1	1
46	3.23	.003077	1	1
47	3.43	.001723	0	0

$M_v = 29.973$ $N = 1000$

$\sigma_v = 4.96853$ $t_i = .201267\, v_i - 6.032569$

$\alpha_{3_v} = .108097$ $\dfrac{N}{\sigma} = 201.267$

TABLE XVIII

Expansion of $(5/6+1/6)^{180}$

Class (1)	Lower Limit (2)	t (3)	Accumulated Percent Freq (4)	Percent Freq (5)	Pred. Freq. (6)	Obs Freq. (7)
11–	10 5	–3.92	.000014	000213	0	0
14–	13 5	–3.32	000227	002128	2	2
17–	16 5	–2.71	002355	.012561	13	12
20–	19.5	–2.11	014916	049031	49	49
23–	22 5	–1 50	.063947	.120876	121	123
26–	25 5	– 90	184823	.203066	203	205
29–	28.5	– .30	387889	239543	239	236
32–	31,5	31	627432	191988	192	192
35–	34 5	91	.819420	112488	112	112
38–	37.5	1.51	931908	.048675	49	49
41–	40.5	2.12	.980583	.015048	15	16
44–	43.5	2 72	.995631	.003627	4	4
47–	46.5	3.33	999258	000742	1	0

$M_y = 29.973$ $N = 1000$

$\sigma_y = 4.96848$ $\frac{l}{\sigma} = 201269$

$\alpha_{3,y} = .105899$

30. As further numerical examples the three illustrated problems used in Section III have been graduated. The complete numerical solution will be found in Tables XVII, XVIII and XIX.

Summary of Section V—

Knowing the three fundamental functions and the total frequency of a distribution, it is possible to obtain predicted or graduated frequencies for that distribution with a surprising degree of accuracy. This is accomplished through the use of tables of the standard ordinates and accumulated percentage areas of the Pearson Type III Curves.

TABLE XIX

Weights of 1000 Female Students

(Original measurements to nearest .1 lb)

Class (1)	Lower Limit (2)	t (3)	Accumulated Percent Freq (4)	Percent Freq. (5)	Pred Freq. (6)	Obs. Freq (7)
70–	69.95	-2 88	000000	.000000	0	2
80–	79 95	-2 29	000000	003358	4	16
90–	89.95	-1.70	.003358	.102159	102	82
100–	99.95	-1 11	105517	.238290	238	231
110–	109.95	- 52	.343807	249585	250	248
120–	119.95	07	.593392	183665	184	196
130–	129.95	.66	.777057	111093	111	122
140–	139.95	1.25	888150	.059338	59	63
150–	149.95	1.84	.947488	.029412	29	23
160–	159.95	2.44	976900	013209	13	5
170–	169.95	3.03	.990109	.005791	6	7
180–	179.95	3.62	.995900	.002445	3	1
190–	189 95	4.21	.998345	.001002	1	2
200–	199.95	4.80	.999347	.000400	0	1
210–	209.95	5 39	.999747	000157	0	1
220–	219.95	5 98	999904	.000096	0	0

$M_v = 118.74$ $\qquad N = 1000$

$\sigma_v = 16.9175$ $\qquad \frac{1}{\sigma} = .0591104$

$\alpha_{3,v} = 976424$

It should be remembered that in advanced statistics moments higher than the third are necessary to characterize a distribution, but from the elementary viewpoint, the Mean, Standard Deviation and Skewness are considered to completely characterize a distribution.

Section VI Bernoulli's Theorem

31 *Factorials* For convenience, the product of the first n consecutive integers is called "factorial n" and is designated by the symbol $\underline{|n}$. Thus

$$\underline{|3} = 1\cdot 2\cdot 3 = 6, \quad \underline{|5} = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120, \quad \frac{\underline{|8}}{\underline{|5}} = 8\cdot 7\cdot 6 = 336.$$

Combinations. The number of combinations, each of r things, that can be formed from n things, is represented by the symbol ${}_nC_r$. Texts on elementary algebra show that

$$(28) \qquad {}_nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} = \frac{n\,(n-1)\,(n-2)\cdots(n-\overline{r-1})}{1\cdot 2\cdot 3\cdots r}$$

For example, suppose we desire to find the number of different committees, each of three persons, that can be selected from five individuals. If we designate the five individuals by the letters A, B, C, D and E, we observe that committees of three may be systematically enumerated as follows:

ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE

The number of committees, which we just enumerated as 10, agrees with the value found by formula (28), for since here $n = 5$, $r = 3$,

$${}_nC_r = {}_5C_3 = \frac{\underline{|5}}{\underline{|3}\cdot\underline{|2}} = \frac{5\cdot 4}{1\cdot 2} = 10$$

Another illustration: The number of different committees, each composed of seven individuals, that can be selected from ten candidates is

$${}_{10}C_7 = \frac{\underline{|10}}{\underline{|7}\,\underline{|3}} = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120$$

and the number of combinations, each of three, that can be formed from ten items is

$${}_{10}C_3 - \frac{\lfloor 10}{\lfloor 9 \; \lfloor 7} = 120$$

It should be noted that ${}_{10}C_7 = {}_{10}C_3$, and in general that

$$ {}_{n}C_{r} = {}_{n}C_{n-r} \tag{29}$$

This follows from the fact that the number of ways of selecting r items from n is equal to the number of ways of rejecting $(n-r)$ from n Thus, every time three are selected from ten, seven are rejected Therefore the number of ways of selecting three from ten, ${}_{10}C_3$, is also equal to ${}_{10}C_7$.

We shall have occasion to refer to the following tabulation of values of ${}_{n}C_{r}$.

TABLE XX

Values of ${}_{n}C_{r}$

N	r												
	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1											
2	1	2											
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1

32 *Binomial Theorem* By repeated multiplication we find that

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

etc

By mathematical induction it can be shown that for positive integer values of n

$$(30)\quad (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots$$

This equation is known as the binomial theorem and may be written more compactly, if n is an integer, in the following form:

$$(31)\quad (a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + {}_nC_3 a^{n-3}b^3 + \cdots$$

Using Table XX, we may write down at once that

$$(a+b)^{12} = a^{12} + 12a^{11}b + 66a^{10}b^2 + 220a^9b^3 + \cdots + 66a^2b^{10} + 12ab^{11} + b^{12}$$

Bernoulli's Series. If p denote the probability that an event will happen in a single trial, and q the probability that it will not happen in that trial, $p+q=1$, then the probability that the event will happen exactly 0, 1, 2, . x times during r trials is given by the respective terms of the binomial expansion

$$(32)\quad (q+p)^r = q^r + {}_rC_1 q^{r-1}p + {}_rC_2 q^{r-2}p^2 + \cdots {}_rC_x q^{r-x}p^x + \cdots$$

To illustrate. If a coin be tossed, we may assume *a priori* that the probability that heads will turn up is $p=1/2$ and the probability that heads will not turn up is $q=1/2$. If an individual tosses the coin twelve times in succession, it is possible that heads may turn up on no occasion, or heads may turn up exactly 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12 times, respectively. By formula (32), these chances are equal respectively to the successive terms of the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$, namely

$$\left(\frac{1}{2}\right)^{12}+{}_{12}C_1\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right)+{}_{12}C_2\,\frac{1}{2}^{10}\left(\frac{1}{2}\right)^{2}+\cdots+{}_{12}C_{11}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{11}+{}_{12}C_{12}\left(\frac{1}{2}\right)^{12}$$

Denoting the probability that heads will turn up on exactly x occasions by P_x, and referring to Table XX for values of ${}_{12}C_x$, we have that

$$P_{12}=\frac{1}{4096},\quad P_{11}=\frac{12}{4096},\quad P_{10}=\frac{66}{4096},\quad P_{9}=\frac{220}{4096}$$

TABLE XXI

Values of the Terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$

Number of Successes (1)	$r=12$, $q=.5$, $p=.5$ Probability P_x (2)	Expected Freq. $4096\,P_x$ (3)	Observed Frequencies (4)
0	1/4096	1	0
1	12/4096	12	7
2	66/4096	66	60
3	220/4096	220	198
4	495/4096	495	430
5	792/4096	792	731
6	924/4096	924	948
7	792/4096	792	847
8	495/4096	495	536
9	220/4096	220	257
10	66/4096	66	71
11	12/4096	12	11
12	1/4096	1	0
Total	1	4096	4096

33 *Expectation* If p denote the probability of success for each of n trials, then pn is defined as the expected number of successes in n trials. For example, we have just shown that the *a priori* probability of throwing heads twelve successive times with a coin is equal to $P_{12}=\frac{1}{4096}$. Therefore if twelve coins be tossed simultaneously on 4096 occasions, we expect that all twelve coins will turn up heads on only one occasion Likewise, the expected number of times that exactly ten heads and two tails would turn up is equal to 4096 $P_{10}=66$, and that exactly half of the coins would turn heads only $4096\cdot P_6 = 924$ times.

It will be seen that the sum of all the probabilities in column (2) is unity This follows from the fact that these values are the several terms of the expansion of $(q+p)^r$, and since $q+p=1$, therefore $(q+p)^r=1$

TABLE XXII

Values of the Terms in the Expansion of $\left(\frac{5}{6}+\frac{1}{6}\right)^{12}$

Number of	$r=12,\quad q=5/6\quad p=1/6$		Observed
Successes (1)	Probability P_x (2)	Expected Freq (3)	Frequencies (4)
0	11216	459	447
1	26918	1103	1145
2	.29609	1213	1181
3	.19739	808	796
4	08883	364	380
5	02843	116	115
6	.00663	27	24
7	.00114	5	7
8	.00014	1	1
9	00001	0	0
10	00000	0	0
11	00000	0	0
12	.00000	0	0
Total	1 00000	4096	4096

A second illustration. Suppose twelve dice are thrown and that only a throw of 6 is to be considered a success By formula (32), therefore, the expansion of

$$\left(\frac{5}{6}+\frac{1}{6}\right)^{12}=\left(\frac{5}{6}\right)^{12}+{}_{12}C_1\left(\frac{5}{6}\right)^{11}\left(\frac{1}{6}\right)+{}_{12}C_2\left(\frac{5}{6}\right)^{10}\left(\frac{1}{6}\right)^{2}+$$

are equal respectively to the probabilities that exactly 0, 1, 2 successes will be obtained in a single throw of the twelve dice, or what is the same thing, in twelve successive throws with a single die.

In this case the probabilities p_x are expressed as decimals, since the expansion contains values of $(6)^{12}$ in the denominators Therefore 6^{12} is the smallest value of N that will produce integer expected frequencies.

34 We shall now attack a more important problem Let us consider a hypothetical group of 100,000 individuals, all of the same age and all exposed to the same hazards of life Moreover, let us assume that the probability that each individual will die within one year is $p = .008$, or that the probability that any specified individual will survive a year is $q = .992$

By formula (32) the terms of the expansion of $(q+p)^{r}$ $(.992 + 008)^{100,000}$, namely, $(992)^{100,000}$ ${}_{100,000}C_1$ $(992)^{99,999}$ $(.008)^1 +$ ${}_{100,000}C_2$ $(.992)^{99,998}$ $(008)^2$ . . . . . ${}_{100,000}C_x$ $(.992)^{100,000-x}$ $(008)^x$. represent the probabilities that exactly 0, 1, 2, . . , x, . . . individuals will die within the year.

The value of $(992)^{100,000}$ is very small Thus $(992)^{100,000} = \left(\frac{992}{1000}\right)^{100,000}$

$\log 992 = 2.9965117$	$\log 992^{100,000} = 299651.17$
$\log 1000 = 3$	$\log 1000^{100,000} = \underline{300000.00}$
	$\log (992)^{100,000} = \overline{349}.17$

Therefore $.992^{100,000} = 000,000,000$ . . 15, where 15 is preceded by 348 zeros The probability that all would die $(008)^{100,000}$ is far less than this value

The values of p_x in Table XXIII are given to the nearest fourth decimal place Thus to six decimal places $P_{690} = .000005$ and $P_0 + P_1 + P_2 + \dots + P_{690} = \sum_0^{690} P_x = .000035$. These values appear in Table XXIII, therefore, as .0000. An inspection of Table XXIII shows that for our hypothetical population

(a) The chance that exactly 800 will die within a year is .0142
(b) The chance that 800 or less will die within a year is .5094.
(c) The chance that 850 or less will die within a year is .9625.
(d) The chance that at least 750 will die within a year is

$$P_{750} + P_{751} + P_{752} + \dots + P_{100,000} = 1 - \sum_0^{749} P_x = 1 - .0355 = .9645$$

Obviously the sum of all terms from P_0 to $P_{100,000}$ is equal to unity. It is interesting to note that although q is relatively much greater than p, nevertheless the values of P_x are very symetrically arranged about their mean For example, the first significant term of P_x is $P_{707} = .0001$, and the last significant term is $P_{896} = .0001$. Thus there are 93 significant terms above and 96 terms below P_{800}. However, there are 707 insignificant terms before P_{707} and 99,104 insignificant terms after P_{896}. We have arbitrarily rejected as insignificant any value less than .0001 Had we taken .0000001 as the limit of significance, we would have found that the limiting significant values of P_x are $P_{668} = P_{944} = .0000001$. Here again the significant ranges above and below the expected P_{800} are almost the same.

In general it may be said that unequal values of q and p when associated with large values of r are reflected in an unequal number of insignificant terms in the upper and lower ranges The significant terms form a distribution which, to the eye, is rather symmetrical.

35 Let us now retrace a few steps. Theoretically, formula (32) enables one to compute the probability that exactly x individuals out of any population of r will die within a year, provided, of course, q and p are known. Actually, however, such computation is very laborious. Thus, it is not easy to show that

$$P_{850} = {}_{100,000}C_{850}\,(.992)^{99150}(.008)^{850} = 0029354$$

TABLE XXIII

Values of P_x and $\sum_{x=0}^{x} P_x$ $P_x = {}_rC_x\, q^{r-x} p^x$ and $r = 100{,}000$,

$q = .999$, $p = .008$

x	P_x	$\sum_0^x P_x$	x	P_x	$\sum_0^x P_x$	x	P_x	$\sum_0^x P_x$
690	.0000	.0000	730	.0006	0062	770	.0081	1474
691	0000	0000	731	.0007	.0069	771	.0084	1558
692	.0000	C000	732	0007	.0077	772	0087	.1645
693	0000	.0001	733	.0008	0085	773	.0091	.1736
694	.0000	0001	734	.0009	0093	774	0094	.1830
695	.0000	.0001	735	0010	0103	775	.0097	.1926
696	.0000	.0001	736	.0010	0113	776	.0100	2026
697	.0000	.0001	737	.0011	.0125	777	.0103	.2128
698	.0000	.0001	738	0012	.0137	778	.0106	.2234
699	.0000	.0001	739	.0013	.0150	779	0108	.2342
700	.0000	0002	740	0014	.0165	780	.0111	.2454
701	.0000	.0002	741	.0016	.0180	781	.0114	.2568
702	.0000	.0002	742	.0017	.0197	782	.0117	.2684
703	0000	.0002	743	.0018	.0215	783	.0119	.2803
704	.0000	.0003	744	0019	.0234	784	.0122	.2925
705	.0000	.0003	745	.0021	.0255	785	.0124	.3049
706	.0000	.0004	746	.0022	.0278	786	.0126	.3175
707	.0001	.0004	747	.0024	0302	787	.0128	.3303
708	.0001	0005	748	.0026	.0327	788	.0130	.3433
709	.0001	0005	749	0027	.0355	789	.0132	.3565
710	.0001	.0006	750	.0029	.0384	790	.0134	.3699
711	.0001	.0007	751	.0031	.0415	791	0135	.3835
712	.0001	.0008	752	.0033	.0448	792	.0137	.3971
713	.0001	.0009	753	.0035	.0484	793	.0138	.4109
714	.0001	.0010	754	.0037	.0521	794	.0139	.4248
715	.0001	.0012	755	.0040	.0561	795	.0140	.4388
716	.0001	.0013	756	.0042	.0603	796	.0141	.4528
717	.0002	.0015	757	.0044	.0647	797	.0141	.4669
718	.0002	.0017	758	.0047	0694	798	.0141	.4811
719	.0002	.0019	759	.0049	.0744	799	0142	.4952
720	.0002	.0021	760	.0052	.0796	800	.0142	.5094
721	.0003	.0023	761	.0055	0850	801	.0141	.5235
722	.0003	.0026	762	.0058	0908	802	.0141	.5377
723	.0003	.0029	763	.0060	.0968	803	.0141	.5517
724	.0003	.0033	764	.0063	.1032	804	.0140	.5657
725	.0004	.0037	765	.0066	.1098	805	.0139	.5796
726	.0004	.0041	766	.0069	.1167	806	.0138	.5934
727	.0005	.0046	767	0072	.1239	807	.0137	.6070
728	.0005	.0051	768	.0075	.1314	808	.0135	.6206
729	.0006	.0056	769	.0078	.1392	809	.0134	.6340

TABLE XXIII (Continued)

x	P_x	$\sum_0^x P_x$	x	P_x	$\sum_0^x P_x$	x	P_x	$\sum_0^x P_x$
810	.0132	.6472	850	.0029	.9625	890	.0001	.9992
811	.0130	.6602	851	.0028	.9652	891	.0001	.9993
812	.0128	.6731	852	.0026	.9678	892	.0001	.9994
813	.0126	.6857	853	.0024	.9702	893	.0001	.9994
814	.0124	.6981	854	.0023	.9725	894	.0001	.9995
815	.0122	.7103	855	.0021	.9746	895	.0001	.9996
816	.0119	.7222	856	.0020	.9766	896	.0001	.9996
817	.0117	.7339	857	.0019	.9785	897	.0000	.9997
818	.0114	.7454	858	.0017	.9802	898	.0000	.9997
819	.0112	.7565	859	.0016	.9818	899	.0000	.9997
820	.0109	.7674	860	.0015	.9833	900	.0000	.9998
821	.0106	.7781	861	.0014	.9847	901	.0000	.9998
822	.0103	.7884	862	.0013	.9860	902	.0000	.9998
823	.0100	.7984	863	.0012	.9872	903	.0000	.9998
824	.0097	.8082	864	.0011	.9883	904	.0000	.9999
825	.0094	.8176	865	.0010	.9893	905	.0000	.9999
826	.0091	.8267	866	.0009	.9902	906	.0000	.9999
827	.0088	.8356	867	.0009	.9911	907	.0000	.9999
828	.0085	.8441	868	.0008	.9919	908	.0000	.9999
829	.0082	.8524	869	.0007	.9926	909	.0000	.9999
830	.0079	.8603	870	.0007	.9933	910	.0000	.9999
831	.0076	.8680	871	.0006	.9939	911	.0000	.9999
832	.0073	.8753	872	.0006	.9945	912		
833	.0071	.8824	873	.0005	.9950	913		
834	.0068	.8891	874	.0005	.9955	914		
835	.0065	.8956	875	.0004	.9959	915		
836	.0062	.9018	876	.0004	.9963	916		
837	.0059	.9077	877	.0004	.9967	917		
838	.0056	.9134	878	.0003	.9970	918		
839	.0054	.9188	879	.0003	.9973	919		
840	.0051	.9239	880	.0003	.9976	920		
841	.0049	.9288	881	.0002	.9978	921		
842	.0046	.9334	882	.0002	.9981	922		
843	.0044	.9378	883	.0002	.9983	923		
844	.0042	.9419	884	.0002	.9984	924		
845	.0039	.9459	885	.0002	.9986	925		
846	.0037	.9496	886	.0001	.9988	926		
847	.0035	.9531	887	.0001	.9989	927		
848	.0033	.9564	888	.0001	.9990	928		
849	.0031	.9595	889	.0001	.9991	929		

It can be done, provided an extensive table of logarithms are available, by using the so-called Stirling's formula

$$\underline{|n} \; \sqrt{2\pi} \;\; n^{n+1/2} \, e^{-n+1/12n \, - \, 1/360n^3 + \cdots}$$

where $\pi = 3.14159 \; 26535 \; 89793 \ldots$
$e = 2.71828 \; 18284 \; 59045$

We shall now proceed to develop a method which will enable us to find approximately the value of any term of the expansion of $(q+p)^r$ and the sum of any number of consecutive terms of this series.

In Section V we made use of the fact that the mean, standard deviation, and skewness may be regarded as satisfactorily describing any distribution. We shall now show that for any distribution whose frequencies are proportional to the terms of the expansion of $(q+p)^r$,

$$(33) \qquad \begin{aligned} M &= rp \\ \sigma &= \sqrt{rp(1-p)} \\ \alpha_3 &= \frac{1-2p}{\sigma} \end{aligned}$$

Thus, for the expected distribution of Table XXI, column (3), since $r = 12$, $p = 1/2$, $q = 1/2$,

$$\begin{aligned} M &= rp = \frac{12}{2} = 6 \\ \sigma &= \sqrt{12 \, \tfrac{1}{2} \, \tfrac{1}{2}} = \sqrt{3} = 1.732 \\ \alpha_3 &= 0 \end{aligned}$$

Similarly, for the expected distribution of Table XXII, column (3), since $r = 12$, $q = 5/6$, $p = 1/6$,

$$\begin{aligned} M &= \frac{12}{6} = 2 \\ \sigma &= \sqrt{12 \cdot \tfrac{1}{6} \, \tfrac{5}{6}} = \sqrt{\tfrac{5}{3}} = 1.291 \\ \alpha_3 &= \frac{1 - 1/3}{1.291} = .516 \end{aligned}$$

Values for these expected distributions may be calculated from the frequencies ${}_{4096}P_x$ in the usual manner. The results will then be found to agree with those obtained as above by means of formulae (33). Since the P_x column in Table XXI is composed of integers they will agree exactly, but since in Table XXII both the probabilities and expected frequencies are approximations, the values of these functions obtained by the two methods may differ slightly. Theoretically those obtained by employing formulae (33) are the more correct.

If, as before, q denote the probability that each individual will die within a year, and P_x the probability that exactly x out of r individuals will die within one year, then the values of P_0, P_1, P_2, . . . are equal to the terms of the expansion of $(q+p)^r$ which are shown in frequency distribution form in Table XXIV.

The total of column (2) is obviously equal to N since the values of f_x are merely the expansion of $N(q+p)^r$. Since $q+p = 1$, therefore $(q+p)^r = 1$, and hence $\Sigma f_x = N$.

If one takes the common factor Nrp out of every term in column (3) of the previous table, it is noted that the sum of this column may be written

$$N\Sigma x f_x = Nrp\left[q^{r-1} + (r-1)q^{r-2}p + \frac{(r-1)(r-2)}{1\ 2}q^{r-3}p^2 + \cdots\right]$$

But the expression within the bracket is merely the expansion of the binomial $(q+p)^{r-1}$. Hence $\Sigma x f_x = Nrp\,[1] = Nrp$ Likewise the sum of the terms in columns (4) and (5) may be factored as follows:

$$\Sigma x(x-1)f_x = Nr(r-1)p^2\left[q^{r-2} + (r-2)q^{r-3}p + \frac{(r-2)(r-3)}{1\ 2}q^{r-4}p^2 + \cdots\right]$$

$$= Nr(r-1)p^2(q+p)^{r-2} = Nr(r-1)p^2$$

$$\Sigma x(x-1)(x-2)f_x = Nr(r-1)(r-2)p^3\left[q^{r-3} + (r-3)q^{r-4}p + \frac{(r-3)(r-4)}{1\cdot 2}q^{r-5}p^2 + \cdot\right]$$

$$= Nr(r-1)(r-2)p^3(q+p)^{r-3} = Nr(r-1)(r-2)p^3$$

TABLE XXIV

x (1)	f_x (2)	$x \cdot f_x$ (3)	$x(x-1)f_x$ (4)	$x(x-1)(x-2)f_x$ (5)
0	Nq^r	0	0	0
1	$Nrq^{r-1}p$	$Nrq^{r-1}p$	0	0
2	$N\frac{r(r-1)}{1\cdot 2}q^{r-2}p^2$	$N\frac{r(r-1)}{1}q^{r-2}p^2$	$Nr(r-1)q^{r-2}p^2$	0
3	$N\frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}q^{r-3}p^3$	$N\frac{r(r-1)(r-2)}{1\cdot 2}q^{r-3}p^3$	$N\frac{r(r-1)(r-2)}{1}q^{r-3}p^3$	$Nr(r-1)(r-2)q^{r-3}p^3$
4	$N\frac{r(r-1)(r-2)(r-3)}{1\cdot 2\cdot 3\cdot 4}q^{r-4}p^4$	$N\frac{r(r-1)(r-2)(r-3)}{1\cdot 2\cdot 3}q^{r-4}p^4$	$N\frac{r(r-1)(r-2)(r-3)}{1\cdot 2}q^{r-4}p^4$	$N\frac{r(r-1)(r-2)(r-3)}{1}q^{r-4}p^4$
	etc.	etc.	etc.	etc.
Total	$N(q+p)^r = N$	$Nrp(q+p)^{r-1} = Nrp$	$Nr(r-1)p^2(q+p)^{r-2} =$ $Nr(r-1)p^2$	$Nr(r-1)(r-2)p^3(q+p)^{r-3} =$ $Nr(r-1)(r-2)p^3$

But we may write

$$x(x-1)f_x = x^2 f_x - x f_x\,.$$

$$\therefore\ \Sigma x(x-1)f_x = \Sigma x^2 f_x - \Sigma x f_x$$

$$x(x-1)(x-2)f_x = x^3 f_x - 3x^2 f_x + 2x f_x\,,$$

$$\therefore\ \Sigma x(x-1)(x-2)f_x = \Sigma x^3 f_x - 3\Sigma x^2 f_x + 2\Sigma x f_x$$

So we have

$$\Sigma f_x = N$$

$$\Sigma x f_x = Nrp$$

$$\Sigma x(x-1)f_x = \Sigma x^2 f_x - \Sigma x f_x = Nr(r-1)p^2$$

$$\Sigma x(x-1)(x-2)f_x = \Sigma x^3 f_x - 3\Sigma x^2 f_x + 2\Sigma x f_x$$

$$= Nr(r-1)(r-2)p^3$$

Therefore

$$\Sigma x^2 f_x = \Sigma x f_x + Nr(r-1)p^2 = Nrp + Nr(r-1)p^2.$$

$$= Nrp + Nr^2p^2 - Nrp^2$$

$$\Sigma x^3 f_x = 3\Sigma x^2 f_x - 2\Sigma x f_x + Nr(r-1)(r-2)p^3$$

$$= 3N(rp + r^2p^2 - rp^2) - 2Nrp + Nr(r-1)(r-2)p^3$$

$$= Nrp + 3Nr^2p^2 - 3Nrp^2 + Nr^3p^3 - 3Nr^2p^3 + 2Nrp^3$$

Hence

$$M_x = \frac{\Sigma x f_x}{\Sigma f_x} = \frac{Nrp}{N} = rp$$

$$\mu_2' = \frac{\Sigma x^2 f_x}{\Sigma f_x} = rp + r^2p^2 - rp^2$$

$$\mu_3' = \frac{\Sigma x^3 f_x}{\Sigma f_x} = rp + 3r^2p^2 - 3rp^2 + r^3p^3 - 3r^2p^3 + 2rp^3$$

$$\mu_2 = \mu_2' - M_x^2 = rp - rp^2 = rp(1-p)$$

$$\mu_3 = \mu_3' - 3M_x\mu_2' + 2M_x^3 = rp - 3rp^2 + 2rp^3$$

$$= rp(1-3p+2p^2) = rp(1-p)(1-2p)$$

The reductions follow since $(q+p) = 1$.

We have finally, that

$$M = rp$$

$$\sigma = \sqrt{\mu_2} = \sqrt{rp(1-p)}$$

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = \frac{rp(1-p)(1-2p)}{\left(\sqrt{rp(1-p)}\right)^3} = \frac{1-2p}{\sigma}$$

Formulae (33) are therefore established.

The equation $M = rp$ shows that for a Bernoulli series the "mean" value is also the "expected" value, since, from our definition of expectation, the expected number of deaths from a group of r individuals is rp.

36. For the distribution of the values of P_x shown in Table XXIII, since $r = 100{,}000$, $q = .992$, $p = .008$

$$M_x = rp = 800$$

$$\sigma = \sqrt{rp(1-p)} = \sqrt{793.6} = 28.1709$$

$$\frac{1}{\sigma} = .0354976$$

$$\alpha_3 = \frac{1-2p}{\sigma} = \frac{.992-.008}{\sigma} = .984(.0354976) = .03493$$

If as before we let $t = \frac{x-M}{\sigma}$, and designate the ordinates of the standard frequency curves by f_t, we can compute any value of P_x with a reasonable degree of accuracy by the formula

$$(34) \qquad P_x = \frac{1}{\sigma} f_t$$

For example: Required the probability that exactly 762 individuals will die within one year in a population of 100,000 for which $p = .008$.

As before, we must first express the number of deaths under consideration in standard units, that is since $M_x = rp = 800$,

$$t = \frac{x-M}{\sigma} = \frac{762-800}{\sigma} = -38(.0354976) = -1.3489$$

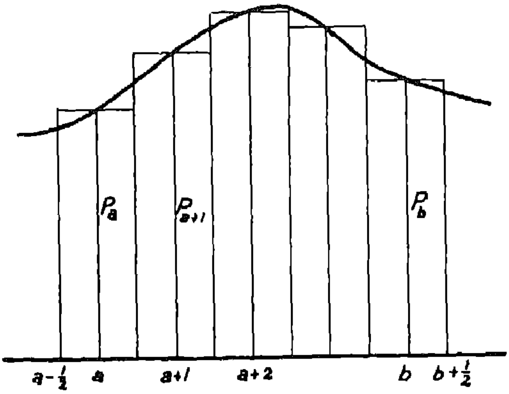
That is, 762 deaths is 38 less than the mean, or $\frac{-38}{\sigma} = -1.3489$ standard units less than the mean.

With $\alpha_3 = 0$ and using the Table of Ordinates of the Pearson Type III Curve, the value of f_t corresponding to $t = -1.35$ is found to be .160383.

$$f_t = .160383$$

$$\therefore \quad P_{762} = \frac{1}{\sigma} f_t = .0354976(.160383) = .005693$$

We shall now consider the following problem: Required the probability that not more than 780 individuals will die within one year, where as before $r = 100{,}000$, $p = .008$. This means that we must obtain the sum of the 781 terms. $P_0 + P_1 + P_2 + \dots + P_{780}$



Suppose we represent the sum of the probabilities $P_a + P_{a+1} + P_{a+2} + \dots + P_b$ by a series of ordinates erected at unit intervals along the x axis, and then construct a series of rectangles having these ordinates as altitudes which bisect the bases of the respective unit bases. Then the area of the first rectangle is $P_a = 1 = P_a$, etc. Thus the sum of the series $P_a + P_{a+1} + \dots + P_b$ is equal to the total area of all the rectangles and is therefore approximately equal to the area under the frequency curve from $a - \frac{1}{2}$ to $b + \frac{1}{2}$. Therefore the sum of all the probabilities, $P_0 + P_1 + \dots + P_{780}$ can be computed readily by calculating by means of the Tables of Areas of the Standard Curves, the per cent of the area of the standard curve lying below $x = 780.5$, that is below $\bar{x} = -19.5$, or $t = \frac{-19.5}{\sigma} = -.6922$. For $\alpha_3 = 0$, the per cent of the area of the frequency curve lying below $t = -.69$ is 24.5097. Since the sum of all probabilities from P_0 to $P_{100,000}$ inclusive is 1, and $P_0 + P_1 + \dots + P_{780}$ represents approximately 24.5097 per cent of the total area under the frequency curve, therefore we estimate that $P_0 + P_1 + P_2 + \dots + P_{780} = 245097$.

By Table XXIII the correct value is .2454, or the error of our approximation is .0003. Using the values $\alpha_3 = 0$, the per cent of the area lying below $t = -.70$ is found to be 24.1964, using straight line interpolation the per cent below $t = -.6922$ is found to be 24.4408. In the same manner, only using $\alpha_3 = .1$, the per cent of the curve lying below $t = -.6922$ is found to be 24.7105. By using straight line interpolation again for the value of α_3, it is found that the per cent of the distribution lying below $t = -.6922$, skewness $= .035$, is 24.5352. The error of our approximation is now zero. In general, however, a sufficient degree of accuracy may be obtained without interpolating for either the value of t or α_3.

Next let it be required to find the probability that less than 840 but more than 780 will die within the year, that is, required the value of $P_{781}+P_{782}+\ldots P_{839}$.

We require therefore the per cent of the area of a standard frequency curve lying between $x = 780.5$ and $x = 839.5$, that is between $\bar{x} = -19.5$ and $\bar{x} = 39.5$ or $t = -.69$ to $t = 1.40$.

As has just been shown, 24.5097 per cent of the area of the curve lies below $t = -.69$. Likewise for $\alpha_3 = 0$ the per cent lying below $t = 1.40$ is 91.9243. Consequently 91.9243% - 24 5097%, or 67.4146%, of the area lies between $t = -.69$ and $t = 1.40$. Therefore the probability that less than 840 but more than 780 will die within the year is .674146.

By Table XXIII, the correct value is .9188 - .2454 = 6734.

Summary of Section VI.

If p represent the probability that an event will happen in a single trial, then the probability that the event will happen either 0, 1, 2, ... times during r trials are given by the respective terms of the expansionn of $(q+p)^r$. The distribution of these probabilities or the corresponding expected frequencies is adequately described by the three fundamental functions as follows:

$$M = rp$$

$$\sigma = \sqrt{rp(1-p)}$$

$$\alpha_3 = \frac{1-2p}{\sigma}$$

The probabilities or expected frequencies may be regarded as a distribution that can be reproduced at will by utilizing the Tables of Pearson's Type III Curves, with the fundamental functions computed from the above formulae. In this way the values of isolated probabilities or the sum of any number of consecutive probabilities may be obtained.

EDITORIAL

FUNDAMENTALS OF THE THEORY OF SAMPLING

III. DISTRIBUTION OF SAMPLE m TH MOMENTS ABOUT THE ORIGIN OF THE PARENT POPULATION

As in section I, we shall be concerned with the $\binom{s}{r}$ possible samples, each consisting of r variates, that can be selected from the parent population of s variates $x_1, x_2, \ldots x_r, \ldots x_s$. The m th moment of each sample, computed in each case about the origin of the parent population, may be written

$$\begin{cases} z_1 = \frac{1}{r}\left\{x_1^m + x_2^m + x_3^m + \cdots + x_r^m\right\} \\ z_2 = \frac{1}{r}\left\{x_2^m + x_3^m + x_4^m + \cdots + x_{r+1}^m\right\} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ z_{\binom{s}{r}} = \frac{1}{r}\left\{x_{s-r+1}^m + x_{s-r+2}^m + x_{s-r+3}^m + \cdots + x_s^m\right\} \end{cases}$$

If we write $\frac{x_i^m}{r} = y_i$, it will be observed that the above distribution may be written

$$\begin{cases} z_1 = y_1 + y_2 + y_3 + \cdots + y_r \\ z_2 = y_2 + y_3 + y_4 + \cdots + y_{r+1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ z_{\binom{s}{r}} = y_{s-r+1} + y_{s-r+2} + y_{s-r+3} + \cdots + y_s \end{cases}$$

and therefore may be regarded as a distribution of the algebraic sums of the respective samples withdrawn from the parent population $y_1, y_2, \ldots y_s$, i. e. $\frac{x_1^m}{r}, \frac{x_2^m}{r}, \ldots \frac{x_s^m}{r}$. Consequently, since

$$\mu'_{n\,y} = \frac{\Sigma y_i^n}{N} = \frac{1}{N}\Sigma\,\frac{x_i^{mn}}{r^n} = \frac{1}{r^n}\mu'_{mn\,x}$$

it follows from formulae 1, 2, . . . of section I that

$$(1)\quad M_x = r\,M_y = \mu'_{m\,x}$$

$$(2)\quad \mu_{2:x} = s\{\rho_1-\rho_2\}\mu_{2:y} = s\{\rho_1-\rho_2\}\{\mu'_{2:y}-M_y^2\}$$

$$= s\{\rho_1-\rho_2\}\left\{\frac{\mu'_{2m:x}}{r^2} - \left(\frac{\mu'_{m\,x}}{r}\right)^2\right\}$$

$$= \frac{s}{r^2}\{\rho_1-\rho_2\}\{\mu'_{2m\,x} - (\mu'_{m:x})^2\}$$

$$(3)\quad \mu_{3:x} = s\{\rho_1-3\rho_2+2\rho_3\}\mu_{3:y} = s\{\rho_1-3\rho_2+2\rho_3\}\{\mu'_{3:y}-3M_y\mu'_{2:y}+2M_y^3\}$$

$$= \frac{s}{r^3}\{\rho_1-3\rho_2+2\rho_3\}\{\mu'_{3m\,x} - 3\mu'_{2m:x}\mu'_{m:x} + 2(\mu'_{m.x})^3\}$$

$$(4)\quad \mu_{4:x} = \frac{s}{r^4}\{\mu'_{4m:x} - 4\mu'_{3m\,x}\mu'_{m:x} + 6\mu'_{2m\,x}(\mu'_{m\,x})^2 - 3(\mu'_{m:x})^4\}$$

$$\{\rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4\} + 3\frac{s^2}{r^4}\{\mu'_{2m:x} - (\mu'_{m:x})^2\}^2\{\rho_2 - 2\rho_3 + \rho_4\}$$

etc.

For the case of sampling from an unlimited supply, we have, permitting s to approach infinity, that corresponding to formulae (18) of section I

$$(5)\begin{cases} M_{\bar{x}} = \mu'_{m.x} \\ \mu_{2,\bar{x}} = \frac{1}{r}\left\{\mu'_{2m,x} - (\mu'_{m,x})^2\right\} \\ \mu_{3,\bar{x}} = \frac{1}{r^2}\left\{\mu'_{3m,x} - 3\mu'_{2m,x}\mu_{m,x} + 2(\mu'_{m,x})^3\right\} \\ \mu_{4,\bar{x}} = \frac{1}{r^3}\left\{\mu'_{4m,x} - 4\mu'_{3m,x}\mu'_{m,x} + 6\mu'_{2m,x}(\mu'_{m,x})^2 - 3(\mu'_{m,x})^4\right\} \\ \qquad + \frac{3r^{(2)}}{r^4}\left\{\mu'_{2m,x} - (\mu'_{m,x})^2 .\right\} \end{cases}$$

etc

The distribution of sample means may be obtained by placing $m = 1$, yielding

$$(6)\begin{cases} M_{\bar{x}} = M_x \\ \mu_{2,\bar{x}} = \frac{1}{r}\mu_{2,x} \\ \mu_{3,\bar{x}} = \frac{1}{r^2}\mu_{3,x} \\ \mu_{4,\bar{x}} = \frac{1}{r^3}\mu_{4,x} + \frac{3(r-1)}{r^3}\mu^2_{2,x} \end{cases}$$

etc.

These results may be written corresponding to formulae (19) of section I,

$$(7)\begin{cases} \mu_{2,\bar{x}} = \frac{1}{r}\mu_{2,x} \\ \mu_{3,\bar{x}} = \frac{1}{r^2}\mu_{3,x} \\ \mu_{4,\bar{x}} - 3\mu^2_{2,\bar{x}} = \frac{1}{r^3}\left\{\mu_{4,x} - 3\mu^2_{2,x}\right\} \\ \mu_{5,\bar{x}} - 10\mu_{3,\bar{x}}\mu_{2,\bar{x}} = \frac{1}{r^4}\left\{\mu_{5,x} - 10\mu_{3,x}\mu_{2,x}\right\} \\ \mu_{6,\bar{x}} - 15\mu_{4,\bar{x}}\mu_{2,\bar{x}} - 10\mu^2_{3,\bar{x}} + 30\mu^3_{2,\bar{x}} = \frac{1}{r^5}\left\{\mu_{6,x} - 15\mu_{4,x}\mu_{2,x} - 10\mu^2_{3,x} + 30\mu^3_{2,x}\right\} \end{cases}$$

etc.

The distribution of sample means withdrawn from an infinite parent population is therefore characterized by means of the semi-invariant relation

$$(8) \qquad \lambda_{n\bar{x}} = \frac{\lambda_{nx}}{r^{n-1}}$$

and the standard semi-invariants by the relation

$$(8\text{-a}) \qquad \gamma_{n\bar{x}} = \frac{\gamma_{nx}}{r^{n/2-1}}$$

An interesting result is obtained by considering the special case of formulae (8) for which $n = 2$, and assuming that the parent population is normal. Since for a normal distribution

$$\mu_{2n+1} = 0$$
$$\mu_{2n} = 1 . 3 . 5 \quad . \; . \quad (2n-1)\sigma^{2n}$$

and for any distribution

$$(9) \left\{ \begin{aligned} \mu_2' &= \mu_2 + M^2 \\ \mu_3' &= \mu_3 + 3M\mu_2 + M^3 \\ \mu_4' &= \mu_4 + 4M\mu_3 + 6M^2\mu_2 + M^4 \end{aligned} \right.$$

etc.

it follows that for a normal distribution

$$(10) \left\{ \begin{aligned} \mu_2' &= \sigma^2 + M^2 \\ \mu_3' &= 3M\sigma^2 + M^3 \\ \mu_4' &= 3\sigma^4 + 6\sigma^2 M^2 + M^4 \\ \mu_5' &= 15M\sigma^4 + 10M^3\sigma^2 + M^5 \end{aligned} \right.$$

etc.

and therefore for the distribution of sample second moments about a fixed point in the case of withdrawals from an unlimited "normal" supply, we have, from (5)

---

1 See formulae 23 and 24, page 117, Vol. I, No. 1, of ANNALS.

$$
(11)\begin{cases}
M_{\bar{x}} = \mu'_{2\cdot x} = \sigma_x^2 + M_x^2 \\[2ex]
\mu_{2\cdot\bar{x}} = \frac{1}{r}\left\{ \mu'_{4\cdot x} - (\mu'_{2\cdot x})^2 \right\} \\[2ex]
\qquad = \frac{2\sigma^2}{r}\left\{ \sigma_x^2 + 2M_x^2 \right\} \\[2ex]
\mu_{3\cdot\bar{x}} = \frac{1}{r^2}\left\{ \mu_{6\cdot x} - 3\mu'_{4\cdot x}\mu'_{2\cdot x} + 2(\mu'_{2\cdot x})^3 \right\} \\[2ex]
\qquad = \frac{8\sigma_x^4}{r^2}\left\{ \sigma_x^2 + 3M_x^2 \right\} \\[2ex]
\mu_{4\cdot\bar{x}} = \frac{48\sigma_x^6}{r^3}\left\{ \sigma_x^2 + 4M_x^2 \right\} + \frac{12\sigma_x^4}{r^2}\left\{ \sigma_x^2 + 2M^2 \right\}^2 \\[2ex]
\mu_{5\cdot\bar{x}} = \frac{384\sigma_x^8}{r^4}\left\{ \sigma_x^2 + 5M_x^2 \right\} + \frac{160\sigma_x^6}{r^3}\left\{ \sigma_x^2 + 2M_x^2 \right\}\left\{ \sigma_x^2 + 3M_x^2 \right\} \\[2ex]
\mu_{6\cdot\bar{x}} = \frac{3840\sigma_x^{10}}{r^5}\left\{ \sigma_x^2 + 6M_x^2 \right\} + \frac{160\sigma_x^8}{r^4}\left\{ 13\sigma_x^4 + 78\sigma_x^2 M_x^2 + 108M_x^4 \right\} \\[2ex]
\qquad + \frac{120\sigma_x^6}{r^3}\left\{ \sigma_x^2 + 2M_x^2 \right\}^3
\end{cases}
$$

In terms of semi-invariants[1]

$$
(12)\begin{cases}
M_{\bar{x}} = \sigma_x^2 + M_x^2 \\[2ex]
\lambda_{2\cdot\bar{x}} = \frac{2\sigma_x^2}{r}\left(\sigma_x^2 + 2M_x^2 \right) \\[2ex]
\lambda_{3\cdot\bar{x}} = \frac{2^2\cdot 2!\,\sigma_x^4}{r^2}\left(\sigma_x^2 + 3M_x^2 \right) \\[2ex]
\lambda_{4\cdot\bar{x}} = \frac{2^3\cdot 3!\,\sigma_x^6}{r^3}\left(\sigma_x^2 + 4M_x^2 \right) \\[2ex]
\lambda_{5\cdot\bar{x}} = \frac{2^4\cdot 4!\,\sigma_x^8}{r^4}\left(\sigma_x^2 + 5M_x^2 \right) \\[2ex]
\lambda_{6\cdot\bar{x}} = \frac{2^5\cdot 5!\,\sigma_x^{10}}{r^5}\left(\sigma_x^2 + 6M_x^2 \right)
\end{cases}
$$

---

[1] Formulae (21), Section I. Page 116, Vol. I, No 1, of ANNALS

Apparently the general expression is

$$(13) \qquad \lambda_{n\,x} = \frac{2^{n-1}(n-1)!\,\sigma_x^{2n}}{r^{n-1}}\left\{1 + n\left(\frac{M_x}{\sigma_x}\right)^2\right\}$$

If the parent population be normal, and if furthermore $M_x = 0$, then

$$(14) \qquad \lambda_{n\cdot x} = \frac{2^{n-1}(n-1)!\,\sigma_x^{2n}}{r^{n-1}}$$

and the standardized semi-invariants would likewise be

$$(15) \qquad \gamma_{n\,x} = \frac{\lambda_{n\,x}}{(\lambda_{2\,x})^{n/2}} = \left(\frac{2}{r}\right)^{\frac{n}{2}-1} \cdot (n-1)!$$

Again, since

$$\gamma_{3\,x} = \alpha_{3\,x} = \frac{2^{3/2}}{r^{1/2}}$$

formula (15) may be written,

$$(16) \qquad \gamma_{n\,x} = \left(\frac{\alpha_3}{2}\right)^{n-2} \cdot (n-1)!$$

On page 196 of Vol I, No 2 of the ANNALS it was shown that the standard moments for Pearson's Type III function

$$y = y_0 \left(1 + \frac{\alpha_3}{2}\,t\right)^{\frac{4}{\alpha_3^2}-1} e^{-\frac{2}{\alpha_3}t}$$

are defined by the recurring relation

$$\alpha_{n+1} = n\left(\alpha_{n-1} + \frac{\alpha_3\,\alpha_n}{2}\right),$$

so that $\alpha_4 = 3\left(1+\frac{\alpha_3^2}{2}\right)$

$$\alpha_5 = 2\alpha_3\left(5+3\,\frac{\alpha_3^2}{2}\right)$$

$$\alpha_6 = 5\left(3+13\,\frac{\alpha_3^2}{2}+6\,\frac{\alpha_3^4}{4}\right)$$

etc.

The standard semi-invariants of Type III are

$$\gamma_4 = \left(\frac{\alpha_3}{2}\right)^2 3!$$

$$\gamma_5 = \left(\frac{\alpha_3}{2}\right)^3 \cdot 4!$$

$$\gamma_6 = \left(\frac{\alpha_3}{2}\right)^4 \cdot 5!$$

etc.

Comparing these results with formula (16) it appears, therefore, that if the parent population be normal *and its mean zero*, the distribution of sample second moments computed about the fixed mean of the parent population will be Pearson't Type III, for r finite. As r approaches infinity, the Type III distribution will approach the Normal Curve as a limit

To illustrate: If from an infinite population of spherical balls whose diameters formed a normal distribution characterized by M_x and σ_x, samples of r balls each were withdrawn, then if the average area be determined for the balls in each sample, the distribution of these areas, from formula (13), would be described by the relation

$$\lambda_{n,x} = \frac{2^{n-1}(n-1)!\,\sigma_x^{2n}}{r^{n-1}}\left\{1+n\left(\frac{M_x}{\sigma_x}\right)^2\right\}$$

and if one could conceive of negative diameters of the balls so that $M_x = 0$, then the distribution of areas would be Type III.

If one were to succeed in finding the function whose n th semi-invariant agrees with the above expression, then the law of distribution for the sample areas would be available. Again by likewise investigating the cases of formulae (5) where $m = 3, 4, 5$, etc., other semi-invariant relations can be found, and these in turn may lead to the discovery of new and important frequency functions. At all events, such sample moments and semi-invariants will generally permit one to express as an infinite series, such as the Gram-Charlier series, the unknown law of distribution

SECTION IV

The problem of the distribution of sample moments about the origin of the parent population[1] is unfortunately often confused with the problem of the distribution of sample moments computed about the means of the respective samples The latter problem is more briefly termed sampling *about the mean* If M_1 and M_2 designate the means of the first two samples respectively, and z_1 and z_2 the second moments of these two samples computed about M_1 and M_2 respectively, then for $m = 2$

$$z_1 = \frac{\sum^{r,1} (x - M_1)^2}{r}$$

$$z_2 = \frac{\sum^{r,2} (x - M_2)^2}{r}$$

where, as before, $\sum^{r,i}$ indicates that the summation extends over the r variates occurring in the i th sample

In order to sum all values of z_i and z_i^n it is necessary to obtain first another expression for the second moment about the mean, which, although of value in algebraic manipulations, is practically of no value in arithmetic computation Thus,

$$z_i = \frac{\sum^{r,i} (x - M_i)^2}{r}$$

$$= \frac{\sum^{r,i} x^2 - 2M_i \sum^{r,i} x + rM_i^2}{r}$$

---

[1] Also referred to as the distribution of sample moments *about a fixed point*

$$= \frac{\overset{r_i}{\Sigma} x^2 - \frac{\left(\overset{r_i}{\Sigma} x\right)^2}{r}}{r}$$

$$= \frac{r\overset{r_i}{\Sigma} x^2 - \left[\overset{r_i}{\Sigma} x^2 + 2\overset{r_i}{\Sigma} x_i x_j\right]}{r^2}$$

$$(17) \qquad z_i = \frac{1}{r^2}\left[(r-1)\overset{r_i}{\Sigma} x^2 - 2\overset{r_i}{\Sigma} x_i x_j\right],$$

where $\overset{r_i}{\Sigma} x_i x_j$ designnates the sum of all the terms formed by taking the products of all the variates in the i th sample two at a time.

Then $$M_z = \frac{\Sigma z_i}{\binom{s}{r}} = \frac{1}{r^2}\left[(r-1)\frac{r}{s}\overset{s}{\Sigma} x^2 - 2\frac{r^{(2)}}{s^{(2)}}\overset{s}{\Sigma} x_i x_j\right]$$

by employing the method employed in section I. The above reduces easily as follows:

$$(18) \qquad \begin{aligned} M_z &= \frac{1}{r^2}\left[(r-1)\frac{r}{s}\overset{s}{\Sigma} x^2 - \frac{r^{(2)}}{s^{(2)}}\left\{\left(\overset{s}{\Sigma} x\right)^2 - \Sigma x^2\right\}\right] \\ &= \frac{s(r-1)}{r(s-1)} \cdot \left\{\frac{\overset{s}{\Sigma} x^2}{s} - \left(\frac{\overset{s}{\Sigma} x}{s}\right)^2\right\} \\ &= \frac{s(r-1)}{r(s-1)} \mu_{2:x} \end{aligned}$$

Whereas the expected value of a sample mean is equal to the mean of the parent population and the expected value of a sample n th moment about a fixed point is equal to the n th moment of the parent population[1], it appears that the expected value of a sample second moment is less than the second moment of the parent population.

A slight digression at this point is desirable. In formula (6) of Section III we found that for the distribution of sample means withdrawn from an infinite parent population,

$$\mu_{2.\bar{x}} = \frac{1}{r}\mu_{2:x}$$

That is, the *standard error of the mean*

---

[1] Formula (1), Section III.

$$(19) \qquad \sigma_M = \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{r}},$$

where σ_x denotes the standard deviation of the infinite parent population By formula (18) above it appears that the expected value of the second sample moment is for $s = \infty$

$$M_{\bar{x}} = \frac{r-1}{r}\mu_{2,x} = \frac{r-1}{r}\sigma_x^2$$

Designating the square root of the expected sample moment by σ'_x, we have that

$$\sigma'_x = \sigma_x\sqrt{\frac{r-1}{r}}, \text{ or } \sigma_x = \sigma'_x\sqrt{\frac{r}{r-1}}$$

and therefore formula (19) may be written

$$(20) \qquad \sigma_M = \frac{\sigma'_x}{\sqrt{r-1}}$$

Since the probable error is defined as 6745 σ, we have that the *probable error of the mean*

$$(21) \qquad P E_M = .6745\frac{\sigma_x}{\sqrt{r}} = 6745\frac{\sigma'_x}{\sqrt{r-1}}$$

It should be observed that the expressions for both the standard and probable errors of the mean are expected values when σ' is employed. If one obtains but a single sample and computes its mean and standard deviation, he still has no accurate knowledge regarding the true value of the standard deviation of the parent population. Consequently even the expression

$$P E._M = .6745\frac{\sigma'_x}{\sqrt{r-1}}$$

is merely an approximation. So far as I know, the true value of the probable error of the mean has never been found—even upon the assumption that the parent population is normal. Since we have shown that for $s = \infty$ the skewness of the samples is only $\frac{1}{\sqrt{r}}$ times the skewness of the parent population, the fact that the parent population is not normal is of no importance compared to the fact that where only functions of the single sample are available, *these* must be substituted as the expected values of the corresponding functions of the unknown parent population

Returning to our problem of describing further the distribution of sample second moments about the mean:

Corresponding to formula (17), one can show by employing symmetric functions that

$$
(22)\qquad \begin{aligned} z_i^2 = \frac{1}{r^4}\Big\{ & (r-1)^2 \overset{r}{\Sigma}{}' x^4 - 4(r-1) \overset{r}{\Sigma}{}' x_i^3 x_j + 2(r^2 - 2r+3) \overset{r}{\Sigma}{}' x_i^2 x_j^2 \\ & - 4(r-3) \overset{r}{\Sigma}{}' x_i^2 x_j x_k + 24 \overset{r}{\Sigma}{}' x_i x_j x_k x_l \Big\} \end{aligned}
$$

and therefore

$$
(23)\qquad \begin{aligned} \mu_{2:x} &= \frac{\Sigma z_i^2}{\binom{s}{r}} - M_x^2 \\ &= \frac{s(r-1)(s-r)}{r^3(s-1)^2(s-2)(s-3)} \cdot \Big\{ (s-1)(rs-s-r-1)\,\mu_{4:x} \\ &\qquad + \left[(3-r)^2 s - 6s + 3r + 3\right] \mu_{2:x}^2 \Big\} \end{aligned}
$$

For $s = \infty$ this becomes

$$
(24)\qquad \begin{aligned} \mu_{2:x} &= \frac{r-1}{r^3}\left[(r-1)\,\mu_{4:x} - (r-3)\,\mu_{2:x}^2\right] \\ &= \frac{(r-1)\sigma^4}{r^3}\left[(r-1)\,\alpha_{4:x} - (r-3)\right] \end{aligned}
$$

In a thesis, C. H. Richardson[1] has shown that when $s = \infty$

---

[1] Submitted in 1927 to the University of Michigan. The balance of this section is a synopsis of one part of this thesis.

$$(25)\quad \mu_{3\cdot x}=\frac{(r-1)\sigma^{6}}{r^{6}}\Big[(r-1)^{2}\alpha_{6\,x}-3(r-1)(r-5)\alpha_{4\,x}$$

$$-2\,(3r^{2}-6r+5)\alpha_{3}^{2}+2\,(r^{2}-12r+15)\Big]$$

$$(26)\quad \mu_{4\cdot x}=\frac{(r-1)\sigma^{8}}{r^{7}}\Big[(r-1)^{3}\alpha_{8\cdot x}-8(r-1)(3r^{2}-6r+7)\,\alpha_{5\cdot x}\,\alpha_{3\,x}$$

$$+(3r^{4}-12r^{3}+42r^{2}-60r+35)\alpha_{4\cdot x}^{2}-4(r-1)^{2}(r-7)\,\alpha_{6\,x}$$

$$-\,6\,(r^{4}-7r^{3}+49r^{2}-105r+70)\,\alpha_{4\,x}$$

$$+16\,(6r^{3}-27r^{2}+50r-35)\,\alpha_{3\,x}^{2}$$

$$+\,3\,(r^{4}-9r^{3}+93r^{2}-255r+210)\Big]$$

$$(27)\quad \mu_{3.x}=\frac{(r-1)\sigma^{10}}{r^{9}}\Big[(r-1)^{4}\alpha_{10\,x}-5(r-1)^{3}(r-9)\,\alpha_{8.x}$$

$$-\,40(r-1)^{2}(r^{2}-2r+3)\,\alpha_{7\,x}\,\alpha_{3\,x}$$

$$+\,10(r-1)(r^{4}-4r^{3}+18r^{2}-28r+21)\,\alpha_{6\,x}\,\alpha_{4\,x}$$

$$-\,10(3r^{5}-27r^{4}+162r^{3}-450r^{2}+595r-315)\alpha_{4\,x}^{2}$$

$$-\,20\,(r-2)(3r^{4}-24r^{3}+80r^{2}-140r+105)\,\alpha_{4\,x}\,\alpha_{3:x}^{2}$$

$$+\,10\,(5r^{5}-64r^{4}+572r^{3}-2070r^{2}+3255r-1890)\,\alpha_{3.x}^{2}$$

$$-\,4\,(5r^{5}-86r^{4}+1050r^{3}-4620r^{2}+8505r-5670)$$

$$-\,10(r-1)\,(r^{4}-7r^{3}+65r^{2}-161r+126)\,\alpha_{6\,x}$$

$$-\,2\,(15r^{4}-60r^{3}+130r^{2}-140r+63)\,\alpha_{5\,x}^{2}$$

$$-80\,(6r^{4}-36r^{3}+97r^{2}-126r+63)\,\alpha_{5\,x}\,\alpha_{3\cdot x}\Big]$$

$$(28)\ \mu_{6,\bar{x}} = \frac{(r-1)\,\sigma^{12}}{r^{11}}\Big[(r-1)^5\,\alpha_{12,x} - 6(r^5-15r^4+50r^3-70r^2+45r-11)\,\alpha_{10,x}$$

$$-\,20(3r^5-15r^4+38r^3-54r^2+39r-11)\,\alpha_{9,x}\,\alpha_{3\,x}$$

$$+\,15(r^6-6r^5+31r^4-84r^3+127r^2-102r+33)\,\alpha_{6,x}\,\alpha_{4\,x}$$

$$-\,15(r^6-9r^5+96r^4-394r^3+729r^2-621r+198)\,\alpha_{8\,x}$$

$$+\,2(5r^6-30r^5+165r^4-460r^3+735r^2-630r+231)\,\alpha^2_{6\,x}$$

$$-\,120(r^6-11r^5+81r^4-294r^3+567r^2-567r+231)\,\alpha_{6,x}\,\alpha_{4,x}$$

$$+\,15(r^7-8r^6+51r^5-258r^4+815r^3-1540r^2+1645r-770)\,\alpha^3_{4,x}$$

$$+\,20(5r^6-75r^5+828r^4-3938r^3+9009r^2-9891r+4158)\,\alpha_{6\,x}$$

$$+\,5(9r^7-159r^6+2436r^5-26130r^4+135885r^3-35941r^2$$

$$+\,474390r-24980)\,\alpha_{4,x}$$

$$-\,5(9r^7-126r^6+1413r^5-11214r^4+47355r^3-107730r^2$$

$$+\,127575r-62370)\,\alpha^2_{4,x}$$

$$-\,24(5r^5-25r^4+70r^3-110r^2-93r-33)\,\alpha_{7\,x}\,\alpha_{5,x}$$

$$+\,480(2r^5-15r^4+52r^3-96r^2+90r-33)\,\alpha_{7\,x}\,\alpha_{3\cdot x}$$

$$-\,40(3r^6-39r^5+206r^4-616r^3+1113r^2-1113r+462)\,\alpha_{6\,x}\,\alpha^2_{3,x}$$

$$+\,24(30r^5-225r^4+820r^3-1610r^2+1638r-693)\,\alpha^2_{5\,x}$$

$$-\,120(3r^6-33r^5+172r^4-530r^3+987r^2-1029r+462)\,\alpha_{5,x}\,\alpha_{4,x}\,\alpha_{3,x}$$

$$+\,40(51r^6-435r^5+1896r^4-5218r^3+9191r^2-9387r+4158)\,\alpha_{5\,x}\,\alpha_{3,x}$$

$$+\,600(3r^6-45r^5+294r^4-1076r^3+2317r^2-2751r+1386)\,\alpha_{4\,x}\,\alpha^2_{3\cdot x}$$

$$-\,80(21r^6-558r^5+5012r^4-22820r^3+57445r^2-76230r$$

$$+\,41580)\,\alpha^2_{3,x}$$

$$+\,40(9r^6-105r^5+564r^4-1830r^3+3745r^2-4445r+2310)\,\alpha^4_{3\cdot x}$$

$$-\,5(3r^7-59r^6+1136r^5-15642r^4+96135r^3-290115r^2$$

$$+\,429030r-249480)\Big]$$

If the parent population be normal, that is if

$$\alpha_{2n+1} = 0$$

$$\alpha_{2n} = \frac{(2n)!}{2^n n!}$$

the preceding formulae yield on reduction

$$(29) \qquad \mu_{2,\bar{x}} = \frac{2(r-1)}{r^2} \mu_{2x}^2$$

$$(30) \qquad \mu_{3\bar{x}} = \frac{8(r-1)}{r^3} \mu_{2x}^3$$

$$(31) \qquad \mu_{4\bar{x}} = \frac{12(r-1)(r+3)}{r^4} \mu_{2x}^4$$

$$(32) \qquad \mu_{6,\bar{x}} = \frac{32(r-1)(5r+7)}{r^6} \mu_{2x}^5$$

$$(33) \qquad \mu_{6\bar{x}} = \frac{40(r-1)(3r^2+46r+47)}{r^6} \mu_{2x}^6$$

These may be written in turn

$$(34) \quad \begin{cases} \alpha_{3\cdot\bar{x}} = \dfrac{4}{\sqrt{2(r-1)}} \\[2ex] \alpha_{4\cdot\bar{x}} = \dfrac{3(r+3)}{r-1} \\[2ex] \alpha_{5\bar{x}} = \dfrac{8(5r+7)}{\sqrt{2}(r-1)^{3/2}} \\[2ex] \alpha_{6,\bar{x}} = \dfrac{5(3r^2+46r+47)}{(r-1)^2} \end{cases}$$

For the corresponding standard semi-invariants

$$\gamma_3 = \alpha_3 = \frac{4}{\sqrt{2(r-1)}}$$

$$\gamma_4 = \alpha_4 = 3 = \frac{12}{r-1} = \left(\frac{\alpha_3}{2}\right)^2 \cdot 3'$$

$$\gamma_5 = \alpha_5 - 10\alpha_3 = \frac{96}{\sqrt{2}\,(r-1)^{3/2}} = \left(\frac{\alpha_3}{2}\right)^3 \cdot 4\,!$$

$$\gamma_6 = \alpha_6 - 15\,\alpha_4 - 10\alpha_3^2 + 30 = \frac{480}{(r-1)^2} = \left(\frac{\alpha_3}{2}\right)^4 \cdot 5\,!$$

These results show that so far as the sixth standard semi-invariant the distribution of sample second moments about the mean is Type III, irrespective of the mean of the parent population.

It is to be regretted that many of the results presented here have never been generalized for moments of any order. The methods presented have been chosen for two reasons: first, they permit one with no knowledge of calculus to achieve somewhat of an understanding into the theory of sampling; and secondly, they yield results of sampling from a finite parent population--a problem of considerable practical importance.

The results of sampling from an infinite population may be obtained more readily and with far greater elegance and rigor by employing the method of semi-invariants.

H. C. C.

	SKEWNESS						
t	0	.1	.2	.3	.4	.5	t
–9.90							–9.90
–9.80							–9.80
–9.70							–9.70
–9.60							–9.60
–9.50							–9.50
–9.40							–9.40
–9.30							–9.30
–9.20							–9.20
–9.10							–9.10
–9.00							–9.00
–8.90							–8.90
–8.80							–8.80
–8.70							–8.70
–8.60							–8.60
–8.50							–8.50
–8.40							–8.40
–8.30							–8.30
–8.20							–8.20
–8.10							–8.10
–8.00							–8.00
–7.90							–7.90
–7.80							–7.80
–7.70							–7.70
–7.60							–7.60
–7.50							–7.50
–7.40							–7.40
–7.30							–7.30
–7.20							–7.20
–7.10							–7.10
–7.00							–7.00
–6.90							–6.90
–6.80							–6.80
–6.70							–6.70
–6.60							–6.60
–6.50							–6.50
–6.40							–6.40
–6.30							–6.30
–6.20							–6.20
–6.10							–6.10
–6.00							–6.00
–5.90							–5.90
–5.80							–5.80
–5.70							–5.70
–5.60							–5.60
–5.50	.000001						–5.50
–5.40	.000001						–5.40
–5.30	.000002						–5.30
–5.20	.000003						–5.20
–5.10	.000005	.000001					–5.10
–5.00	.000007	.000001					–5.00

t	SKEWNESS						t
	6	7	8	.9	1.0	1.1	
-9.90							-9.90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9.10
-9.00							-9.00
-8.90							-8.90
-8.80							-8.80
-8.70							-8.70
-8.60							-8.60
-8.50							-8.50
-8.40							-8.40
-8.30							-8.30
-8.20							-8.20
-8.10							-8.10
-8.00							-8.00
-7.90							-7.90
-7.80							-7.80
-7.70							-7.70
-7.60							-7.60
-7.50							-7.50
-7.40							-7.40
-7.30							-7.30
-7.20							-7.20
-7.10							-7.10
-7.00							-7.00
-6.90							-6.90
-6.80							-6.80
-6.70							-6.70
-6.60							-6.60
-6.50							-6.50
-6.40							-6.40
-6.30							-6.30
-6.20							-6.20
-6.10							-6.10
-6.00							-6.00
-5.90							-5.90
-5.80							-5.80
-5.70							-5.70
-5.60							-5.60
-5.50							-5.50
-5.40							-5.40
-5.30							-5.30
-5.20							-5.20
-5.10							-5.10
-5.00							-5.00

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
-4.90	.000012	.000002					-4.90
-4.80	.000019	.000003					-4 80
-4 70	000030	.000006					-4.70
-4.60	.000047	.000011	.000001				-4.60
-4.50	.000072	.000019	.000002				-4.50
-4 40	.000110	.000032	.000005				-4.40
-4 30	.000166	.000055	000009				-4.30
-4.20	.000248	.000090	000019	000001			-4 20
-4.10	.000366	000147	.000036	000003			-4.10
-4.00	.000535	000235	.000066	.000008			-4.00
-3 90	.000775	.000370	.000120	000018			-3.90
-3.80	.001109	.000573	000212	000040	000001		-3 80
-3.70	.001572	.000874	.000364	000087	.000005		-3 70
-3.60	.002203	.001313	.000609	.000177	.000017		-3.60
-3.50	.003054	.001941	000995	000344	.000049		-3.50
-3.40	.004190	.002825	001586	000641	.000128	.000003	-3.40
-3 30	.005684	.004049	002470	.001146	000302	.000017	-3.30
-3.20	.007629	.005717	003760	.001974	000661	000069	-3.20
-3 10	.010127	.007950	.005598	003277	.001344	.000234	-3 10
-3.00	.013296	010890	008154	.005256	.002561	000662	-3 00
-2.90	.017262	.014698	011630	008159	.004601	.001623	-2 90
-2 80	.022163	.019545	.016246	012275	.007834	.003540	-2 80
-2 70	.028136	.025608	022239	017924	.012700	.006991	-2 70
-2.60	.035316	.033062	029844	.025438	.019681	.012686	-2.60
-2.50	.043821	.042064	039276	035127	.029254	021390	-2.50
-2.40	.053747	.052738	.050708	.047251	.041831	.033818	-2.40
-2.30	.065152	.065158	.064244	061970	.057693	.050508	-2.30
-2 20	.078044	079330	.079895	.079313	.076923	071689	-2.20
-2 10	.092366	095172	097548	.099136	099346	.097186	-2 10
-2.00	.107982	112500	.116952	.121097	.124495	126364	-2.00
-1.90	124670	.131014	.137704	144643	.151607	.158125	-1.90
-1 80	142110	.150291	159241	169016	.179627	190976	-1 80
-1 70	159883	.169790	.180854	.193275	207285	223136	-1 70
-1 60	.177473	188856	201707	.216340	233159	.252688	-1.60
-1 50	.194276	206743	.220876	.237054	255782	277745	-1.50
-1.40	.209618	222645	237390	254249	.273749	296608	-1 40
-1 30	222779	235726	.250291	266832	.285824	.307909	-1 30
-1.20	233023	.245172	.258688	.273853	.291033	.310712	-1.20
-1.10	.239637	.250235	.261823	.274582	288736	304576	-1.10
-1.00	.241971	.250281	.259120	.268557	278673	.289564	-1 00
- 90	.239477	.244839	250233	.255628	.260978	.266223	- .90
- .80	.231753	233638	.235076	235963	.236165	235512	- .80
- .70	.218578	.216634	.213840	.210049	.205082	.198720	- .70
- .60	.199935	.194029	.186988	178657	.168852	.157358	- .60
- .50	.176033	.166270	.155237	142796	128795	113055	- .50
- .40	.147308	.134036	119517	103653	086338	067454	- .40
- .30	.114416	.098208	.080927	062522	042938	022119	- .30
- .20	.078209	.059827	.040672	.020733	.000000	-.021537	- .20
- .10	.039695	.020043	.000000	-020417	-.041192	-.062305	- 10
.00	.000000	-.019943	-.039861	-.059729	-.079523	-.099218	.00

t	SKEWNESS						t
	.6	.7	8	.9	1.0	1.1	
−4 90							−4 90
−4.80							−4 80
−4 70							−4 70
−4 60							−4 60
−4 50							−4 50
−4 40							−4 40
−4 30							−4.30
−4 20							−4.20
−4 10							−4 10
−4.00							−4 00
−3 90							−3 90
−3.80							−3.80
−3 70							−3.70
−3 60							−3 60
−3.50							−3 50
−3 40							−3 40
−3.30							−3.30
−3 20							−3 20
−3 10	000001						−3.10
−3 00	.000019						−3.00
−2 90	000137						−2 90
−2.80	000628						−2 80
−2 70	.002066	.000036					−2.70
−2 60	005395	000536					−2.60
−2 50	011851	.002918					−2 50
−2.40	022763	009573	000364				−2.40
−2 30	.039266	.023160	005119				−2 30
−2.20	.062013	.045673	.021161	.000121			−2 20
−2.10	090955	.077731	052860	013697			−2.10
−2 00	125234	118306	100022	.059683			−2.00
−1 90	.163228	164867	.158494	.133204	061132		−1.90
−1 80	202703	.213833	221819	.219728	185902	.031381	−1.80
−1 70	241060	261146	.282948	304226	.316115	.277318	−1.70
−1 60	275619	.302858	335559	375014	.421770	470105	−1 60
−1.50	303895	335595	374866	424825	.490506	580401	−1 50
−1 40	.323832	356881	.397950	450518	520464	618649	−1.40
−1 30	333974	365290	403738	.452231	515552	602164	−1 30
−1.20	333548	360455	392746	.432376	.482398	547863	−1.20
−1 10	.322482	342963	366717	.394732	428455	470099	−1 10
−1.00	.301350	314173	.328212	343691	.360894	380193	−1.00
− .90	271274	.276005	.280235	283702	286020	286608	− .90
− .80	.233788	.230710	.225910	.218898	.209014	195355	− 80
− 70	.190694	.180672	168238	.152867	.133891	110444	− .70
− 60	.143917	128225	109914	.088542	063567	034327	− 60
− .50	095375	.075522	.053226	028174	000000	−.031726	− 50
− .40	046869	.024439	.000000	− 026629	−.055654	− 087304	− 40
− 30	.000000	−.023484	− 048406	− 074843	−.102879	−.132606	− 30
− 20	−.043888	−.067064	−.091075	− 115932	−.141646	− 168230	− .20
− 10	− 083738	− 105473	− 127489	− 149766	−.172285	−.195025	− 10
.00	−.118789	− 138212	− 157465	− 176524	− 195367	− 213972	.00

t	SKEWNESS						t
	0	.1	.2	.3	.4	.5	
00	.000000	-.019943	-.039861	-059729	-079523	-.099218	00
10	-039695	-058941	-077764	-096151	-114088	-.131563	10
20	-078209	-095827	-112693	-128817	-144210	-158882	.20
30	-.114416	-.129600	-.143805	-.157072	-.169444	-180957	30
.40	-147308	-.159421	-.170454	-.180482	-.189571	-197783	.40
.50	-176033	-184644	-192211	-198827	-204577	-209535	50
.60	-.199935	-204840	-208859	-212093	-214627	-.216536	60
70	-.218573	-.219795	-.220388	-220442	-220032	-219219	70
.80	-231753	-229507	-.226967	-224189	-221217	-.218089	.80
.90	-239477	-.234167	-.228926	-223762	-218684	-213694	.90
1 00	-.241971	-.234132	-226714	-.219676	-212981	-206595	1.00
1 10	-.239637	-.229889	-.220873	-212494	-204674	-.197346	1 10
1.20	-.233023	-222022	-.211994	-202799	-.194322	-.186470	1.20
1 30	-.222779	-.211174	-200694	-191168	-.182458	-.174450	1.30
1.40	-209618	-.198009	-.187582	-178152	-.169571	-.161717	1.40
1 50	-.194276	-183183	-.173236	-164256	-.156099	-.148647	1 50
1.60	-.177473	-.167315	-.158186	-.149931	-.142423	-135559	1.60
1.70	-.159883	-150967	-.142899	-135563	-.128862	-122712	1 70
1 80	-.142110	-134628	-.127773	-.121475	-115674	-.110315	1 80
1.90	-124670	-.118709	-113132	-107924	-103063	-.098523	1.90
2 00	-.107982	-.103534	-.099229	-095105	-091177	-.087449	2.00
2 10	-092366	-.089346	-.086249	-083156	-080118	-077165	2 10
2 20	-078044	-.076312	-.074312	-072165	-069947	-067712	2.20
2 30	-065152	-064527	-.063487	-062176	-060692	-.059103	2.30
2 40	-053747	-.054029	-053795	-053200	-.052351	-.051327	2 40
2 50	-043821	-044806	-045221	-045215	-044900	-044359	2.50
2.60	-035316	-036811	-.037720	-.038180	-.038300	-038160	2.60
2 70	-028136	-.029964	-031226	-032039	-032498	-032681	2.70
2 80	-022163	-024172	-.025661	-026723	-027436	-027870	2.80
2.90	-017262	-019326	-.020937	-.022158	-023050	-023670	2 90
3 00	-.013296	-015318	-.016964	-.018268	-019273	-020023	3.00
3 10	-010127	-012037	-013651	-014978	-016042	-016874	3 10
3 20	-007629	-009379	-010911	-012214	-.013293	-.014168	3 20
3 30	-.005684	-007247	-008665	-009908	-010969	-011854	3 30
3 40	-004190	-005554	-006837	-007996	-.009013	-009884	3 40
3 50	-003054	-004222	-005361	-006422	-.007376	-008214	3.50
3 60	-002203	-.003184	-004178	-005132	-006013	-.006805	3.60
3 70	-.001572	-.002382	-.003236	-004082	-004883	-.005620	3 70
3 80	-.001109	-.001768	-002492	-003232	-003951	-004627	3 80
3 90	-000775	-001303	-.001908	-002547	-003185	-.003799	3.90
4.00	-000535	-000952	-001452	-.001999	-.002559	-003110	4.00
4 10	-.000366	-000691	-001100	-001561	-002049	-002539	4 10
4.20	-.000248	-000498	-.000828	-.001215	-.001635	-.002067	4 20
4 30	-000166	-000356	-.000620	-.000941	-.001301	-.001679	4 30
4.40	-000110	-.000253	-000462	-000726	-001031	-001360	4 40
4 50	-000072	-000178	-000342	-000558	-.000815	-001099	4 50
4 60	-.000047	-000125	-.000252	-000428	-.000642	-.000886	4 60
4.70	-000030	-.000087	-.000185	-000326	-000505	-.000713	4.70
4 80	-.000019	-000060	-000135	-.000248	-.000396	-000572	4 80
4 90	-000012	-000041	-.000098	-000188	-.000309	-000458	4 90

t	SKEWNESS						t
	6	7	8	.9	1.0	1 1	
.00	-118789	-138212	-157465	-.176524	-195367	-213972	00
10	-.148563	-165078	-.181098	-.196613	-.211618	-226104	.10
20	-172846	-186112	-198692	-210598	-221843	-.232441	20
30	-191649	-201555	-210707	-219137	-.226877	-.233955	30
40	-.205174	-211799	-.217704	-222936	-227536	-231545	40
.50	-213769	-217340	-.220302	-.222707	-224598	-.226019	50
.60	-.217887	-218739	-219142	-219142	-218780	-.218092	60
70	-218055	-216587	-214854	-212890	-210726	-208386	70
.80	-.214834	-.211479	-.208044	-204548	-201006	-197431	80
.90	-208794	-.203987	-.199272	-.194648	-.190114	-185671	.90
1 00	-200493	-194649	-189043	-183655	-.178470	-173473	1.00
1 10	-190456	-183956	-177806	-171972	-.166424	-161137	1 10
1 20	-.179166	-172343	-165948	-159935	-154263	-.148900	1 20
1 30	-167053	-.160191	-153798	-147823	-142218	-136946	1.30
1 40	-154494	-147820	-141627	-135859	-130469	-125416	1 40
1 50	-141806	-135497	-129653	-.124221	-119153	-.114410	1.50
1 60	-129254	-123437	-118049	-113040	-.108368	-103997	1 60
1 70	-.117047	-111807	-.106944	-102415	-098185	-094223	1 70
1 80	-105348	-100733	-.096431	-.092411	-088645	-.085109	1 80
1 90	-.094278	-090302	-086573	-083069	-079770	-076660	1.90
2 00	-083916	-080572	-.077405	-074407	-.071565	-068870	2 00
2 10	-074314	-071572	-068944	-066428	-.064021	-.061721	2 10
2 20	-065493	-063313	-061186	-059120	-.057120	-.055189	2 20
2 30	-.057456	-.055786	-054116	-.052462	-050835	-.049243	2 30
2 40	-050187	-048971	-047709	-.046425	-045135	-043849	2 40
2 50	-043655	-042835	-041933	-040977	-039985	-.038973	2 50
2.60	-.037824	-.037342	-036750	-036078	-035348	-.034578	2 60
2 70	-032648	-032449	-032119	-031691	-031187	-.030627	2 70
2 80	-028079	-028110	-027999	-.027776	-.027464	-027084	2 80
2 90	-024066	-024280	-024346	-.024293	-024143	-023915	2.90
3 00	-020558	-020913	-021120	-.021204	-021187	-021087	3.00
3.10	-017505	-017965	-.018279	-.018472	-018563	-018568	3 10
3 20	-.014860	-015393	-015787	-016063	-016238	-016329	3.20
3 30	-012578	-.013156	-013606	-013944	-014184	-014342	3.30
3 40	-010616	-011218	-011703	-.012084	-012373	-012582	3 40
3 50	-008935	-009543	-010046	-010455	-010778	-011026	3.50
3 60	-007501	-008100	-008609	-009032	-.009377	-009652	3.60
3 70	-006280	-.006862	-007364	-007791	-008148	-008440	3 70
3 80	-005246	-005800	-.006288	-.006711	-007071	-007374	3.80
3 90	-004371	-004893	-005361	-005773	-006130	-.006436	3.90
4.00	-.003634	-.004120	-004563	-004959	-005309	-005612	4 00
4.10	-003014	-003463	-.003878	-004255	-004592	-.004889	4.10
4.20	-002495	-002906	-003291	-003646	-.003969	-004256	4 20
4 30	-.002061	-002434	-002789	-003121	-003426	-.003702	4.30
4.40	-.001699	-002035	-002360	-.002669	-002955	-003218	4 40
4 50	-.001397	-001699	-001995	-002279	-002547	-002795	4 50
4 60	-.001147	-001416	-.001684	-.001944	-002193	-002425	4.60
4 70	-000940	-001179	-.001420	-001657	-001886	-002104	4 70
4 80	-000769	-000979	-001195	-.001411	-001621	-001823	4 80
4 90	-000628	-000813	-001005	-001200	-001393	-001579	4 90

t	SKEWNESS						t
	0	.1	.2	.3	.4	.5	
5.00	-.000007	-.000028	-.000071	-.000142	-.000241	-.000366	5.00
5.10	-.000005	-.000019	-.000051	-.000106	-.000187	-.000292	5.10
5.20	-.000003	-.000013	-.000036	-.000080	-.000145	-.000232	5.20
5.30	-.000002	-.000008	-.000026	-.000060	-.000112	-.000184	5.30
5.40	-.000001	-.000006	-.000018	-.000044	-.000087	-.000146	5.40
5.50	-.000001	-.000004	-.000013	-.000033	-.000067	-.000116	5.50
5.60		-.000002	-.000009	-.000024	-.000051	-.000091	5.60
5.70		-.000002	-.000006	-.000018	-.000039	-.000072	5.70
5.80		-.000001	-.000004	-.000013	-.000030	-.000057	5.80
5.90		-.000001	-.000003	-.000010	-.000023	-.000044	5.90
6.00			-.000002	-.000007	-.000017	-.000035	6.00
6.10			-.000001	-.000005	-.000013	-.000027	6.10
6.20			-.000001	-.000004	-.000010	-.000021	6.20
6.30			-.000001	-.000003	-.000008	-.000017	6.30
6.40				-.000002	-.000006	-.000013	6.40
6.50				-.000001	-.000004	-.000010	6.50
6.60				-.000001	-.000003	-.000008	6.60
6.70				-.000001	-.000002	-.000006	6.70
6.80				-.000001	-.000002	-.000005	6.80
6.90					-.000001	-.000004	6.90
7.00					-.000001	-.000003	7.00
7.10					-.000001	-.000002	7.10
7.20					-.000001	-.000002	7.20
7.30						-.000001	7.30
7.40						-.000001	7.40
7.50						-.000001	7.50
7.60						-.000001	7.60
7.70							7.70
7.80							7.80
7.90							7.90
8.00							8.00
8.10							8.10
8.20							8.20
8.30							8.30
8.40							8.40
8.50							8.50
8.60							8.60
8.70							8.70
8.80							8.80
8.90							8.90
9.00							9.00
9.10							9.10
9.20							9.20
9.30							9.30
9.40							9.40
9.50							9.50
9.60							9.60
9.70							9.70
9.80							9.80
9.90							9.90

t	SKEWNESS						t
	.6	7	8	.9	1.0	1 1	
5 00	-000512	-.000674	-000845	-001020	-.001195	-.001367	5 00
5 10	-000417	-000557	-000709	-000866	-001025	-001182	5.10
5.20	-000339	-000461	-000594	-000735	-000878	-001022	5.20
5.30	-000275	-.000380	-000497	-000623	-000752	-.000883	5.30
5.40	-000223	-.000314	-000416	-.000527	-.000644	-000763	5 40
5.50	-.000180	-000258	-.000348	-000446	-000551	-.000658	5 50
5 60	-000145	-000212	-.000290	-000377	-.000471	-000568	5 60
5.70	-.000117	-000174	-000242	-000319	-.000402	-.000490	5 70
5 80	-000094	-000143	-000202	-.000269	-000343	-000422	5 80
5 90	-.000076	-.000117	-000168	-000227	-000293	-000364	5 90
6 00	-000061	-000096	-000140	-000191	-000250	-000313	6 00
6 10	-.000049	-000079	-000116	-000161	-000213	-.000269	6 10
6.20	-.000039	-.000064	-.000097	-000136	-000181	-.000232	6 20
6.30	-000031	-.000052	-.000080	-000114	-000154	-000199	6 30
6 40	-.000025	-000043	-000066	-.000096	-000131	-000171	6 40
6 50	-000020	-000035	-.000055	-000081	-.000112	-.000147	6 50
6 60	-000016	-000028	-000046	-.000068	-000095	-000126	6 60
6 70	-.000013	-000023	-000038	-000057	-000081	-000109	6 70
6 80	-000010	-000019	-000031	-.000048	-000069	-000093	6.80
6 90	-000008	-000015	-000026	-000040	-000058	-000080	6 90
7 00	-.000006	-000012	-.000021	-000034	-.000049	-000069	7 00
7 10	-000005	-000010	-000018	-000028	-000042	-000059	7 10
7 20	-.000004	-000008	-000014	-000024	-000035	-000050	7 20
7 30	-000003	-000007	-000012	-.000020	-000030	-.000043	7.30
7 40	-000003	-000005	-.000010	-000016	-000025	-.000037	7 40
7 50	-000002	-000004	-000008	-.000014	-000022	-000032	7 50
7.60	-000002	-.000003	-.000007	-000012	-000018	-.000027	7 60
7 70	-.000001	-000003	-000005	-000010	-000015	-.000023	7 70
7 80	-000001	-000002	-000005	-000008	-000013	-.000020	7 80
7 90	-.000001	-.000002	-000004	-.000007	-000011	-000017	7.90
8 00	-000001	-000001	-.000003	-000006	-000009	-000014	8.00
8 10		-000001	-000002	-000005	-.000008	-000012	8 10
8 20		-000001	-.000002	-.000004	-.000007	-000011	8 20
8 30		-000001	-000002	-000003	-000006	-.000009	8.30
8 40		-000001	-000001	-000003	-000005	-.000008	8.40
8 50			-.000001	-000002	-000004	-.000007	8 50
8 60			-000001	-000002	-000003	-000006	8.60
8.70			-000001	-.000002	-000003	-000005	8.70
8 80			-000001	-000001	-.000002	-.000004	8.80
8.90			-000001	-000001	-.000002	-000003	8.90
9.00				-000001	-000002	-.000003	9.00
9 10				-.000001	-000001	-000003	9.10
9.20				-000001	-.000001	-000002	9 20
9 30				-.000001	-000001	-000002	9 30
9.40					-000001	-.000002	9 40
9.50					-.000001	-.000001	9.50
9 60					-.000001	-.000001	9.60
9 70					-.000001	-.000001	9.70
9 80						-.000001	9.80
9 90						-.000001	9.90

	SKEWNESS						
t	0	.1	2	3	.4	.5	t
10.00							10.00
10.10							10.10
10.20							10.20
10.30							10.30
10.40							10.40
10.50							10.50
10.60							10.60
10.70							10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

t	SKEWNESS .6	.7	.8	.9	1.0	1.1	t
10.00						-.000001	10.00
10.10						-.000001	10.10
10.20							10.20
10.30							10.30
10.40							10.40
10.50							10.50
10.60							10.60
10.70							10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

t	SKEWNESS						t
	0	.1	.2	.3	4	.5	
-9.90							-9.90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9.10
-9.00							-9.00
-8.90							-8.90
-8.80							-8.80
-8.70							-8.70
-8.60							-8.60
-8.50							-8.50
-8.40							-8.40
-8.30							-8.30
-8.20							-8.20
-8.10							-8.10
-8.00							-8.00
-7.90							-7.90
-7.80							-7.80
-7.70							-7.70
-7.60							-7.60
-7.50							-7.50
-7.40							-7.40
-7.30							-7.30
-7.20							-7.20
-7.10							-7.10
-7.00							-7.00
-6.90							-6.90
-6.80							-6.80
-6.70							-6.70
-6.60							-6.60
-6.50							-6.50
-6.40							-6.40
-6.30							-6.30
-6.20							-6.20
-6.10							-6.10
-6.00							-6.00
-5.90							-5.90
-5.80	000001						-5.80
-5.70	.000001						-5.70
-5.60	.000002						-5.60
-5.50	.000003						-5.50
-5.40	.000005						-5.40
-5.30	.000009	.000001					-5.30
-5.20	.000014	.000002					-5.20
-5.10	.000022	000003					-5.10
-5.00	.000036	.000006					-5.00

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
-9.90							-9.90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9.10
-9.00							-9.00
-8.90							-8.90
-8.80							-8.80
-8.70							-8.70
-8.60							-8.60
-8.50							-8.50
-8.40							-8.40
-8.30							-8.30
-8.20							-8.20
-8.10							-8.10
-8.00							-8.00
-7.90							-7.90
-7.80							-7.80
-7.70							-7.70
-7.60							-7.60
-7.50							-7.50
-7.40							-7.40
-7.30							-7.30
-7.20							-7.20
-7.10							-7.10
-7.00							-7.00
-6.90							-6.90
-6.80							-6.80
-6.70							-6.70
-6.60							-6.60
-6.50							-6.50
-6.40							-6.40
-6.30							-6.30
-6.20							-6.20
-6.10							-6.10
-6.00							-6.00
-5.90							-5.90
-5.80							-5.80
-5.70							-5.70
-5.60							-5.60
-5.50							-5.50
-5.40							-5.40
-5.30							-5.30
-5.20							-5.20
-5.10							-5.10
-5.00							-5.00

	SKEWNESS						
t	0	.1	.2	.3	.4	.5	t
-4.90	000056	.000011	.000001				-4 90
-4.80	.000087	000020	.000002				-4 80
-4.70	000134	.000036	000004				-4.70
-4.60	.000204	.000061	.000008				-4.60
-4 50	.000308	.000103	.000017				-4 50
-4.40	000458	.000171	.000034	.000001			-4.40
-4 30	000674	.000279	000066	000004			-4.30
-4.20	000981	000446	000124	.000011			-4 20
-4.10	.001411	.000702	000227	.000029			-4.10
-4 00	002007	.001085	.000404	000068	.000001		-4.00
-3 90	.002823	.001650	.000698	000151	000005		-3 90
-3.80	003924	.002465	.001174	000318	000020		-3 80
-3.70	005390	.003622	.001921	000638	000065		-3.70
-3 60	.007318	.005233	003063	.001216	.000190		-3 60
-3 50	.009818	.007435	.004759	002213	000493	000008	-3 50
-3 40	013012	.010387	.007210	.003855	.001157	000056	-3.40
-3.30	.017036	014272	010658	.006445	002482	.000259	-3 30
-3.20	.022029	.019282	.015376	010360	004921	000914	-3 20
-3 10	028127	.025617	.021661	.016039	009087	.002622	-3 10
-3.00	035455	.033464	.029805	.023959	.015732	.006377	-3.00
-2 90	044108	.042977	.040072	034578	.025682	.013553	-2 90
-2.80	.054142	.054254	052649	.048276	.039717	.025751	-2.80
-2 70	.065547	067309	067607	065269	.058421	.044473	-2.70
-2.60	.078238	082041	084853	.085530	.082005	.070719	-2.60
-2 50	092024	098208	.104081	.108701	.110147	.104574	-2 50
-2.40	.106598	115397	.124739	134041	.141874	.144908	-2 40
-2 30	121523	133016	.146010	.160398	.175522	189277	-2 30
-2.20	136222	.150286	166813	.186228	.208791	.234060	-2 20
-2 10	149984	.166250	.185835	.209664	.238900	274831	-2.10
-2 00	.161973	179808	.201589	228635	.262822	306883	-2 00
-1 90	171257	.189761	212507	.241024	.277610	325833	-1 90
-1 80	.176848	194883	217043	.244859	.280668	328187	-1 80
-1 70	177753	193997	.213810	.238508	270098	.311788	-1.70
-1 60	173036	186074	201707	.220856	244914	276085	-1 60
-1 50	161897	170327	.180042	.191456	205187	.222196	-1.50
-1 40	.143738	146302	.148634	.150619	.152083	.152768	-1 40
-1 30	118244	.113955	.107884	.099440	087784	071682	-1 30
-1.20	.085442	073710	.058793	039758	.015318	-016353	-1 20
-1 10	.045749	.026480	002942	-025961	-061696	-.106262	-1 10
-1.00	.000000	-026345	-.057582	-094785	-.139336	-193043	-1.00
- .90	-.050556	-082945	-.120304	-.163523	-213693	-.272169	- .90
- .80	-104289	-.141156	-.182513	-.228950	-.281149	-.339887	- .80
- 70	-.159250	-198588	-.241433	-288031	-.338624	-.393425	- .70
- 60	-213264	-.252765	-.294407	-.338118	-.383756	-.431081	- 60
- .50	-264049	-301276	-.339069	-.377114	-415005	-.452220	- .50
- .40	-309347	-.341929	-.373489	-.403584	-.431690	-.457186	- .40
- .30	-347063	-372891	-.396290	-.416811	-433951	-.447153	- .30
- 20	-375401	-392803	-.406717	-.416794	-.422664	-.423938	- .20
- .10	-392983	-.400862	-.404661	-.404202	-.399307	-389805	- 10
.00	-.398942	-.396865	-390638	-380276	-.365806	-347261	.00

t	SKEWNESS						t
	6	.7	8	9	10	11	
-4.90							-4.90
-4 80							-4 80
-4.70							-4 70
-4.60							-4 60
-4 50							-4.50
-4 40							-4.40
-4.30							-4 30
-4 20							-4 20
-4 10							-4.10
-4 00							-4.00
-3 90							-3.90
-3.80							-3.80
-3 70							-3.70
-3 60							-3 60
-3 50							-3.50
-3 40							-3.40
-3 30							-3.30
-3 20	000008						-3 20
-3.10	.000036						-3.10
-3 00	.000442						-3 00
-2 90	002377						-2.90
-2 80	008378						-2 80
-2.70	.021975	001288					-2 70
-2 60	.046701	011088					-2.60
-2 50	084682	040658					-2.50
-2.40	.135492	097040	014364				-2.40
-2 30	.195696	.178156	093291				-2.30
-2 20	.259189	273180	235123	015680			-2.20
-2 10	318205	366237	.398395	290543			-2.10
-2 00	.364653	440957	.537617	618006			-2.00
-1.90	391464	.484203	.620767	826786	1 055709		-1 90
-1 80	393656	488248	.633769	.878055	1.344215	2.171541	-1 80
-1 70	368970	.451364	578139	792281	1 211776	2.318708	-1 70
-1 60	318022	377196	.466054	611436	881883	1 511052	-1 60
-1 50	244037	273391	315228	379697	.490506	.715683	-1 50
-1 40	152272	.149950	.144709	134579	.115659	.079111	-1.40
-1 30	.049275	.017638	-.028037	-.096151	- 202538	-.380314	-1 30
-1 20	-.057908	- 113328	- 188820	-294517	- 447941	-.681730	-1.20
-1 10	- 162444	-.234218	- 327426	- 450951	- 618879	- 854725	-1 10
-1.00	- 258300	-.338340	- 437616	- 562403	- 721788	- 929360	-1 00
- 90	-.340641	- 421241	- 516683	- 630449	- 767053	-.932389	- 90
- 80	-.406052	-.480646	-.564776	-.659627	- 766385	- 886072	- 80
- 70	- 452596	- 516205	-.584160	- 656102	-.731248	- 808124	- .70
- 60	-.479724	-.529131	- 578496	- 626664	-.671992	- 712150	- 60
- .50	- 488097	-.521789	- 552221	- 578022	- 597445	-.608256	- .50
- .40	- 479345	-.497304	-.510042	- 516352	- 514799	-.503678	- .40
- 30	- 455795	-.459186	-.456554	-.447036	- 429670	-.403375	- 30
- 20	-.420207	-.411038	-.395979	- 374550	-.346246	-.310536	- .20
- 10	- 375528	- 356312	- 332002	- 302445	- 267495	-.227013	- 10
.00	- 324689	- 298144	- 267691	- 233404	-.195367	-.153671	00

t	SKEWNESS						t
	.0	.1	2	3	.4	5	
00	-398942	-.396865	-.390638	-380276	-.365806	-.347261	.00
.10	-392983	-381208	-.365722	-346712	-.324368	-.298881	10
.20	-375401	-354845	-.331450	-305516	-.277327	-247150	.20
.30	-347063	-.319212	-.289703	-258865	-.226991	-.194347	30
.40	-.309347	-.276121	-242569	-.208963	-.175528	-.142459	.40
.50	-.264049	-227632	-.192211	-.157924	-.124871	-.093127	.50
.60	-213264	-175927	-.140741	-.107668	-.076652	-.047627	.60
.70	-.159250	-.123170	-.090112	-059849	-.032168	-006874	70
.80	-.104289	-.071396	-.042031	-015803	007628	.028559	.80
.90	-050556	-.022408	.002100	023470	.042119	058401	.90
1.00	000000	022298	.041221	.057307	.070994	.082638	1 00
1.10	045749	.061582	.074619	.085362	094207	.101468	1.10
1.20	.085442	.094674	.101920	.107574	.111937	.115246	1 20
1 30	118244	.121175	.123055	124117	.124535	.124440	1.30
1.40	143738	.141025	138219	135360	132477	129592	1.40
1 50	.161897	.154462	.147816	.141807	136322	.131273	1 50
1 60	.173036	.161963	152411	.144058	136668	.130063	1.60
1.70	.177753	.164185	152670	.142759	.134125	126521	1 70
1 80	.176848	161901	.149315	138570	129283	.121170	1 80
1 90	.171257	.155944	.143079	132127	.122694	114484	1 90
2.00	161973	147151	134668	124028	114859	106882	2.00
2.10	149984	136328	122839	114809	106219	.098724	2.10
2.20	.136222	.124207	.113878	.104940	097149	.090313	2 20
2.30	.121523	111433	.102586	.094814	087962	081894	2 30
2.40	106598	098547	.091279	.084754	078907	.073662	2 40
2 50	.092024	.085981	.080285	075012	.070177	065763	2 50
2 60	078238	074064	.069851	065773	.061913	058305	2 60
2 70	.065547	063024	060152	057169	054212	051357	2 70
2.80	054142	.053008	.051295	049280	.047134	044963	2 80
2.90	044108	044085	043335	.042146	.040707	039139	2.90
3.00	035455	.036269	036285	.035776	.034933	.033885	3.00
3.10	028127	.029527	030122	030152	.029797	029186	3.10
3.20	.022029	023795	024799	.025238	.025271	025017	3 20
3 30	017036	018986	020254	.020987	021315	.021343	3 30
3.40	.013012	015004	016415	017341	017883	018129	3.40
3.50	009818	011746	.013204	014242	014928	.015334	3 50
3.60	.007318	009111	.010544	011628	012402	012917	3 60
3.70	.005390	.007004	008361	009440	.010254	010839	3 70
3.80	.003924	.005336	.006585	007621	.008441	009061	3 80
3.90	.002823	004031	005151	006120	006918	.007548	3 90
4.00	002007	003019	004003	004890	.005646	.006266	4.00
4.10	.001411	.002242	.003091	003887	.004590	.005184	4.10
4.20	000981	001652	.002372	003075	.003716	004275	4.20
4.30	.000674	.001207	.001810	.002421	.002997	.003515	4 30
4.40	.000458	.000875	.001372	.001897	.002409	.002881	4.40
4 50	.000308	.000629	001034	001480	001929	002355	4.50
4.60	.000204	000449	.000775	001150	001539	.001919	4 60
4 70	000134	000318	.000578	.000889	001224	001560	4.70
4 80	000087	.000224	.000428	.000685	.000970	001265	4 80
4 90	.000056	.000156	000316	000525	000767	.001023	4.90

t	SKEWNESS						t
	6	.7	.8	.9	1.0	1 1	
00	- 324689	-.298144	- 267691	- 233404	-.195367	-.153671	00
10	-.270442	- 239243	-.205476	--.169332	- 131001	- 090672	.10
20	- 215242	- 181842	- 147179	-.111467	- 074908	- 037693	.20
.30	-.161173	- 127682	-.094065	-.060496	- 027127	005906	30
.40	- 109915	-.078031	-.046919	-.016670	.012641	040955	.40
.50	- 062736	-.033730	- 006120	020094	.044920	.068375	50
.60	- 020517	004757	028276	050122	.070377	089119	.60
70	016219	.037278	.056460	073908	.089753	104118	.70
80	.047251	063932	078805	092047	103816	.114254	80
.90	.072612	085008	095804	.105187	113319	120340	.90
1 00	092535	100929	108025	113993	.118980	123110	1 00
1 10	.107400	112210	.116068	.119113	121463	.123215	1.10
1.20	.117687	119411	.120537	121163	121369	.121222	1.20
1.30	.123938	123108	.122015	120710	.119234	.117620	1 30
1 40	.126718	123868	121049	118266	115524	.112825	1 40
1 50	.126593	.122229	.118138	114285	110642	107185	1 50
1 60	124106	118689	.113730	.109161	.104928	100987	1.60
1 70	119760	113697	108217	103232	.098668	094466	1 70
1.80	114013	.107646	101936	096780	092094	.087811	1.80
1 90	.107270	.100878	.095170	090039	085395	081170	1 90
2 00	.099878	.093681	088156	083198	078722	074658	2 00
2 10	092132	086292	081084	076409	072189	068360	2.10
2 20	084273	.078905	.074104	.069787	065884	.062340	2 20
2 30	.076495	071668	067332	.063418	059870	056640	2 30
2 40	068947	064697	.060854	057366	.054190	051288	2 40
2 50	061741	.058075	054730	051672	048870	046297	2 50
2 60	.054956	051859	049000	046362	.043925	041672	2 60
2 70	.048642	046084	043686	.041446	039357	.037410	2 70
2 80	042828	040765	038796	036927	.035163	033501	2 80
2 90	037523	035908	034327	.032797	031330	.029931	2 90
3 00	032722	.031503	030268	029043	027845	.026685	3 00
3.10	.028411	027535	026603	025648	024689	023743	3 10
3 20	024565	023980	023311	022590	.021843	021085	3 20
3 30	021156	020815	020367	.019848	019284	018692	3.30
3.40	018152	.018009	.017747	.017399	016991	016543	3 40
3 50	.015519	015535	015424	.015218	014942	014618	3.50
3 60	013222	.013362	013372	013283	013118	012897	3 60
3 70	011229	.011461	011566	011570	011497	011363	3.70
3 80	.009506	.009804	009981	.010060	010060	009998	3.80
3.90	.008024	008366	008596	.008731	008789	008785	3 90
4 00	006753	007122	.007387	007564	007668	007711	4.00
4.10	005668	006049	.006336	.006543	006680	.006760	4.10
4 20	.004745	005126	005425	.005651	005813	.005919	4 20
4 30	.003961	004334	004636	.004873	005051	005178	4.30
4 40	003299	.003658	.003955	004196	004384	.004525	4 40
4 50	.002742	.003080	003369	003608	003800	.003950	4 50
4 60	.002273	002589	002865	.003098	003291	.003446	4 60
4 70	.001880	002173	.002433	.002657	.002847	.003003	4.70
4 80	.001552	001820	002062	.002276	002460	002614	4 80
4.90	.001278	.001522	001746	.001947	.002124	.002274	4.90

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
5.00	.000036	.000108	.000232	.000401	.000604	.000825	5.00
5.10	.000022	.000074	.000169	.000305	.000474	.000664	5.10
5.20	.000014	.000051	.000123	.000232	.000371	.000533	5.20
5.30	.000009	.000034	.000089	.000175	.000290	.000427	5.30
5.40	.000005	.000023	.000064	.000132	.000226	.000341	5.40
5.50	.000003	.000015	.000046	.000099	.000175	.000272	5.50
5.60	.000002	.000010	.000032	.000074	.000136	.000216	5.60
5.70	.000001	.000007	.000023	.000055	.000105	.000172	5.70
5.80	.000001	.000004	.000016	.000041	.000081	.000136	5.80
5.90		.000003	.000011	.000030	.000062	.000108	5.90
6.00		.000002	.000008	.000022	.000048	.000085	6.00
6.10		.000001	.000006	.000016	.000036	.000067	6.10
6.20		.000001	.000004	.000012	.000028	.000053	6.20
6.30			.000003	.000009	.000021	.000041	6.30
6.40			.000002	.000006	.000016	.000032	6.40
6.50			.000001	.000005	.000012	.000025	6.50
6.60			.000001	.000003	.000009	.000020	6.60
6.70			.000001	.000002	.000007	.000015	6.70
6.80				.000002	.000005	.000012	6.80
6.90				.000001	.000004	.000009	6.90
7.00				.000001	.000003	.000007	7.00
7.10				.000001	.000002	.000006	7.10
7.20					.000002	.000004	7.20
7.30					.000001	.000003	7.30
7.40					.000001	.000003	7.40
7.50					.000001	.000002	7.50
7.60					.000001	.000002	7.60
7.70						.000001	7.70
7.80						.000001	7.80
7.90						.000001	7.90
8.00						.000001	8.00
8.10							8.10
8.20							8.20
8.30							8.30
8.40							8.40
8.50							8.50
8.60							8.60
8.70							8.70
8.80							8.80
8.90							8.90
9.00							9.00
9.10							9.10
9.20							9.20
9.30							9.30
9.40							9.40
9.50							9.50
9.60							9.60
9.70							9.70
9.80							9.80
9.90							9.90

	SKEWNESS						
t	.6	.7	.8	.9	1.0	1.1	t
5.00	.001051	.001270	.001476	.001664	.001832	.001977	5.00
5.10	.000862	.001059	.001247	.001421	.001578	.001717	5.10
5.20	.000706	.000881	.001051	.001212	.001359	.001491	5.20
5.30	.000577	.000732	.000885	.001032	.001169	.001293	5.30
5.40	.000471	.000607	.000745	.000878	.001004	.001121	5.40
5.50	.000383	.000503	.000626	.000747	.000863	.000971	5.50
5.60	.000312	.000416	.000525	.000634	.000740	.000840	5.60
5.70	.000253	.000344	.000440	.000538	.000635	.000727	5.70
5.80	.000205	.000284	.000369	.000456	.000544	.000629	5.80
5.90	.000166	.000234	.000309	.000387	.000466	.000543	5.90
6.00	.000134	.000193	.000258	.000327	.000399	.000469	6.00
6.10	.000108	.000158	.000215	.000277	.000341	.000405	6.10
6.20	.000087	.000130	.000180	.000234	.000291	.000349	6.20
6.30	.000070	.000107	.000150	.000198	.000249	.000301	6.30
6.40	.000056	.000087	.000125	.000167	.000212	.000260	6.40
6.50	.000045	.000072	.000104	.000141	.000181	.000224	6.50
6.60	.000036	.000058	.000086	.000119	.000154	.000193	6.60
6.70	.000029	.000048	.000072	.000100	.000132	.000166	6.70
6.80	.000023	.000039	.000059	.000084	.000112	.000143	6.80
6.90	.000018	.000032	.000049	.000071	.000095	.000123	6.90
7.00	.000015	.000026	.000041	.000059	.000081	.000105	7.00
7.10	.000012	.000021	.000034	.000050	.000069	.000091	7.10
7.20	.000009	.000017	.000028	.000042	.000059	.000078	7.20
7.30	.000007	.000014	.000023	.000035	.000050	.000067	7.30
7.40	.000006	.000011	.000019	.000029	.000042	.000057	7.40
7.50	.000005	.000009	.000016	.000025	.000036	.000049	7.50
7.60	.000004	.000007	.000013	.000021	.000030	.000042	7.60
7.70	.000003	.000006	.000011	.000017	.000026	.000036	7.70
7.80	.000002	.000005	.000009	.000015	.000022	.000031	7.80
7.90	.000002	.000004	.000007	.000012	.000019	.000027	7.90
8.00	.000001	.000003	.000006	.000010	.000016	.000023	8.00
8.10	.000001	.000003	.000005	.000008	.000013	.000019	8.10
8.20	.000001	.000002	.000004	.000007	.000011	.000017	8.20
8.30	.000001	.000002	.000003	.000006	.000010	.000014	8.30
8.40	.000001	.000001	.000003	.000005	.000008	.000012	8.40
8.50		.000001	.000002	.000004	.000007	.000010	8.50
8.60		.000001	.000002	.000003	.000006	.000009	8.60
8.70		.000001	.000002	.000003	.000005	.000008	8.70
8.80		.000001	.000001	.000002	.000004	.000007	8.80
8.90			.000001	.000002	.000003	.000006	8.90
9.00			.000001	.000002	.000003	.000005	9.00
9.10			.000001	.000001	.000002	.000004	9.10
9.20			.000001	.000001	.000002	.000003	9.20
9.30				.000001	.000002	.000003	9.30
9.40				.000001	.000001	.000003	9.40
9.50				.000001	.000001	.000002	9.50
9.60				.000001	.000001	.000002	9.60
9.70					.000001	.000002	9.70
9.80					.000001	.000001	9.80
9.90					.000001	.000001	9.90

	SKEWNESS						
t	0	.1	.2	.3	.4	.5	t
10.00							10.00
10.10							10.10
10.20							10.20
10.30							10.30
10.40							10.40
10.50							10.50
10.60							10.60
10.70							10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
10.00					.000001	.000001	10.00
10.10						.000001	10.10
10.20						.000001	10.20
10.30						.000001	10.30
10.40						.000001	10.40
10.50							10.50
10.60							10.60
10.70							10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

	SKEWNESS						
t	.0	.1	.2	.3	.4	.5	t
-9.90							-9.90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9.10
-9.00							-9.00
-8.90							-8.90
-8.80							-8.80
-8.70							-8.70
-8.60							-8.60
-8.50							-8.50
-8.40							-8.40
-8.30							-8.30
-8.20							-8.20
-8.10							-8.10
-8.00							-8.00
-7.90							-7.90
-7.80							-7.80
-7.70							-7.70
-7.60							-7.60
-7.50							-7.50
-7.40							-7.40
-7.30							-7.30
-7.20							-7.20
-7.10							-7.10
-7.00							-7.00
-6.90							-6.90
-6.80							-6.80
-6.70							-6.70
-6.60							-6.60
-6.50							-6.50
-6.40							-6.40
-6.30							-6.30
-6.20							-6.20
-6.10							-6.10
-6.00	.000001						-6.00
-5.90	.000002						-5.90
-5.80	.000004						-5.80
-5.70	.000006						-5.70
-5.60	.000010						-5.60
-5.50	.000016	.000002					-5.50
-5.40	.000026	.000003					-5.40
-5.30	.000042	.000006					-5.30
-5.20	.000067	.000012					-5.20
-5.10	.000105	.000021					-5.10
-5.00	.000164	.000038	.000003				-5.00

	SKEWNESS						
t	.6	7	.8	.9	1.0	1.1	t
-9.90							-9.90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9.10
-9.00							-9.00
-8.90							-8.90
-8.80							-8.80
-8.70							-8.70
-8.60							-8.60
-8.50							-8.50
-8.40							-8.40
-8.30							-8.30
-8.20							-8.20
-8.10							-8.10
-8.00							-8.00
-7.90							-7.90
-7.80							-7.80
-7.70							-7.70
-7.60							-7.60
-7.50							-7.50
-7.40							-7.40
-7.30							-7.30
-7.20							-7.20
-7.10							-7.10
-7.00							-7.00
-6.90							-6.90
-6.80							-6.80
-6.70							-6.70
-6.60							-6.60
-6.50							-6.50
-6.40							-6.40
-6.30							-6.30
-6.20							-6.20
-6.10							-6.10
-6.00							-6.00
-5.90							-5.90
-5.80							-5.80
-5.70							-5.70
-5.60							-5.60
-5.50							-5.50
-5.40							-5.40
-5.30							-5.30
-5.20							-5.20
-5.10							-5.10
-5.00							-5.00

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
-4 90	.000251	.000067	.000006				-4.90
-4.80	000381	.000116	.000014				-4.80
-4.70	.000572	000196	.000029				-4 70
-4 60	000847	.000326	.000060				-4.60
-4.50	.001241	.000531	.000120				-4.50
-4.40	.001795	.000850	.000230	000016			-4 40
-4 30	.002567	001336	000427	.000043			-4 30
-4.20	.003624	.002060	.000768	000109			-4.20
-4.10	.005054	.003119	001340	000255			-4.10
-4.00	006959	004637	002267	000563	000018		-4.00
-3.90	009460	006766	003724	.001168	000073		-3 90
-3 80	012692	009694	005942	002293	000251		-3 80
-3.70	.016801	.013632	009213	004267	.000740		-3 70
-3.60	.021940	018816	013889	007556	001917		-3.60
-3.50	.028253	.025486	020368	012760	004442	.000178	-3.50
-3 40	035863	.033867	.029060	.020595	.009325	.000949	-3.40
-3 30	044851	.044140	040347	031834	017934	.003585	-3 30
-3 20	055235	.056401	.054515	.047196	031871	.010507	-3.20
-3.10	066940	070617	071673	067197	052709	.025309	-3.10
-3 00	079773	.086578	091666	091968	.081588	052096	-3 00
-2.90	093389	.103848	.113982	121059	.118728	.094158	-2.90
-2.80	.107270	.121725	137680	.153273	162975	152364	-2.80
-2 70	120705	.139216	161342	.186568	.211485	223811	-2.70
-2 60	.132787	.155031	183074	.218055	263099	301307	-2 60
-2.50	.142417	.167610	.200569	.244130	.301543	.373931	-2 50
-2 40	.148341	.175189	211232	.260747	330213	428654	-2.40
-2 30	.149198	175896	.212378	263811	.338890	452620	-2.30
-2 20	.143601	167894	.201481	.249661	.321821	.435578	-2 20
-2 10	.130235	149545	.176464	215583	.275271	.371893	-2 10
-2 00	.107982	119605	.135996	160271	198270	.261753	-2 00
-1 90	076048	077409	.079756	084193	.093031	.111349	-1 90
-1.80	.034106	.023049	008637	-.010236	-035083	-.067918	-1 80
-1.70	-.017587	-042496	-.075148	-118678	-177972	-260996	-1 70
-1.60	-.078088	-117286	-.168089	-.235127	-.325592	-.451173	-1 60
-1.50	-145707	-198427	-265531	-.352360	-.466992	-.622148	-1.50
-1 40	-218003	-282164	-361956	-462545	-.591433	-759871	-1 40
-1.30	-.291841	-364068	-451376	-557937	-689459	-853914	-1.30
-1.20	-.363516	-.439306	-.527799	-631571	-753783	-898261	-1 20
-1.10	-428951	-.502977	-561923	-.677890	-.779897	-.891507	-1.10
-1.00	-.483941	-.550479	-.531037	-693231	-766350	-.836520	-1.00
- .90	-524454	-.577893	-.629285	-.676116	-714713	-.739705	- .90
- .80	-546938	-.582320	-.610227	-.627340	-.629237	-.610023	- .80
- 70	-548630	-.562167	-.563714	-.549840	-516310	-457901	- .70
- 60	-.527828	-.517322	-.491807	-448387	-.383756	-.294180	- .60
- 50	-484090	-.449217	-398197	-.329133	-.240099	-129206	- 50
- 40	-.418355	-360770	-.287898	-.199071	-093846	027898	- .40
- .30	-.332952	-256192	-166859	-065468	047137	169710	- .30
- 20	-231497	-140701	-041502	.064673	.176110	290779	- .20
- 10	-.118689	-.020144	081750	.185082	.287793	387675	- .10
.00	.000000	099416	.196913	290583	.378529	.458881	.00

t	SKEWNESS						t
	6	.7	8	9	1 0	1.1	
-4 90							-4.90
-4.80							-4.80
-4 70							-4 70
-4.60							-4.60
-4.50							-4.50
-4 40							-4 40
-4 30							-4 30
-4 20							-4.20
-4.10							-4 10
-4.00							-4.00
-3.90							-3.90
-3.80							-3 80
-3.70							-3 70
-3 60							-3 60
-3 50							-3.50
-3 40							-3.40
-3.30							-3 30
-3 20	.000428						-3.20
-3 10	.001108						-3.10
-3 00	.008908						-3.00
-2.90	034461						-2 90
-2 80	.091644						-2.80
-2 70	186439	037335					-2 70
-2 60	.311828	179031					-2.60
-2 50	447156	.424944					-2 50
-2 40	.563261	699116	412736				-2 40
-2 30	630460	.904484	1 154771				-2.30
-2.20	.626233	.968743	1 606671	1.308533			-2.20
-2 10	.540369	.864479	1 580220	3 463876			-2 10
-2 00	376624	607660	1 150250	2.823381			-2 00
-1 90	.151178	.244293	489790	1 298782	6 008392		-1 90
-1 80	- 111123	- 166161	- 226346	-.233321	.314604	26 296986	-1 80
-1.70	-.381520	- 565200	- 865030	-1.409161	-2.599172	-6.749225	-1 70
-1 60	- 631967	- 905133	-1 346380	-2.132746	-3.776760	-8.464687	-1.60
-1 50	- 838850	-1.154030	-1.637910	-2.439060	-3 924047	-7 246598	-1 50
-1 40	- 985394	-1 296626	-1.743088	-2.417049	-3 508310	-5.465547	-1.40
-1 30	-1 062686	-1 332423	-1 688081	-2 167842	-2.830276	-3 758660	-1 30
-1 20	-1 069482	-1 272258	-1.510562	-1 783859	-2.076035	-2.320434	-1 20
-1.10	-1 011123	-1.134368	-1.251236	-1.340054	-1.354128	-1 190123	-1 10
-1 00	- 897900	-.940660	-.948168	- 891892	- 721788	-.347336	-1 00
- .90	-.743216	-.713610	-.633538	- 476810	-.203287	249654	- 90
- 80	-.561803	-.473971	-.332221	- 117191	197402	.647237	- 80
- .70	-.368188	-.239300	-.061661	176251	488027	.889156	- 70
- .60	- 175509	-.023216	.167459	.401253	.682370	1.013589	- .60
- .50	005280	.164776	.350128	.561253	.796593	1 052338	- 50
- .40	165834	.319034	.485754	663162	846984	1 031034	- 40
- 30	.300524	.437271	.576943	.715691	848660	.969782	- 30
- .20	.406299	.519904	.628401	.728128	814903	.883974	- .20
- 10	.482370	569371	646021	.709516	.756908	.785099	- 10
.00	529797	.589476	.636159	668143	.683784	681502	00

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
.00	000000	099416	.196913	290583	378529	458881	00
.10	.118689	212123	298828	.377333	446308	504560	10
.20	.231497	.312723	383442	.442932	.490655	526245	.20
.30	.332952	.396892	.447991	486405	512433	526493	.30
.40	.418355	.461478	.491064	.508096	.513583	508539	.40
.50	.484090	.504634	.512563	509471	496829	.475980	.50
.60	.527828	.525849	513571	.492878	465389	.432494	.60
.70	.548630	.525878	.496156	.461276	.422704	.381621	.70
80	.546938	.506577	.463119	417974	.372202	.326592	80
90	.524454	.470682	417735	366379	.317109	.270225	.90
1.00	.483941	.421543	.363493	.309789	260310	.214859	1 00
1.10	.428951	.362844	.303855	.251224	.204259	.162335	1.10
1 20	.363516	.298335	.242061	.193306	.150934	.114009	1.20
1.30	.291841	.231592	180974	.138185	101825	.070791	1 30
1 40	.218003	.165822	122976	.087512	.057959	033194	1 40
1 50	145707	.103725	069917	.042440	.019940	001404	1.50
1 60	.078088	047402	023097	.003664	-.011988	-024665	1.60
1 70	.017587	-.001667	-.016702	-.028531	-037884	-045300	1 70
1 80	-034106	-.042615	-.049166	-054198	-058037	-060934	1.80
1.90	-076048	-075129	-.074377	-.073658	-072905	-.072092	1.90
2 00	-107982	-.099370	-092732	-.087429	-.083057	-.079351	2.00
2.10	-130235	-115877	-101088	-.096161	-089123	-.083300	2 10
2.20	-.143601	-.125462	-.111534	-.100572	-.091752	-.084515	2.20
2.30	-.149198	-.129110	-113618	-.101402	-.091579	-.083537	2.30
2.40	-.148341	-.127891	-.111985	-.099375	-.089202	-.080863	2 40
2 50	-.142417	-.122878	-107484	-.095167	-085165	-.076931	2.50
2 60	-.132787	-.115092	-100896	-089387	-.079949	-.072122	2.60
2 70	-.120705	-.105455	-.092916	-.082565	-073964	-.066758	2 70
2.80	-.107270	-.094763	-.084134	-.075151	-.067554	-061105	2.80
2 90	-.093389	-.083674	-.075038	-067508	-060996	-.055373	2.90
3.00	-.079773	-072706	-066010	-.059925	-054507	-049728	3.00
3.10	-.066940	-.062242	-.057337	-.052617	-048251	-.044293	3.10
3.20	-.055235	-.052547	-049222	-.045738	-.042343	-039155	3 20
3 30	-044851	-.043785	-.041793	-039388	-036861	-.034374	3.30
3 40	-035863	-.036031	-.035120	-.033623	-031849	-.029982	3.40
3 50	-028253	-029301	-029225	-.028467	-.027326	-025994	3.50
3 60	-.021940	-023558	-024094	-023915	-.023291	-022409	3 60
3 70	-016801	-.018734	-.019688	-.019942	-.019729	-.019217	3.70
3.80	-012692	-014741	-015951	-.016514	-.016613	-.016396	3 80
3 90	-.009460	-011481	-012818	-.013583	-.013911	-.013924	3.90
4.00	-.006959	-.008854	-010220	-.011101	-.011586	-.011771	4.00
4.10	-.005054	-006762	-008087	-.009017	-.009601	-.009908	4.10
4.20	-.003624	-.005116	-.006352	-.007281	-.007917	-.008306	4.20
4.30	-.002567	-.003835	-.004954	-.005845	-006498	-.006936	4.30
4.40	-.001795	-002849	-003837	-.004667	-.005309	-005770	4 40
4 50	-001241	-.002098	-.002952	-.003707	-.004320	-.004783	4 50
4 60	-.000847	-.001532	-.002257	-002929	-.003500	-.003952	4.60
4 70	-.000572	-.001109	-.001714	-.002303	-.002824	-.003254	4.70
4 80	-.000381	-000797	-.001294	-.001802	-.002270	-002671	4.80
4 90	-.000251	-000567	-000971	-.001403	-001818	-.002185	4 90

t	SKEWNESS						t
	6	7	8	9	1 0	1.1	
00	529797	.589476	636159	668143	.683784	681502	00
10	551037	584817	605109	611252	602702	579038	.10
.20	549488	560304	558735	.544926	519119	481635	20
30	529089	520796	.502242	474090	437027	391754	30
40	493963	.470827	440067	402580	359212	310764	40
50	448147	.414435	375840	333257	287486	239240	50
60	395381	355065	312408	268145	.222899	177193	60
70	338975	295532	.251902	208570	165921	124257	70
80	.281727	238034	195818	155291	116596	079819	80
.90	.225895	.184190	145114	108629	.074664	.043126	90
1.00	173207	135106	100308	068570	039660	013358	1 00
1.10	124896	091451	.061567	.034861	010997	-.010321	1 10
1 20	.081756	.053530	.028789	007081	-011981	-028724	1.20
1 30	.044204	021359	001684	-015290	-029950	-042616	1 30
1 40	012349	-005260	-020175	-032830	-.043577	-052703	1.40
1 50	-013939	-.026683	-037291	-.046132	-053497	-059624	1.50
1 60	-.034971	-043367	-.050211	-055782	-060300	-063943	1 60
1 70	-051180	-.055831	-059489	-062338	-.064522	-066158	1 70
1 80	-063078	-.064616	-065666	-.066316	-066641	-066699	1 80
1 90	-.071211	-070262	-069251	-068187	-067079	-065933	1 90
2 00	-.076134	-073284	-.070716	-.068369	-066198	-064171	2 00
2 10	-078386	-074165	-.070483	-067227	-064311	-061674	2 10
2 20	-.078470	-073341	-.068926	-065077	-061682	-058657	2 20
2 30	-.076847	-071200	-.066369	-062188	-058529	-.055295	2 30
2 40	-073926	-.068078	-063089	-.058785	-055032	-051731	2 40
2 50	-.070062	-064265	-.059318	-055052	-051338	-048078	2 50
2.60	-.065560	-060001	-055245	-051139	-047563	-044424	2 60
2 70	-060671	-055484	-051027	-047167	-043798	-040837	2 70
2 80	-.055600	-.050872	-046785	-043228	-040113	-037366	2.80
2.90	-050510	-046291	-.042614	-.039394	-036558	-034049	2 90
3.00	-045527	-041835	-.038585	-035715	-.033172	-.030910	3 00
3 10	-040742	-037573	-034748	-.032229	-.029979	-.027964	3 10
3 20	-036222	-033553	-.031138	-028959	-026994	-025220	3 20
3 30	-032010	-.029807	-027777	-.025918	-.024223	-022679	3.30
3 40	-028130	-026350	-024674	-.023113	-021670	-020340	3 40
3.50	-024591	-.023190	-021833	-020542	-019329	-018198	3.50
3 60	-021394	-.020323	-019249	-.018200	-017196	-.016243	3 60
3 70	-018527	-017741	-016913	-016078	-015259	-.014468	3 70
3 80	-015976	-015430	-014814	-014165	-013509	-012861	3 80
3 90	-013720	-.013373	-.012936	-012447	-.011933	-011411	3 90
4.00	-011738	-.011552	-011264	-.010911	-010519	-010107	4 00
4 10	-010005	-.009948	-009783	-009543	-.009255	-.008937	4.10
4.20	-.008499	-008542	-.008475	-008328	-008127	-007890	4.20
4.30	-.007196	-.007314	-.007324	-.007253	-007125	-.006955	4 30
4 40	-.006073	-006246	-006315	-.006305	-006236	-006123	4 40
4.50	-005110	-005320	-.005433	-.005471	-005449	-.005383	4.50
4 60	-.004288	-.004520	-.004665	-.004738	-004754	-004726	4 60
4.70	-.003588	-003832	-.003998	-004097	-.004142	-004145	4.70
4.80	-002994	-.003241	-003419	-003537	-003604	-.003631	4 80
4.90	-.002492	-002736	-.002919	-003049	-.003132	-003177	4 90

t	SKEWNESS 0	1	2	.3	.4	.5	t
5.00	-.000164	-.000401	-.000724	-.001088	-.001450	-.001783	5.00
5.10	-.000105	-.000281	-.000537	-.000840	-.001153	-.001450	5.10
5.20	-.000067	-.000196	-.000396	-.000645	-.000914	-.001177	5.20
5.30	-.000042	-.000135	-.000291	-.000494	-.000722	-.000952	5.30
5.40	-.000026	-.000092	-.000212	-.000377	-.000568	-.000768	5.40
5.50	-.000016	-.000063	-.000154	-.000286	-.000446	-.000619	5.50
5.60	-.000010	-.000042	-.000111	-.000216	-.000349	-.000497	5.60
5.70	-.000006	-.000028	-.000080	-.000163	-.000272	-.000398	5.70
5.80	-.000004	-.000019	-.000057	-.000122	-.000212	-.000318	5.80
5.90	-.000002	-.000013	-.000041	-.000091	-.000164	-.000254	5.90
6.00	-.000001	-.000008	-.000029	-.000068	-.000127	-.000202	6.00
6.10		-.000005	-.000020	-.000051	-.000098	-.000160	6.10
6.20		-.000003	-.000014	-.000037	-.000075	-.000127	6.20
6.30		-.000002	-.000010	-.000028	-.000058	-.000101	6.30
6.40		-.000001	-.000007	-.000020	-.000044	-.000079	6.40
6.50		-.000001	-.000005	-.000015	-.000034	-.000063	6.50
6.60			-.000003	-.000011	-.000026	-.000049	6.60
6.70			-.000002	-.000008	-.000020	-.000039	6.70
6.80			-.000002	-.000006	-.000015	-.000030	6.80
6.90			-.000001	-.000004	-.000011	-.000024	6.90
7.00			-.000001	-.000003	-.000009	-.000018	7.00
7.10				-.000002	-.000006	-.000014	7.10
7.20				-.000002	-.000005	-.000011	7.20
7.30				-.000001	-.000004	-.000009	7.30
7.40				-.000001	-.000003	-.000007	7.40
7.50				-.000001	-.000002	-.000005	7.50
7.60					-.000002	-.000004	7.60
7.70					-.000001	-.000003	7.70
7.80					-.000001	-.000002	7.80
7.90					-.000001	-.000002	7.90
8.00						-.000001	8.00
8.10						-.000001	8.10
8.20						-.000001	8.20
8.30						-.000001	8.30
8.40							8.40
8.50							8.50
8.60							8.60
8.70							8.70
8.80							8.80
8.90							8.90
9.00							9.00
9.10							9.10
9.20							9.20
9.30							9.30
9.40							9.40
9.50							9.50
9.60							9.60
9.70							9.70
9.80							9.80
9.90							9.90

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
5.00	-.002069	-.002304	-.002488	-.002624	-.002719	-.002777	5.00
5.10	-.001714	-.001937	-.002117	-.002256	-.002357	-.002425	5.10
5.20	-.001417	-.001625	-.001798	-.001936	-.002041	-.002116	5.20
5.30	-.001169	-.001361	-.001526	-.001660	-.001765	-.001844	5.30
5.40	-.000962	-.001138	-.001292	-.001421	-.001525	-.001606	5.40
5.50	-.000790	-.000950	-.001093	-.001215	-.001317	-.001397	5.50
5.60	-.000647	-.000792	-.000923	-.001038	-.001135	-.001215	5.60
5.70	-.000530	-.000659	-.000779	-.000886	-.000978	-.001055	5.70
5.80	-.000432	-.000547	-.000656	-.000755	-.000842	-.000916	5.80
5.90	-.000353	-.000454	-.000552	-.000643	-.000724	-.000795	5.90
6.00	-.000287	-.000376	-.000464	-.000547	-.000622	-.000689	6.00
6.10	-.000233	-.000311	-.000389	-.000465	-.000534	-.000597	6.10
6.20	-.000189	-.000257	-.000326	-.000395	-.000459	-.000517	6.20
6.30	-.000153	-.000212	-.000273	-.000335	-.000393	-.000447	6.30
6.40	-.000124	-.000174	-.000229	-.000284	-.000337	-.000387	6.40
6.50	-.000100	-.000144	-.000191	-.000240	-.000288	-.000334	6.50
6.60	-.000080	-.000118	-.000160	-.000203	-.000247	-.000289	6.60
6.70	-.000065	-.000097	-.000133	-.000172	-.000211	-.000249	6.70
6.80	-.000052	-.000079	-.000111	-.000145	-.000180	-.000215	6.80
6.90	-.000042	-.000065	-.000092	-.000123	-.000154	-.000185	6.90
7.00	-.000033	-.000053	-.000077	-.000103	-.000131	-.000160	7.00
7.10	-.000027	-.000044	-.000064	-.000087	-.000112	-.000138	7.10
7.20	-.000021	-.000036	-.000053	-.000073	-.000095	-.000118	7.20
7.30	-.000017	-.000029	-.000044	-.000062	-.000081	-.000102	7.30
7.40	-.000014	-.000024	-.000037	-.000052	-.000069	-.000088	7.40
7.50	-.000011	-.000019	-.000030	-.000044	-.000059	-.000075	7.50
7.60	-.000009	-.000016	-.000025	-.000037	-.000050	-.000065	7.60
7.70	-.000007	-.000013	-.000021	-.000031	-.000043	-.000056	7.70
7.80	-.000005	-.000010	-.000017	-.000026	-.000036	-.000048	7.80
7.90	-.000004	-.000008	-.000014	-.000022	-.000031	-.000041	7.90
8.00	-.000003	-.000007	-.000012	-.000018	-.000026	-.000035	8.00
8.10	-.000003	-.000005	-.000010	-.000015	-.000022	-.000030	8.10
8.20	-.000002	-.000004	-.000008	-.000013	-.000019	-.000026	8.20
8.30	-.000002	-.000004	-.000007	-.000011	-.000016	-.000022	8.30
8.40	-.000001	-.000003	-.000005	-.000009	-.000014	-.000019	8.40
8.50	-.000001	-.000002	-.000004	-.000007	-.000011	-.000016	8.50
8.60	-.000001	-.000002	-.000004	-.000006	-.000010	-.000014	8.60
8.70	-.000001	-.000002	-.000003	-.000005	-.000008	-.000012	8.70
8.80	-.000001	-.000001	-.000002	-.000004	-.000007	-.000010	8.80
8.90		-.000001	-.000002	-.000004	-.000006	-.000009	8.90
9.00		-.000001	-.000002	-.000003	-.000005	-.000008	9.00
9.10		-.000001	-.000001	-.000003	-.000004	-.000006	9.10
9.20		-.000001	-.000001	-.000002	-.000004	-.000005	9.20
9.30			-.000001	-.000002	-.000003	-.000005	9.30
9.40			-.000001	-.000001	-.000003	-.000004	9.40
9.50			-.000001	-.000001	-.000002	-.000003	9.50
9.60			-.000001	-.000001	-.000002	-.000003	9.60
9.70				-.000001	-.000002	-.000002	9.70
9.80				-.000001	-.000001	-.000002	9.80
9.90				-.000001	-.000001	-.000002	9.90

	SKEWNESS						
t	0	1	.2	.3	.4	.5	t
10.00							10.00
10.10							10.10
10.20							10.20
10.30							10.30
10.40							10.40
10.50							10.50
10.60							10.60
10.70							10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

t	SKEWNESS						t
	6	7	8	9	10	11	
10.00					-000001	-000002	10.00
10.10					-000001	-000001	10.10
10.20					-000001	-000001	10.20
10.30					-000001	-000001	10.30
10.40						-000001	10.40
10.50						-000001	10.50
10.60						-000001	10.60
10.70						-000001	10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

t	SKEWNESS						t
	.0	.1	.2	.3	.4	.5	
-9.90							-9.90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9.10
-9.00							-9.00
-8.90							-8.90
-8.80							-8.80
-8.70							-8.70
-8.60							-8.60
-8.50							-8.50
-8.40							-8.40
-8.30							-8.30
-8.20							-8.20
-8.10							-8.10
-8.00							-8.00
-7.90							-7.90
-7.80							-7.80
-7.70							-7.70
-7.60							-7.60
-7.50							-7.50
-7.40							-7.40
-7.30							-7.30
-7.20							-7.20
-7.10							-7.10
-7.00							-7.00
-6.90							-6.90
-6.80							-6.80
-6.70							-6.70
-6.60							-6.60
-6.50							-6.50
-6.40							-6.40
-6.30							-6.30
-6.20							-6.20
-6.10							-6.10
-6.00	.000007						-6.00
-5.90	.000011						-5.90
-5.80	.000018						-5.80
-5.70	.000030						-5.70
-5.60	.000049						-5.60
-5.50	.000079	.000011					-5.50
-5.40	.000126	.000021					-5.40
-5.30	.000198	.000040					-5.30
-5.20	.000307	.000071					-5.20
-5.10	.000470	.000126					-5.10
-5.00	.000711	.000219	.000023				-5.00

t	SKEWNESS						t
	6	7	.8	.9	10	11	
-9.90							-9.90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9.10
-9.00							-9.00
-8.90							-8.90
-8.80							-8.80
-8.70							-8.70
-8.60							-8.60
-8.50							-8.50
-8.40							-8.40
-8.30							-8.30
-8.20							-8.20
-8.10							-8.10
-8.00							-8.00
-7.90							-7.90
-7.80							-7.80
-7.70							-7.70
-7.60							-7.60
-7.50							-7.50
-7.40							-7.40
-7.30							-7.30
-7.20							-7.20
-7.10							-7.10
-7.00							-7.00
-6.90							-6.90
-6.80							-6.80
-6.70							-6.70
-6.60							-6.60
-6.50							-6.50
-6.40							-6.40
-6.30							-6.30
-6.20							-6.20
-6.10							-6.10
-6.00							-6.00
-5.90							-5.90
-5.80							-5.80
-5.70							-5.70
-5.60							-5.60
-5.50							-5.50
-5.40							-5.40
-5.30							-5.30
-5.20							-5.20
-5.10							-5.10
-5.00							-5.00

t	SKEWNESS						t
	.0	.1	.2	3	.4	.5	
-4 90	.001062	000373	000050				-4 90
-4.80	001567	000621	000106				-4.80
-4.70	.002283	.001014	000216				-4.70
-4 60	.003283	001623	.000424				-4 60
-4.50	.004660	.002547	000801				-4.50
-4 40	006526	.003919	.001461	000165			-4 40
-4.30	009015	005910	002576	.000414			-4 30
-4.20	.012280	008735	004392	000965			-4 20
-4.10	.016488	.012652	007249	002098			-4 10
-4.00	.021814	.017953	011585	.004275	.000270		-4.00
-3.90	.028424	.024952	.017935	008198	000964		-3.90
-3.80	.036456	.033952	.026904	014839	002892		-3 80
-3 70	.045994	.045210	039109	.025429	007498		-3.70
-3.60	057030	058872	.055089	011352	.017134		-3 60
-3.50	.069433	074908	075174	.063929	035043	003363	-3 50
-3.40	.082896	.093028	099316	094085	064923	.014067	-3 40
-3.30	.096898	.112599	126916	131914	109964	042678	-3 30
-3 20	.110664	132568	156637	.176212	171459	101860	-3.20
-3.10	.123133	.151405	.186283	224065	247286	201253	-3 10
-3.00	.132955	.167089	212736	.270631	.330741	340245	-3.00
-2.90	138504	.177145	232029	309206	.410203	502693	-2 90
-2.80	.137931	.178748	239565	331680	470001	.656676	-2.80
-2.70	129262	.168916	.230499	329387	.492520	.760190	-2.70
-2 60	110533	.144769	200274	294297	468964	771242	-2 60
-2 50	.079973	.103864	.145268	.220393	.364364	659133	-2 50
-2.40	.036225	.044574	.063483	.105036	.197539	413477	-2.40
-2 30	-.021415	-033521	-.044820	-049946	-033760	.048415	-2 30
-2 20	-.092745	-129292	-176631	-237650	-313972	-398860	-2 20
-2.10	-176458	-239792	-325968	-446158	-618703	-874547	-2 10
-2.00	-.269955	-360149	-483969	-659320	-917573	-1 317793	-2.00
-1.90	-.369279	-.483634	-639366	-.858210	-1 178217	-1 671021	-1.90
-1 80	-469154	-601945	-779314	-1.023098	-1 370447	-1.888901	-1 80
-1 70	-.563157	-705722	-890510	-1.135663	-1 470408	-1.944452	-1.70
-1 60	-644051	-.785244	-960510	-1 181176	-1 463555	-1.831597	-1 60
-1.50	-704252	-.831283	-979070	-1 150310	-1 346367	-1 564257	-1.50
-1 40	-736420	-.836023	-939371	-1.040370	-1 126539	-1 172695	-1 40
-1 30	-734126	-793947	-838954	-.855748	-821731	-698106	-1.30
-1 20	-692545	-702593	-680247	-607581	-.457192	-186561	-1 20
-1 10	-609093	-.563089	-451878	-.312649	-062669	316737	-1.10
-1.00	-.483941	-380366	-221798	.008309	330924	.772172	-1 00
- .90	-320340	-163025	.050847	.332640	.694644	1 149001	- .90
- 80	-124683	.077163	.329834	.637935	1 004102	1 427081	- 80
- 70	.093707	326094	596972	904033	1 241282	1.597145	- .70
- 60	.323095	.568419	.834958	1.114676	1 395475	1 659900	- 60
- 50	.550102	.788781	1 028829	1.258656	1.463382	1 624302	- .50
- .40	.760699	973094	1 167154	1.330391	1 448486	1 505356	- .40
- 30	.941303	1.109698	1 208438	1 329919	1.359877	1 321796	- .30
- 20	1.079904	1 190311	1 253522	1.262385	1.210756	1.093886	- 20
- .10	1 167080	1.210653	1 201474	1.137122	1 016802	841546	- 10
00	1 196827	1 170712	1 093148	.966479	.794593	.582903	.00

t	SKEWNESS						t
	6	7	8	9	1 0	1.1	
-4.90							-4 90
-4 80							-4 80
-4 70							-4 70
-4 60							-4 60
-4 50							-4 50
-4 40							-4 40
-4 30							-4 30
-4 20							-4.20
-4.10							-4 10
-4 00							-4 00
-3.90							-3 90
-3 80							-3 80
-3 70							-3 70
-3 60							-3 60
-3 50							-3 50
-3 40							-3 40
-3 30							-3 30
-3 20	020824						-3 20
-3 10	028526						-3 10
-3 00	147081						-3 00
-2 90	395781						-2 90
-2 80	.759352						-2 80
-2.70	1 124906	812211					-2.70
-2 60	1.347521	2.017497					-2 60
-2 50	1.309026	2 763730					-2 50
-2 40	962273	2 549975	7 177420				-2 40
-2 30	343284	1 433681	6 605834				-2 30
-2 20	- 445103	-.193679	2 155290	47 637462			-2 20
-2 10	-1 265630	-1 862554	-2 531723	3 046070			-2 10
-2 00	-1 981650	-3 194250	-5 763751	-12.89341			-2 00
-1 90	-2 485037	-3 971531	-7 147354	-16 21020	-75 41929		-1 90
-1.80	-2 712264	-4 138399	-6.952061	-13 86402	-40 95566	-1703 342	-1 80
-1 70	-2 648307	-3 762105	-5 690392	-9 514587	-19 03718	-55 10019	-1 70
-1 60	-2 320871	-2 983216	-3 883787	-5 027707	-5 674726	4 547146	-1 60
-1 50	-1 788663	-1 969634	-1 954736	-1 253452	1 962024	16 717548	-1 50
-1 40	-1 127406	- 882058	- 194341	1 521847	5 860034	17 978712	-1 40
-1.30	-.416546	147390	1 230284	3 306926	7 396594	15 872761	-1 30
-1 20	271442	1 024888	2 251319	4 247538	7 511627	12 840068	-1 20
-1.10	878276	1 695880	2 871243	4 536417	6 834117	9 805846	-1.10
-1 00	1.363543	2.140438	3 136250	4 364950	5 774305	7.121962	-1 00
- 90	1 705324	2 365735	3 114882	3.899959	4 590501	4.896272	- 90
- .80	1 898524	2 397668	2 882507	3 275111	3 437122	3 128725	- 80
- .70	1.951749	2.273015	2 511075	2 590410	2 398938	1 773397	- 70
- 60	1 883508	2 032867	2 063195	1 915738	1.515225	.768222	- 60
- 50	1 718347	1 717661	1 589403	1 296049	.796593	049368	- .50
- 40	1 483373	1 363810	1 127644	756864	236529	-.442085	- 40
- 30	1 205400	1 001743	704131	309331	- 180839	-.757437	- 30
- 20	908852	655086	334973	- 045445	-.475651	-.939707	- .20
- 10	614408	.340679	028099	- 312925	- 669095	-1 023989	- 10
00	338310	.069165	- 214782	- 502446	-.781467	-1 038291	00

	SKEWNESS						
t	.0	.1	2	.3	4	.5	t
00	1 196827	1 170712	1 093148	.966479	.794593	.582903	.00
.10	1 167080	1 074614	938367	764534	560223	333295	10
20	1.079904	930144	.749299	.545828	.328197	104721	20
.30	.941303	.747971	.539336	324239	110656	-094324	30
40	.760699	.540662	321978	.112068	-083069	-258707	40
.50	.550102	321613	.109835	-080603	-.246603	-.386303	.50
.60	.323095	.103981	-.086178	-246284	-.376418	-477487	.60
.70	093707	-.100271	-257417	-.380193	-.471536	-534582	.70
.80	-.124683	-.281142	-397824	-480139	-533110	-561286	.80
.90	-.320340	-.431125	-503997	-.546236	-.563934	-562147	.90
1.00	-.483941	-545473	-575047	-580507	-567948	-542106	1 00
1.10	-.609093	-.622220	-.612288	-586411	-.549764	-.506123	1 10
1.20	-692545	-.661971	-.618802	-.568366	-514257	-.458890	1.20
1.30	-734126	-667522	-.598957	-.531300	-466219	-.404633	1.30
1 40	-.736420	-.643356	-.557907	-.480250	-410109	-.346993	1 40
1 50	-704252	-595087	-501121	-420037	-.349867	-288967	1.50
1.60	-.644051	-.528901	-.433987	-355027	-288813	-232901	1.60
1.70	-563157	-451048	-.361482	-288969	-229601	-.180529	1 70
1.80	-469154	-367406	-287948	-224909	-.174230	-.133032	1.80
1.90	-369279	-.283159	-216948	-.165165	-124086	-091107	1.90
2.00	-269955	-.202581	-151207	-111359	-080012	-.055060	2.00
2.10	-176458	-128927	-.095701	-064481	-.042394	-024879	2.10
2.20	-.092745	-.064425	-.042334	-.024976	-.011244	-.000317	2 20
2.30	-021415	-010335	-000806	.007156	.013705	019042	2.30
2.40	.036225	.032924	.032034	.032252	.032923	.033717	2 40
2.50	.079973	.065624	.056680	.050897	.047011	.044288	2 50
2.60	.110533	088555	073917	063826	056636	051348	2.60
2.70	.129262	.102860	.084711	.071857	062491	.055483	2 70
2.80	.137931	.109882	.090113	.075826	.065250	057241	2 80
2.90	.138504	.111035	091177	.076544	065547	.057123	2.90
3 00	.132955	107697	088906	074760	063956	.055574	3.00
3.10	.123133	101141	.084210	071145	.060979	.052981	3 10
3 20	.110664	092480	077881	066276	057048	.049668	3 20
3.30	.096898	082650	070581	060636	052522	045906	3 30
3 40	.082896	.072394	.062845	054615	.047688	.041910	3.40
3 50	.069433	.062277	.055085	048518	.042774	.037852	3 50
3.60	.057030	052701	.047604	.042570	.037951	033860	3 60
3.70	.045994	043925	.040610	.036935	.033342	.030027	3.70
3.80	036456	.036097	.034233	.031718	.029031	.026420	3 80
3.90	.028424	029271	.028537	.026980	.025070	.023080	3.90
4.00	.021814	023440	.023543	022747	021485	020029	4.00
4.10	016488	018546	.019233	.019020	018283	017274	4.10
4.20	012280	014507	.015566	.015781	.015455	.014813	4.20
4.30	009015	.011224	.012487	012997	012982	012634	4.30
4.40	006526	008592	009933	.010630	010842	.010722	4.40
4.50	.004660	006510	.007837	.008636	.009004	.009056	4.50
4 60	.003283	004884	.006136	006972	.007438	007615	4.60
4.70	002283	003629	004768	.005594	.006113	.006375	4.70
4.80	.001567	002671	.003678	.004463	005000	005316	4.80
4.90	.001062	.001948	002818	.003541	.004071	.004416	4.90

t	SKEWNESS						t
	.6	.7	8	.9	1.0	1.1	
.00	.338310	069165	-.214782	-.502446	-.781467	-1.038291	.00
10	.092212	- 154102	- 396402	-.625246	- 831115	-1 004521	.10
.20	- 116564	- 328001	-.522394	-.693075	- 833942	- 939499	.20
.30	- 284508	- 454578	-.600057	-.717270	-.803289	- 855884	.30
.40	- 411246	- 538066	- 637395	- 708191	-.750026	-.762993	.40
.50	-.498817	- 584031	- 642421	- 674916	- 682779	-.667508	.50
.60	-.550963	- 598678	- 622683	-.625131	-.608205	-.574058	60
70	-.492063	- 588315	- 584965	-.565132	- 531294	- 485703	.70
80	-.568715	- 558962	- 535134	- 499924	-.455656	- 404319	80
90	- 545052	- 516090	- 478086	- 433353	-.383781	- 330911	.90
1 00	-.506662	- 464476	-.417768	- 368259	- 317280	- 265855	1.00
1 10	-.458242	- 408129	- 357245	-.306646	- 257083	-.251529	1.10
1.20	-.403880	-.350288	- 298798	- 249826	-.203605	- 160241	1.20
1 30	- 346995	- 293466	- 244031	-.198567	- 156889	- 118778	1 30
1 40	- 290326	- 239514	- 193988	- 153216	- 116712	-.084036	1.40
1 50	- 235962	-.189710	- 149257	- 113803	-.082677	- 055313	1.50
1 60	- 185405	- 144850	- 110070	-.080131	- 054280	- 031903	1 60
1 70	- 139641	- 105337	-.076391	- 051849	-.030957	-.013117	1.70
1 80	- 099226	-.071267	- 047991	- 028508	-.012128	.001690	1.80
1 90	-.064367	- 042504	- 024507	- 009608	002780	.013113	1 90
2 00	- 035003	- 018750	- 005495	005372	.014313	021689	2 00
2 10	- 010873	.000404	.009531	016944	.022980	.027897	2 10
2.20	.008421	.015432	021068	.025602	029244	032158	2.20
2.30	.023361	026834	029607	031802	.033518	034838	2.30
2 40	.034472	.035112	035610	035960	036170	036251	2.40
2 50	042290	.040751	039506	038450	037517	.036664	2.50
2 60	.047337	.044203	.041681	039598	037833	036305	2.60
2.70	.050105	.045878	042480	039690	037353	035361	2.70
2 80	051042	046145	042202	038969	036272	033988	2.80
2 90	050547	045324	041104	037639	034753	032315	2.90
3 00	.048971	043689	039402	035873	032930	.030444	3.00
3.10	.046611	041474	037279	033811	030911	.028459	3.10
3.20	043716	038869	.034881	031567	028784	026424	3 20
3.30	.040491	036031	032329	029231	026616	.024391	3 30
3.40	.037099	033084	029716	026875	.024461	022396	3 40
3.50	.033670	.030124	.027115	024552	022358	020470	3 50
3.60	.030299	027226	.024581	022303	.020336	.018631	3 60
3 70	027059	024443	.022154	.020157	018416	016894	3 70
3.80	.024000	021812	019861	018134	016610	015266	3.80
3.90	021152	019356	017719	016245	.014927	.013752	3.90
4 00	.018534	.017090	015738	014497	.013370	012353	4.00
4.10	.016154	015018	013921	001289	.011938	011067	4 10
4.20	.014009	.013141	.012268	.011425	.010629	.009890	4 20
4 30	012093	011452	.010773	010095	009439	008819	3.30
4.40	.010393	.009942	009429	008894	.008361	.007847	4.40
4 50	008896	008600	008227	.007815	.007390	006968	4.50
4 60	.007585	007415	.007158	006850	006517	006176	4.60
4.70	006443	006373	006211	005990	.005736	005465	4.70
4.80	.005455	.005462	.005375	.005226	.005038	004828	4 80
4.90	004603	004667	.004641	004551	.004418	004258	4.90

t	SKEWNESS						t
	0	.1	.2	.3	4	.5	
5.00	.000711	.001408	002144	002794	003300	003654	5.00
5.10	.000470	001009	.001621	.002193	002664	003014	5.10
5.20	.000307	.000717	001217	.001713	002142	002477	5.20
5.30	.000198	000505	000909	001331	001715	002029	5.30
5.40	.000126	.000353	.000674	001030	001368	.001657	5.40
5.50	.000079	.000245	000497	000793	.001087	.001349	5.50
5.60	.000049	.000168	000365	000608	.000861	.001095	5.60
5.70	.000030	000115	000266	.000464	000680	.000887	5.70
5.80	000018	.000078	.000193	000352	.000535	.000716	5.80
5.90	000011	000052	000139	000267	000419	000577	5.90
6.00	000007	000035	.000100	.000202	000328	.000463	6.00
6.10		000023	.000071	.000151	.000256	.000371	6.10
6.20		.000015	.000051	000113	000199	000297	6.20
6.30		.000010	.000036	000084	.000154	.000237	6.30
6.40		.000006	.000025	000063	.000119	.000188	6.40
6.50		000004	000018	.000046	.000092	000150	6.50
6.60			000012	000034	000070	.000119	6.60
6.70			.000009	.000025	000054	.000094	6.70
6.80			000006	.000018	.000041	.000074	6.80
6.90			.000004	000014	000032	.000058	6.90
7.00			000003	000010	000024	.000046	7.00
7.10			.000002	.000007	000018	000036	7.10
7.20			000001	.000005	000014	000028	7.20
7.30			.000001	.000004	000010	000022	7.30
7.40			.000001	.000003	000008	.000017	7.40
7.50				000002	000006	000013	7.50
7.60				000001	.000004	.000010	7.60
7.70				.000001	.000003	000008	7.70
7.80				.000001	000003	000006	7.80
7.90					.000002	.000005	7.90
8.00					000001	000004	8.00
8.10					.000001	.000003	8.10
8.20					.000001	.000002	8.20
8.30					.000001	000002	8.30
8.40						000001	8.40
8.50						000001	8.50
8.60						000001	8.60
8.70						000001	8.70
8.80							8.80
8.90							8.90
9.00							9.00
9.10							9.10
9.20							9.20
9.30							9.30
9.40							9.40
9.50							9.50
9.60							9.60
9.70							9.70
9.80							9.80
9.90							9.90

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
5.00	.003872	.003977	.003997	.003954	.003867	.003750	5.00
5.10	.003247	.003381	.003436	.003430	.003380	.003299	5.10
5.20	.002715	.002867	.002947	.002970	.002949	.002897	5.20
5.30	.002265	.002426	.002522	.002567	.002570	.002542	5.30
5.40	.001884	.002048	.002155	.002215	.002236	.002227	5.40
5.50	.001563	.001725	.001838	.001908	.001943	.001950	5.50
5.60	.001294	.001450	.001565	.001642	.001686	.001705	5.60
5.70	.001068	.001217	.001330	.001410	.001462	.001489	5.70
5.80	.000880	.001019	.001129	.001210	.001266	.001299	5.80
5.90	.000724	.000852	.000956	.001037	.001095	.001133	5.90
6.00	.000594	.000711	.000809	.000887	.000946	.000987	6.00
6.10	.000486	.000592	.000684	.000758	.000816	.000859	6.10
6.20	.000397	.000493	.000577	.000648	.000704	.000747	6.20
6.30	.000324	.000409	.000486	.000552	.000606	.000649	6.30
6.40	.000264	.000339	.000409	.000470	.000522	.000563	6.40
6.50	.000215	.000281	.000344	.000400	.000449	.000489	6.50
6.60	.000174	.000232	.000289	.000340	.000386	.000424	6.60
6.70	.000141	.000192	.000242	.000289	.000331	.000367	6.70
6.80	.000114	.000158	.000203	.000245	.000284	.000318	6.80
6.90	.000092	.000130	.000170	.000208	.000244	.000275	6.90
7.00	.000074	.000107	.000142	.000176	.000209	.000238	7.00
7.10	.000060	.000088	.000118	.000149	.000179	.000205	7.10
7.20	.000048	.000072	.000099	.000126	.000153	.000177	7.20
7.30	.000039	.000059	.000082	.000107	.000131	.000153	7.30
7.40	.000031	.000048	.000069	.000090	.000112	.000132	7.40
7.50	.000025	.000040	.000057	.000076	.000095	.000114	7.50
7.60	.000020	.000032	.000047	.000064	.000081	.000098	7.60
7.70	.000016	.000026	.000039	.000054	.000069	.000085	7.70
7.80	.000013	.000022	.000033	.000045	.000059	.000073	7.80
7.90	.000010	.000018	.000027	.000038	.000050	.000063	7.90
8.00	.000008	.000014	.000022	.000032	.000043	.000054	8.00
8.10	.000006	.000012	.000019	.000027	.000036	.000046	8.10
8.20	.000005	.000009	.000015	.000023	.000031	.000040	8.20
8.30	.000004	.000008	.000013	.000019	.000026	.000034	8.30
8.40	.000003	.000006	.000011	.000016	.000022	.000029	8.40
8.50	.000002	.000005	.000009	.000013	.000019	.000025	8.50
8.60	.000002	.000004	.000007	.000011	.000016	.000022	8.60
8.70	.000002	.000003	.000006	.000009	.000014	.000019	8.70
8.80	.000001	.000003	.000005	.000008	.000012	.000016	8.80
8.90	.000001	.000002	.000004	.000007	.000010	.000014	8.90
9.00	.000001	.000002	.000003	.000006	.000008	.000012	9.00
9.10	.000001	.000001	.000003	.000005	.000007	.000010	9.10
9.20		.000001	.000002	.000004	.000006	.000009	9.20
9.30		.000001	.000002	.000003	.000005	.000007	9.30
9.40		.000001	.000002	.000003	.000004	.000006	9.40
9.50		.000001	.000001	.000002	.000004	.000005	9.50
9.60			.000001	.000002	.000003	.000005	9.60
9.70			.000001	.000002	.000003	.000004	9.70
9.80			.000001	.000001	.000002	.000003	9.80
9.90			.000001	.000001	.000002	.000003	9.90

t	SKEWNESS						t
	.6	.7	8	9	1.0	1.1	
10.00							10.00
10.10							10.10
10.20							10.20
10.30							10.30
10.40							10.40
10.50							10.50
10.60							10.60
10.70							10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

t	SKEWNESS						t
	6	.7	8	.9	1 0	1.1	
10.00				.000001	000002	000002	10 00
10.10				.000001	000001	000002	10.10
10 20				000001	.000001	000002	10.20
10 30				000001	000001	.000002	10.30
10 40					000001	.000001	10.40
10.50					.000001	000001	10.50
10 60					.000001	000001	10 60
10.70						.000001	10.70
10 80						000001	10 80
10 90						000001	10 90
11 00						.000001	11 00
11 10							11.10
11.20							11 20
11 30							11 30
11 40							11.40
11.50							11.50
11 60							11.60
11 70							11.70
11 80							11.80
11.90							11 90
12.00							12 00
12 10							12.10
12 20							12 20
12 30							12 30
12.40							12 40
12 50							12.50
12.60							12 60
12.70							12.70
12 80							12.80
12 90							12.90
13 00							13.00
13.10							13 10
13 20							13.20
13.30							13 30
13.40							13.40
13.50							13 50
13.60							13 60
13.70							13.70
13 80							13 80
13 90							13.90
14 00							14.00
14 10							14 10
14.20							14 20
14.30							14.30
14.40							14 40
14 50							14.50
14 60							14.60
14.70							14.70
14 80							14.80
14.90							14.90

t	SKEWNESS						t
	.0	.1	2	.3	.4	.5	
-9.90							-9 90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9 10
-9.00							-9 00
-8.90							-8.90
-8 80							-8 80
-8 70							-8.70
-8 60							-8.60
-8.50							-8.50
-8.40							-8.40
-8 30							-8.30
-8 20							-8.20
-8 10							-8.10
-8 00							-8.00
-7.90							-7.90
-7 80							-7.80
-7 70							-7.70
-7.60							-7 60
-7.50							-7 50
-7.40							-7.40
-7.30							-7.30
-7.20							-7 20
-7 10							-7 10
-7 00							-7.00
-6.90							-6.90
-6 80							-6 80
-6.70							-6 70
-6.60							-6.60
-6 50							-6 50
-6.40							-6 40
-6 30							-6 30
-6.20							-6.20
-6 10							-6 10
-6.00	000035						-6.00
-5 90	000057						-5 90
-5.80	.000093						-5.80
-5 70	000149						-5.70
-5.60	000237						-5.60
-5 50	.000372	000073					-5.50
-5 40	.000575	.000134					-5 40
-5 30	000879	.000238					-5.30
-5.20	001326	.000415					-5.20
-5.10	001974	000709					-5 10
-5 00	.002899	.001185	000185				-5.00

	SKEWNESS						
t	6	7	8	.9	1.0	1 1	t
-9.90							-9.90
-9.80							-9.80
-9.70							-9.70
-9.60							-9.60
-9.50							-9.50
-9.40							-9.40
-9.30							-9.30
-9.20							-9.20
-9.10							-9.10
-9.00							-9.00
-8.90							-8.90
-8.80							-8.80
-8.70							-8.70
-8.60							-8.60
-8.50							-8.50
-8.40							-8.40
-8.30							-8.30
-8.20							-8.20
-8.10							-8.10
-8.00							-8.00
-7.90							-7.90
-7.80							-7.80
-7.70							-7.70
-7.60							-7.60
-7.50							-7.50
-7.40							-7.40
-7.30							-7.30
-7.20							-7.20
-7.10							-7.10
-7.00							-7.00
-6.90							-6.90
-6.80							-6.80
-6.70							-6.70
-6.60							-6.60
-6.50							-6.50
-6.40							-6.40
-6.30							-6.30
-6.20							-6.20
-6.10							-6.10
-6.00							-6.00
-5.90							-5.90
-5.80							-5.80
-5.70							-5.70
-5.60							-5.60
-5.50							-5.50
-5.40							-5.40
-5.30							-5.30
-5.20							-5.20
-5.10							-5.10
-5.00							-5.00

t	SKEWNESS						t
	.0	1	2	3	.4	.5	
-4 90	004199	001939	000387				-4 90
-4 80	005998	.003107	000775				-4 80
-4 70	008445	.004872	001494				-4 70
-4 60	.011715	007477	.002771				-4 60
-4 50	016008	011228	004953				-4 50
-4.40	021533	016492	.008532	001582			-4 40
-4.30	028497	.023686	014175	003653			-4 30
-4 20	037077	033246	022723	007821			-4 20
-4.10	047385	045577	035150	015601			-4 10
-4 00	059421	060973	052467	029110	003698		-4.00
-3.90	.073016	079514	075548	050964	.011373		-3.90
-3 80	087768	.100934	104865	083927	029557		-3 80
-3 70	.102971	124473	140154	130216	.066481		-3 70
-3.60	.117547	148727	180029	.190495	131709		-3 60
-3 50	.130002	.171523	.221615	262656	232800	054837	-3 50
-3 40	.138396	189842	260269	340701	370353	.176319	-3.40
-3 30	.140362	199841	.289515	.413951	532885	.418003	-3 30
-3 20	.133185	.196991	301268	467184	693686	782995	-3 20
-3 10	113952	176376	286440	481803	.811755	1.204809	-3 10
-3 00	079773	.133159	235965	438269	837828	1 548187	-3.00
-2 90	.028105	.063208	142169	.319573	724933	1 646581	-2 90
-2 80	-.042873	-.036154	.000387	.115261	441138	1 361310	-2 80
-2.70	-.133814	-.165339	189409	174737	-.018815	637016	-2 70
-2.60	-.243764	- 321742	- 421241	- 537330	- 629273	- 468718	-2 60
-2 50	- 369738	- 499140	- 682317	- 945332	-1 319247	-1 791490	-2 50
-2 40	- 506424	-.687411	-.953041	-1 358873	-2.008265	-3 097791	-2 40
-2 30	- 646042	-.872660	-1 208037	-1.729252	-2 591569	-4 140341	-2 30
-2 20	-.778443	-1 037858	-1 418202	-2.004833	-2.971517	-4 713838	-2 20
-2 10	- 891503	-1 163872	-1 353678	-2.138166	-3 071825	-4.696711	-2 10
-2 00	- 971838	-1 231855	-1 587421	-2 093193	-2.851085	-4 070717	-2 00
-1.90	-1 005825	-1 223902	-1 498933	-1 851350	-2 310517	-2.917258	-1 90
-1 80	- 980903	-1 126606	-1 277630	-1 415493	-1 493788	-1 394949	-1.80
-1 70	- 887018	- 932582	- 925326	- 810963	- 480907	293881	-1 70
-1 60	- 718128	- 642320	- 457386	-.083541	.623876	1 939386	-1.60
-1.50	-.473549	- 265293	097713	705429	1.706836	3 356043	-1 50
-1.40	- 158975	179815	700455	1 486002	2 659884	4 405508	-1 40
-1 30	212999	.665911	1.303840	2.187685	3 393671	5 008519	-1 30
-1.20	623011	1.159322	1 857979	2 747403	3 846966	5 146248	-1 20
-1 10	1.045802	1.622521	2.220861	3 116326	3 991435	4 853130	-1 10
-1 00	1.451824	2 017519	2.635041	3 264706	3.831751	4.204047	-1.00
- .90	1.809510	2 309533	2.788437	3 184231	3 401691	3 298929	- .90
- 80	2 088004	2 470542	2.760718	2.887791	2.757297	2 247383	- .80
- .70	2 260116	2 482291	2 552957	2 406880	1 968436	1.155361	- .70
- 60	2 305168	2.338384	2.181623	1.787195	1 110037	.115043	- 60
- 50	2 211410	2 045193	1 676618	1.083101	254115	-.801603	- 50
- 40	1 977699	1 621472	1 077996	351751	- 536640	-1.545714	- 40
- 30	1 614197	1.096704	.438916	- 352453	-1 213257	-2.091401	- 30
- .20	1.141970	.508372	- 214335	- 982478	-1 742756	-2 433363	- 20
- .10	591463	- 101530	- 815746	-1.501985	-2 108462	-2.583054	- .10
00	.000000	- 690340	-1.334227	-1 887190	-2 308710	-2 564153	00

	SKEWNESS						
t	.6	.7	.8	.9	1 0	1 1	t
–4.90							–4 90
–4.80							–4.80
–4.70							–4 70
–4 60							–4.60
–4 50							–4 50
–4.40							–4.40
–4 30							–4 30
–4 20							–4.20
–4.10							–4 10
–4 00							–4 00
–3.90							–3 90
–3.80							–3 80
–3 70							–3.70
–3.60							–3.60
–3 50							–3 50
–3 40							–3 40
–3.30							–3.30
–3 20	.842257						–3 20
–3 10	.588693						–3 10
–3.00	1 849903						–3 00
–2 90	3 201920						–2 90
–2 80	3 878637						–2.80
–2 70	3 179643	11 313829					–2 70
–2 60	1 068006	11 097192					–2 60
–2 50	–1.918392	2 984160					–2 50
–2.40	–4 953572	–7 118615	30 500628				–2 40
–2.30	–7.249070	–14.50980	–32.96668				–2.30
–2 20	–8.283837	–17 22664	–49 96355	–761 6006			–2.20
–2 10	–7 894204	–15 50871	–41.08784	–260 2257			–2.10
–2 00	–6 243303	–10 76401	–23 00499	–80.70170			–2 00
–1 90	–3 717972	–4 695233	5 185098	3 299597	424 36017		–1 90
–1 80	802582	1 237087	8.199275	27 923815	274 10548	151298 68	–1 80
–1 70	2 033508	6 046683	16 148280	46 253906	170 84286	1305 4169	–1.70
–1 60	4 414962	9.245487	19 275092	42 139264	101 07146	238 57941	–1 60
–1 50	6 100725	10.754799	18 822519	32 911782	54 936664	46 225475	–1 50
–1 40	6 990201	10.774953	16.111270	22.634124	25 290674	–10 47424	–1.40
–1.30	7 105003	9.657556	12 273179	13 324788	6 986257	–28 00302	–1.30
–1 20	6 557023	7 802318	8 156143	5 816354	–3 652442	–31 23638	–1 20
–1 10	5 512210	5 585612	4 322899	283832	–9 225001	–28 90925	–1 10
–1.00	4.156897	3 318942	1.094040	–3 433856	–11 54861	–24.60908	–1 00
– .90	2 670779	1.231482	–1 394055	–5 643201	–11 87573	–19 91430	– 90
– .80	1.208217	– .530420	–3.132533	–6.687731	–11 05453	–15.51779	– 80
– 70	– .112213	–1 893330	–4 191323	–6.890713	–9 646454	–11 69445	– 70
– .60	–1.211125	–2 841685	–4 681987	–6 528740	–8 012009	–8 516467	– .60
– .50	–2 046433	–3 401428	–4 730771	–5 823350	–6 372745	–5 959217	– .50
– .40	–2 608007	–3 624745	–4 461008	–4 952497	–4 855811	–3.955431	– 40
– 30	–2 910524	–3 577221	–3 982722	–4 006892	–3 525641	–2 423242	– 30
– .20	–2.985855	–3.327957	–3 387561	–3 098185	–2 406236	–1 280497	– 20
– .10	–2.875929	–2.942594	–2 747366	–2.267305	–1.496634	– .451666	– 10
.00	–2 626655	–2 478942	–2.115071	–1 542070	– .781467	.129293	00

	SKEWNESS						
t	0	.1	2	3	.4	5	t
.00	000000	-.690340	-1 334227	-1 887190	-2 308710	-2 564153	.00
.10	-.591463	-1 218512	-1 740024	-2 127279	-2 354389	-2.407989	10
20	-1 141970	-1 652927	-2 017921	-2.223557	-2 265822	-2 149357	.20
.30	-1 614197	-1.969413	-2.158675	-2 187627	-2 069421	-1 823040	.30
40	-1.977699	-2 154258	-2 167343	-2 038926	-1 794478	-1.461172	40
50	-2 211410	-2 204630	-2 057225	-1 801983	-1 470374	--1 091458	.50
.60	-2.305168	-2.127944	-1.848575	-1 503696	-1 124365	- 736196	60
.70	-2 260116	-1.940314	-1 566096	-1 170883	- 780005	- .411955	70
80	-2 088004	-1 664529	-1.236393	- 828287	- 456215	- 129776	80
90	-1 809510	-1 327212	- .885638	- 497116	- 166914	104257	.90
1 00	-1 451824	- .956507	- .537636	- .194147	078882	288242	1 00
1 10	-1 045802	- 579507	- 212395	068637	276614	.423568	1 10
1.20	- 623011	- 220265	074750	.283975	425507	514034	1 20
1 30	- .212999	101683	313476	.448891	527781	565021	1 30
1.40	158975	372194	498348	564042	587834	582781	1.40
1.50	.473549	.582793	628327	.632916	.611467	.573849	1.50
1.60	718128	.730424	706022	661016	605202	544590	1.60
1.70	.887018	816786	736813	.655060	.575716	500885	1 70
1.80	.980903	.847379	727921	622290	529406	.447918	1 80
1.90	1 005825	.830419	687550	569897	472079	390073	1.90
2 00	.971838	775717	.624121	504581	408762	330901	2 00
2.10	.891503	693655	539812	432255	343601	.273145	2.10
2 20	.778443	594315	.459374	357869	.279854	.218807	2.20
2.30	.646042	.486812	.371323	.285342	.219924	169238	2.30
2 40	506424	.378852	286324	217579	.165452	.125239	2 40
2 50	369738	276486	207918	.156549	117416	087165	2 50
2.60	.243764	.184060	138447	103403	.076253	055028	2 60
2 70	.133814	104303	079202	058616	041973	028582	2 70
2 80	042873	038519	030597	.022127	014271	007408	2 80
2 90	-.028105	- 013156	- 007636	- 006517	-.007374	-.009025	2 90
3 00	- 079773	- 051472	- 036256	-.028035	- 023621	- 021303	3 00
3.10	- 113952	- 077795	-.056342	- 043304	-.035192	- 030026	3.10
3 20	- 133185	-.093851	- 069144	- 053270	- 042824	- 035779	3.20
3.30	-.140362	- 101518	- 075965	- 058881	- 047229	-.039110	3 30
3 40	- 138396	- 102660	- 078070	-.061034	-.049067	-.040516	3 40
3 50	- 130002	- 099010	- 076622	-.060545	- 048929	- 040438	3 50
3 60	- 117547	- 092091	- 072649	- 058132	-.047331	-.039253	3 60
3 70	- 102971	- 083181	-.067016	- 054399	- 044709	- 037282	3.70
3 80	-.087768	-.073300	- 060432	-.049848	- 041420	- 034789	3.80
3.90	-.073016	- 063224	- 053446	- 044873	-.037754	- 031983	3 90
4.00	-.059421	-.053504	-.046473	- 039778	-.033935	-.029033	4.00
4.10	-.047385	-.044506	-.039802	- 034788	-.030132	- 026066	4.10
4 20	- 037077	-.036442	- 033627	-.030057	-.026465	- 023173	4 20
4 30	-.028497	-.029405	- 028056	- 025686	-.023019	-.020423	4.30
4 40	- 021533	-.023405	- 023139	- 021731	-.019844	-.017856	4.40
4 50	-.016008	-.018390	- 018880	- 018216	- 016969	- 015501	4.50
4 60	- 011715	- 014274	-.015250	- 015138	- 014402	- 013367	4.60
4 70	- 008445	- 010951	-.012201	- 012480	- 012138	-.011456	4.70
4.80	-.005998	- 008308	-.009674	- 010210	-.010164	-.009763	4.80
4 90	-.004199	-.006235	-.007605	- 008294	- 008459	- 008277	4.90

t	SKEWNESS 6	7	8	9	10	11	t
.00	-2 626655	-2.478942	-2 115071	-1 542070	- 781467	129293	00
10	-2 283191	-1 984713	-1 526917	- .933664	- .238010	.518231	.10
20	-1 886598	-1 496838	-1.005251	- .441900	.159245	761246	20
30	-1 471780	-1 041901	- .561462	- .059315	435740	895647	.30
40	-1 066503	- 637251	- 198726	225751	615190	951124	.40
50	- 691263	- 292484	085577	426932	718714	950936	.50
60	- 359748	- 011045	297857	558300	764493	913026	60
70	- 225920	208228	446717	633354	767747	851009	.70
.80	146078	369701	.541745	664415	740914	775038	.80
90	318539	479757	592641	662309	693932	692521	.90
1.00	.441406	.545840	608622	.636249	634561	608726	1 00
1 10	.520183	.575713	.598052	593860	.568717	577633	1.10
1 20	561361	576925	568240	541282	500789	.450516	1.20
1 30	571777	556451	525374	483328	433925	379892	1.30
1 40	558132	520476	474525	.423655	370284	.316137	1.40
1 50	526650	474288	419728	364948	311256	.259494	1.50
1.60	482864	422255	364084	309088	257638	209867	1 60
1 70	431504	367861	309884	257311	.209788	166927	1 70
1 80	376465	.313781	258737	210344	167746	130204	1 80
1.90	320823	.261975	211694	168529	.131325	099148	1.90
2 00	.266905	213795	169358	.131919	.100192	.073173	2 00
2 10	216372	170089	131987	100359	073921	.051692	2 10
2 20	170315	.131304	099584	073557	052040	034138	2 20
2 30	129359	097575	071962	051132	034060	019978	2.30
2 40	.093758	068803	048813	032656	019500	008722	2 40
2 50	063480	044730	029747	.017680	007898	-000073	2 50
2 60	038290	024986	014337	005763	-.001173	-006804	2 60
2 70	017813	009140	002140	-003520	-.008104	-011818	2 70
2 80	.001587	-003268	-007278	-.010566	-013247	-015421	2 80
2 90	-010892	-012701	-014329	-015735	-016912	-.017873	2 90
3 00	-.020139	-019603	-019397	-019351	-019371	-019401	3 00
3 10	-026654	-024392	-022826	-021702	-020858	-020195	3 10
3 20	-030906	-027448	-024896	-023035	-021575	-020415	3 20
3 30	-033325	-029108	-025964	-023565	-021691	-.020194	3 30
3 40	-034291	-.029671	-026174	-023474	-021349	-019643	3 40
3 50	-034138	-029390	-025751	-022914	-020666	-018853	3 50
3.60	-.033150	-028481	-024861	-022012	-019739	-017898	3 60
3 70	-031565	-027125	-023639	-020872	-018646	-016835	3 70
3 80	-029578	-025466	-022198	-019576	-.017452	-015712	3 80
3.90	-.027346	-.023624	-020624	-018192	-016204	-.014566	3.90
4 00	-024995	-021691	-018989	-016772	-.014943	-013424	4 00
4.10	-022619	-019739	-017345	-015355	-013696	-012307	4 10
4 20	-020290	-017822	-.015732	-013971	-012486	-.011232	4 20
4 30	-018059	-.015979	-014182	-012643	-011329	-.010208	4 30
4 40	-015959	-014236	-012712	-011384	-.010235	-009243	4 40
4 50	-014014	-012611	-011337	-010205	-009210	-.008341	4 50
4.60	-012233	-.011113	-010064	-.009111	-008258	-007504	4 60
4.70	-010621	-009747	-008897	-008104	-007381	-.006732	4.70
4 80	-009176	-008511	-007834	-007183	-006577	-006023	4 80
4 90	-007890	-007401	-006873	-006347	-005844	-005377	4.90

	SKEWNESS						
t	0	.1	.2	.3	.4	.5	t
5.00	-.002899	-.004632	-.005930	-.006692	-.007000	-.006981	5.00
5.10	-.001974	-.003406	-.004587	-.005365	-.005761	-.005861	5.10
5.20	-.001326	-.002481	-.003522	-.004275	-.004717	-.004899	5.20
5.30	-.000879	-.001790	-.002685	-.003386	-.003843	-.004078	5.30
5.40	-.000575	-.001279	-.002032	-.002667	-.003117	-.003381	5.40
5.50	-.000372	-.000906	-.001528	-.002090	-.002516	-.002793	5.50
5.60	-.000237	-.000637	-.001141	-.001629	-.002023	-.002298	5.60
5.70	-.000149	-.000443	-.000847	-.001263	-.001620	-.001885	5.70
5.80	-.000093	-.000306	-.000625	-.000972	-.001292	-.001541	5.80
5.90	-.000057	-.000210	-.000458	-.000749	-.001026	-.001256	5.90
6.00	-.000035	-.000142	-.000334	-.000572	-.000812	-.001020	6.00
6.10		-.000096	-.000242	-.000436	-.000641	-.000826	6.10
6.20		-.000064	-.000174	-.000330	-.000504	-.000668	6.20
6.30		-.000043	-.000125	-.000249	-.000395	-.000538	6.30
6.40		-.000028	-.000089	-.000188	-.000308	-.000432	6.40
6.50		-.000018	-.000063	-.000140	-.000240	-.000346	6.50
6.60		-.000012	-.000045	-.000105	-.000186	-.000277	6.60
6.70			-.000031	-.000078	-.000144	-.000221	6.70
6.80			-.000022	-.000058	-.000111	-.000176	6.80
6.90			-.000015	-.000043	-.000086	-.000140	6.90
7.00			-.000011	-.000031	-.000066	-.000111	7.00
7.10			-.000007	-.000023	-.000050	-.000087	7.10
7.20			-.000005	-.000017	-.000038	-.000069	7.20
7.30			-.000003	-.000012	-.000029	-.000054	7.30
7.40			-.000002	-.000009	-.000022	-.000043	7.40
7.50				-.000006	-.000017	-.000034	7.50
7.60				-.000005	-.000013	-.000026	7.60
7.70				-.000003	-.000010	-.000021	7.70
7.80				-.000002	-.000007	-.000016	7.80
7.90				-.000002	-.000005	-.000012	7.90
8.00				-.000001	-.000004	-.000010	8.00
8.10					-.000003	-.000007	8.10
8.20					-.000002	-.000006	8.20
8.30					-.000002	-.000005	8.30
8.40					-.000001	-.000004	8.40
8.50						-.000003	8.50
8.60						-.000002	8.60
8.70						-.000002	8.70
8.80						-.000001	8.80
8.90						-.000001	8.90
9.00						-.000001	9.00
9.10							9.10
9.20							9.20
9.30							9.30
9.40							9.40
9.50							9.50
9.60							9.60
9.70							9.70
9.80							9.80
9.90							9.90

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
5.00	-.006755	-.006411	-.006010	-.005592	-.005180	-.004788	5.00
5.10	-.005760	-.005534	-.005239	-.004913	-.004580	-.004256	5.10
5.20	-.004893	-.004761	-.004553	-.004305	-.004040	-.003775	5.20
5.30	-.004141	-.004083	-.003946	-.003763	-.003557	-.003343	5.30
5.40	-.003493	-.003491	-.003411	-.003282	-.003125	-.002955	5.40
5.50	-.002937	-.002977	-.002942	-.002857	-.002741	-.002608	5.50
5.60	-.002461	-.002531	-.002531	-.002482	-.002400	-.002298	5.60
5.70	-.002057	-.002147	-.002173	-.002151	-.002097	-.002022	5.70
5.80	-.001714	-.001817	-.001862	-.001862	-.001830	-.001777	5.80
5.90	-.001425	-.001534	-.001592	-.001609	-.001595	-.001559	5.90
6.00	-.001181	-.001292	-.001358	-.001387	-.001388	-.001367	6.00
6.10	-.000977	-.001086	-.001157	-.001195	-.001206	-.001197	6.10
6.20	-.000806	-.000911	-.000984	-.001027	-.001046	-.001047	6.20
6.30	-.000663	-.000763	-.000835	-.000882	-.000907	-.000914	6.30
6.40	-.000545	-.000638	-.000708	-.000756	-.000785	-.000798	6.40
6.50	-.000447	-.000532	-.000599	-.000648	-.000679	-.000696	6.50
6.60	-.000365	-.000443	-.000506	-.000554	-.000587	-.000606	6.60
6.70	-.000298	-.000368	-.000427	-.000473	-.000506	-.000528	6.70
6.80	-.000243	-.000306	-.000360	-.000404	-.000437	-.000459	6.80
6.90	-.000198	-.000254	-.000303	-.000344	-.000376	-.000399	6.90
7.00	-.000160	-.000210	-.000255	-.000293	-.000324	-.000346	7.00
7.10	-.000130	-.000173	-.000214	-.000249	-.000278	-.000300	7.10
7.20	-.000105	-.000143	-.000180	-.000212	-.000239	-.000261	7.20
7.30	-.000085	-.000118	-.000150	-.000180	-.000205	-.000226	7.30
7.40	-.000069	-.000097	-.000126	-.000153	-.000176	-.000195	7.40
7.50	-.000055	-.000080	-.000105	-.000129	-.000151	-.000169	7.50
7.60	-.000044	-.000066	-.000088	-.000109	-.000129	-.000146	7.60
7.70	-.000036	-.000054	-.000073	-.000093	-.000111	-.000126	7.70
7.80	-.000029	-.000044	-.000061	-.000078	-.000095	-.000109	7.80
7.90	-.000023	-.000036	-.000051	-.000066	-.000081	-.000094	7.90
8.00	-.000018	-.000029	-.000042	-.000056	-.000069	-.000081	8.00
8.10	-.000015	-.000024	-.000035	-.000047	-.000059	-.000070	8.10
8.20	-.000012	-.000020	-.000029	-.000040	-.000050	-.000060	8.20
8.30	-.000009	-.000016	-.000024	-.000033	-.000043	-.000052	8.30
8.40	-.000007	-.000013	-.000020	-.000028	-.000037	-.000045	8.40
8.50	-.000006	-.000011	-.000017	-.000024	-.000031	-.000039	8.50
8.60	-.000005	-.000009	-.000014	-.000020	-.000026	-.000033	8.60
8.70	-.000004	-.000007	-.000011	-.000017	-.000023	-.000029	8.70
8.80	-.000003	-.000006	-.000009	-.000014	-.000019	-.000025	8.80
8.90	-.000002	-.000005	-.000008	-.000012	-.000016	-.000021	8.90
9.00	-.000002	-.000004	-.000006	-.000010	-.000014	-.000018	9.00
9.10	-.000002	-.000003	-.000005	-.000008	-.000012	-.000016	9.10
9.20	-.000001	-.000002	-.000004	-.000007	-.000010	-.000013	9.20
9.30	-.000001	-.000002	-.000004	-.000006	-.000008	-.000011	9.30
9.40	-.000001	-.000002	-.000003	-.000005	-.000007	-.000010	9.40
9.50	-.000001	-.000001	-.000002	-.000004	-.000006	-.000008	9.50
9.60		-.000001	-.000002	-.000003	-.000005	-.000007	9.60
9.70		-.000001	-.000002	-.000003	-.000004	-.000006	9.70
9.80		-.000001	-.000001	-.000002	-.000004	-.000005	9.80
9.90		-.000001	-.000001	-.000002	-.000003	-.000005	9.90

	SKEWNESS						
t	.0	.1	.2	.3	4	5	t
10.00							10.00
10.10							10.10
10.20							10.20
10.30							10.30
10.40							10.40
10.50							10.50
10.60							10.60
10.70							10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

t	SKEWNESS						t
	6	.7	8	9	1.0	1 1	
10.00			- 000001	-.000002	- 000003	- 000004	10.00
10.10			-.000001	- 000001	- 000002	-.000003	10.10
10.20			- 000001	- 000001	-.000002	-.000003	10.20
10.30			- 000001	- 000001	- 000002	- 000002	10.30
10.40				- 000001	- 000001	- 000002	10.40
10.50				-.000001	- 000001	-.000002	10 50
10 60				-.000001	-.000001	- 000002	10.60
10.70					-.000001	- 000001	10 70
10 80					- 000001	- 000001	10.80
10 90					- 000001	-.000001	10.90
11.00						-.000001	11 00
11.10						- 000001	11 10
11.20						- 000001	11 20
11 30						- 000001	11.30
11.40							11 40
11.50							11 50
11.60							11 60
11.70							11 70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12 30							12 30
12.40							12 40
12.50							12.50
12.60							12 60
12 70							12 70
12 80							12.80
12 90							12 90
13.00							13.00
13.10							13 10
13.20							13.20
13 30							13 30
13 40							13.40
13 50							13.50
13.60							13.60
13 70							13 70
13.80							13 80
13.90							13.90
14.00							14 00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14 60							14.60
14 70							14.70
14.80							14.80
14.90							14.90

	SKEWNESS						
t	.0	.1	.2	.3	.4	.5	t
–9.90							–9.90
–9.80							–9.80
–9.70							–9.70
–9.60							–9.60
–9.50							–9.50
–9.40							–9.40
–9.30							–9.30
–9.20							–9.20
–9.10							–9.10
–9.00							–9.00
–8.90							–8.90
–8.80							–8.80
–8.70							–8.70
–8.60							–8.60
–8.50							–8.50
–8.40							–8.40
–8.30							–8.30
–8.20							–8.20
–8.10							–8.10
–8.00							–8.00
–7.90							–7.90
–7.80							–7.80
–7.70							–7.70
–7.60							–7.60
–7.50							–7.50
–7.40							–7.40
–7.30							–7.30
–7.20							–7.20
–7.10							–7.10
–7.00							–7.00
–6.90							–6.90
–6.80							–6.80
–6.70							–6.70
–6.60							–6.60
–6.50							–6.50
–6.40							–6.40
–6.30							–6.30
–6.20							–6.20
–6.10							–6.10
–6.00	000175						–6.00
–5.90	.000282						–5.90
–5.80	.000447						–5.80
–5.70	.000700						–5.70
–5.60	.001081						–5.60
–5.50	.001648	000447					–5.50
–5.40	.002477	000786					–5.40
–5.30	.003672	.001349					–5.30
–5.20	.005364	002265					–5.20
–5.10	.007720	.003718					–5.10
–5.00	.010942	.005964	.001401				–5.00

	SKEWNESS						
t	.6	.7	.8	.9	1.0	1.1	t
–9.90							–9.90
–9.80							–9.80
–9.70							–9.70
–9.60							–9.60
–9.50							–9.50
–9.40							–9.40
–9.30							–9.30
–9.20							–9.20
–9.10							–9.10
–9.00							–9.00
–8.90							–8.90
–8.80							–8.80
–8.70							–8.70
–8.60							–8.60
–8.50							–8.50
–8.40							–8.40
–8.30							–8.30
–8.20							–8.20
–8.10							–8.10
–8.00							–8.00
–7.90							–7.90
–7.80							–7.80
–7.70							–7.70
–7.60							–7.60
–7.50							–7.50
–7.40							–7.40
–7.30							–7.30
–7.20							–7.20
–7.10							–7.10
–7.00							–7.00
–6.90							–6.90
–6.80							–6.80
–6.70							–6.70
–6.60							–6.60
–6.50							–6.50
–6.40							–6.40
–6.30							–6.30
–6.20							–6.20
–6.10							–6.10
–6.00							–6.00
–5.90							–5.90
–5.80							–5.80
–5.70							–5.70
–5.60							–5.60
–5.50							–5.50
–5.40							–5.40
–5.30							–5.30
–5.20							–5.20
–5.10							–5.10
–5.00							–5.00

t	SKEWNESS						t
	.0	.1	2	3	4	5	
-4.90	015267	009347	002766				-4 90
-4.80	020954	014312	005236				-4 80
-4 70	028274	.021396	009515				-4.70
-4.60	.037474	.031216	016605				-4 60
-4.50	048736	044414	027810				-4 50
-4 40	.062116	.061567	044852	013863			-4.40
-4.30	.077464	.083049	069421	029156			-4 30
-4 20	094326	.108841	.103173	056718			-4.20
-4 10	.111837	.138280	.147080	102430			-4.10
-4 00	128611	.169792	200769	172160	015017		-4 00
-3.90	.142641	200610	261691	269650	117666		-3 90
-3.80	.151238	226551	324272	393475	259945		-3 80
-3 70	.151024	.241913	379259	.533619	495046		-3 70
-3.60	.138020	239563	413503	668647	822075		-3 60
-3.50	.107844	211314	410484	764762	1 200746	.742856	-3.50
-3.40	.056066	148638	351775	.778218	1 531759	1 764169	-3 40
-3.30	-.021297	.043742	219572	.661208	1.674232	3 080073	-3.30
-3.20	-.127125	-.109010	000159	372046	1 472443	4 108961	-3 20
-3.10	-.262416	- 311448	- 312050	-.112822	805005	4.095235	- 3 10
-3.00	- 425457	-.559902	- 710518	- 786892	- 364035	2 484221	-3 00
-2.90	- 611017	- 843721	-1 173459	-1 605106	-1 949117	- 757276	-2.90
-2 80	-.809701	-1.144331	-1.662462	-2.481732	-3 736776	-5.045593	-2.80
-2.70	-1 007607	-1.435140	-2.123635	-3.296576	-5 414832	-9.343178	-2 70
-2.60	-1.186451	-1 682587	-2 491694	-3.909747	-6 720425	-12 49085	-2 60
-2.50	-1.324208	-1.848487	-2.696991	-4.183196	-7 073686	-13 56574	-2.50
-2 40	-1.396544	-1.893676	-2 674851	-4 005451	-6.533877	-12 13850	-2 40
-2.30	-1.378834	-1 782727	-2.376055	-3 314841	-4 966528	-8 363173	-2.30
-2 20	-1.248851	-1 489294	-1 776879	-2.116465	-2.502962	-2 900863	-2 20
-2 10	-.989868	-1 001128	-.886935	- 489062	.567018	3 273070	-2 10
-2 00	-.593901	-.326029	246817	1 420125	3.844994	9 107213	-2 00
-1 90	-.064672	508204	1 541012	3 412172	6 893098	13.691811	-1 90
-1.80	.580144	1 450348	2 882212	5 262389	9 306193	16.410943	-1.80
-1 70	1.307853	2 429501	4 138184	6 751069	10.775134	17 010500	-1.70
-1.60	2.071248	3 360003	5 172815	7 693951	11 128417	15 586539	-1.60
-1.50	2 810937	4.149259	5 862695	7 957495	10 348409	12 514059	-1 50
-1.40	3.459534	4 707433	6 113044	7 525126	8.562045	8 342962	-1 40
-1.30	3 947529	4 957908	5 870642	6 402253	6 010840	3 687149	-1.30
-1 20	4.210340	4.847185	5 131842	4 709910	3 007841	- 872959	-1 20
-1.10	4 195847	4.352868	3.786317	2 618573	- 109997	-4.861982	-1 10
-1.00	3 871531	3.488518	2.403342	335208	-3.026213	-7 950511	-1.00
- .90	3.230259	2.304551	639885	-1 922775	-5.480152	-9.966922	- 90
- 80	2 293820	884850	-1.192420	-3 952789	-7.289797	-10 88572	- .80
- .70	1 113544	- 660680	-2 935015	-5.588315	-8 361193	-10.80001	- 70
- 60	-.232372	-2.206781	-4.441265	-6 713777	-8 685678	-9.885928	- 60
- .50	-1.644805	-3.625505	-5 591378	-7.271912	-8.327546	-8.365606	- .50
- .40	-3 012214	-4.799309	-6.303511	-7 263647	-7.405560	-6 473884	- 40
- 30	-4 222257	-5 632987	-6.364810	-6 741491	-6 071759	-4.431674	- .30
- .20	-5 171124	-6 063022	-6 308038	-5 798132	-4.490653	-2 427427	- 20
- .10	-5.776254	-6 063275	-5.656056	-4 552304	-2 821126	-.606621	- 10
.00	-5.984135	-5 646455	-4.665204	-3.133923	-1 202514	931715	00

t	SKEWNESS						t
	.6	7	.8	.9	1.0	1.1	
-4.90							-4.90
-4 80							-4.80
-4 70							-4 70
-4.60							-4.60
-4 50							-4 50
-4.40							-4 40
-4.30							-4.30
-4 20							-4 20
-4.10							-4 10
-4.00							-4 00
-3 90							-3 90
-3.80							-3 80
-3.70							-3 70
-3 60							-3 60
-3 50							-3 50
-3 40							-3 40
-3 30							-3 30
-3 20	26.875960						-3 20
-3.10	8 895339						-3 10
-3 00	14 844781						-3 00
-2 90	11 870834						-2 90
-2.80	.511734						-2 80
-2 70	-14 54133	49 586240					-2.70
-2.60	-26 74182	-50 43212					-2.60
-2.50	-31.55201	-100.9999					-2 50
-2 40	-27 79824	-93 03413	-897.1775				-2 40
-2 30	-17.22889	-51 64289	-371 6563				-2 30
-2 20	-3.200061	-3 279440	-6 531180	14261 298			-2 20
-2 10	10 702406	35.142528	156 15605	2561 9104			-2.10
-2 00	21 648976	56.825503	190 10374	1209 5822			-2 00
-1 90	28 031135	62 079701	138 10092	540 76774	-1793 994		-1 90
-1.80	29 481130	54 921276	106 57407	232 57201	-1241 483	-21843131	-1 80
-1 70	26.608535	40.473714	53.588006	5 612894	-846 0655	-27894 86	-1.70
-1 60	20 618004	23.393943	11 107951	-75.76662	-565 6321	-3569 339	-1.60
-1 50	12 932907	7 147980	-17 91647	-102 2365	-368.8605	-953 1063	-1 50
-1 40	4 898190	-6 141522	-34.40810	-100 0908	-232 5508	-304 3152	-1.40
-1 30	-2 409460	-15.52843	-40 94149	-84.79357	-139.5989	-81 95704	-1 30
-1 20	-8 267851	-20 94228	-40 46917	-65 13526	-77 45935	4.106084	-1 20
-1.10	-12.31332	-22 86993	-35 67140	-45 81429	-36 98463	36 888914	-1 10
-1 00	-14.49033	-22 08158	-28 68817	-29 06763	-11 54861	46.646847	-1.00
- .90	-14.97304	-19.42548	-21.06770	-15 70089	3 611887	46.164258	- 90
- 80	-14 08005	-15.69277	-13 82424	-5 732500	11.890584	41 340786	- 80
- .70	-12 19660	-11 54227	-7.541844	1.210768	15.680242	35 022093	- 70
- .60	-9 712425	-7.470641	-2.484734	5.659817	16 646714	28 587090	- 60
- 50	-6.978085	-3.813356	1.301813	8 169192	15.931861	22 674434	- .50
- .40	-4 279153	-.763934	3 909282	9 248079	14 303076	17.539966	- 40
- 30	-1 825508	1.598081	5 503477	9 329346	12.262853	13.241811	- 30
- .20	247988	3 277084	6.280186	8 761196	10 128576	9 739410	- .20
- 10	1 873235	4.333462	6.435881	7 811120	8.090226	6.947384	10
.00	3 036428	4 859952	6 150084	6 675819	6.251738	4 764787	.00

t	SKEWNESS .0	.1	.2	.3	.4	.5	t
.00	-5.984135	-5.646455	-4.665204	-3.133923	-1.202514	931715	00
10	-5.776254	-4.861356	-3.438799	-1 670337	.254500	2.132985	10
20	-5.171124	-3 786395	-2.090353	- .274980	1 472275	2.981242	.20
30	-4.222257	-2 520401	- 731916	.960727	2.406462	3 491248	.30
.40	-3.012214	-1 171892	.536015	1 971948	3.043064	3.699785	.40
.50	-1.644805	.151841	1 632165	2 721665	3 393319	3 657272	50
.60	- 232372	1 354605	2.499227	3.199050	3 487452	3 420390	.60
.70	1.113544	2.359129	3.105617	3.415727	3.368150	3.046137	70
.80	2 293820	3.112206	3.444507	3 400697	3.084467	2 587429	80
90	3 230259	3.586870	3.530678	3 194705	2.686604	2 090218	.90
1 00	3.871531	3.781737	3.395867	2.844710	2.221834	1.591942	1 00
1.10	4 195847	3 717924	3 083342	2 398951	1.731618	1.121050	1 10
1.20	4.210340	3 434203	2 642373	1.902952	1 250054	.697352	1 20
1.30	3.947529	2 981144	2.123169	1.396604	802892	.332911	1 30
1.40	3.459534	2 414999	1.572664	.912359	.407952	033260	1.40
1.50	2.810937	1 792005	1.031410	.474437	.075673	- 201246	1 50
1.60	2.071248	1.163611	531626	098868	- 189775	- 374091	1.60
1.70	1 307853	572967	.096362	- 205823	- 389234	-.491358	1 70
1.80	580144	.052784	- 260397	-.437511	- 527257	-.560670	1 80
1.90	- 064672	- 375458	-.532887	-.599134	- 610882	- 590313	1 90
2 00	-.593901	- 701132	-.722234	-.697299	-.648554	-.588568	2 00
2 10	-.989868	-.913199	- 809084	- 740958	-.649236	- 563231	2.10
2.20	-1.248851	-1 048345	-.880804	- 740237	- 621724	-.521293	2 20
2.30	-1.378834	-1 088823	- 872200	- 705481	-.574151	-.468771	2 30
2.40	-1.396544	-1 060287	- 821890	-.646523	-.513679	- 410636	2.40
2.50	-1 324208	-.979805	- 742355	-.572181	- 446329	-.350831	2.50
2 60	-1.186451	-.864201	- 644933	-.489957	- 376936	- 292338	2 60
2 70	-1.007607	- 728816	- 539310	-.405909	-.309189	- 237290	2 70
2 80	-.809701	- 586683	- 433276	-.324649	-.245729	-.187092	2 80
2.90	- 611017	- 448101	- 332680	- 249447	-.188292	-.142554	2 90
3.00	-.425457	-.320545	-.241545	-.182391	- 137856	-.104005	3 00
3 10	- 262416	- 208830	-.162278	-.124581	-.094796	- 071429	3.10
3.20	- 127125	-.115450	-.095950	-.076334	-.059035	-.044545	3 20
3 30	- 021297	- 041016	- 042586	- 037380	-.030168	- 022905	3 30
3 40	056066	015276	-.001451	- 007041	- 007579	- 005958	3 40
3 50	.107844	055191	028687	.015614	009469	.006895	3.50
3.60	138020	081059	.049341	031649	.021765	016260	3.60
3.70	.151024	.095440	.062130	.042162	.030093	.022718	3 70
3.80	.151238	100881	.068649	.048214	035196	026809	3.80
3.90	.142641	.099734	070375	.050783	037750	.029018	3 90
4.00	.128611	094057	.068614	.050736	038353	.029771	4.00
4.10	.111837	.085556	.064474	048815	037518	.029431	4 10
4.20	.094326	075579	.058859	045636	.035673	028304	4.20
4.30	.077464	065141	052475	041691	033167	026640	4 30
4.40	.062116	.054955	045856	037366	.030277	.024639	4 40
4.50	048736	045489	.039380	.032946	027214	.022458	4.50
4 60	.037474	037012	.033302	.028639	024137	.020216	4 60
4.70	028274	029644	.027773	024583	021160	018002	4.70
4.80	020954	023399	022872	.020863	018359	.015877	4.80
4.90	015267	.018220	.018617	017524	015780	013884	4.90

	SKEWNESS						
t	.6	7	.8	.9	1 0	1.1	t
00	3.036428	4 859952	6 150084	6 675819	6 251738	4 764787	.00
10	3 764079	4 963167	5 576252	5 493291	4.660385	3 090633	10
20	4 109449	4 750215	4 838542	4 354939	3.327388	1 831652	20
30	4.140621	4 319865	4 032352	3 316558	2 242176	.905655	.30
40	3 930837	3 757417	3 227075	2 407733	1.382071	203484	40
50	3 551296	3 132436	2 469942	1.639509	718714	- 216483	50
.60	3 066280	2.498523	1.790194	1.010443	222193	-.519178	60
70	2 500095	1.894408	1 203125	.511222	- 136437	-.704349	70
80	1 986914	1 345841	.713700	.128065	- 383710	- 802822	80
.90	1 468665	.867818	319643	-.154839	-.543055	-.838819	.90
1.00	997981	466872	013947	- 353672	-.634561	-.831126	1.00
1.10	588480	.143223	- 213162	-.483906	-.675048	- 799963	1.10
1 20	246555	- 107330	-.372754	- 559657	- 678305	- 738617	1.20
1 30	- 027004	- 291823	- 476136	- 593360	- 655415	-.672661	1.30
1.40	- 235490	- 418856	- 534137	- 595648	- 615113	- 602070	1 40
1 50	- 384920	- 497630	- 556659	- 575384	-.564151	-.530962	1.50
1 60	-.482913	-.537238	- 552429	- 539782	- 507631	- 462145	1 60
1 70	- 537788	-.546198	-.528908	- 494574	-.449308	- 397442	1 70
1 80	-.557885	- 532154	-.492291	- 444204	- 391863	- 337936	1 80
1 90	- 551088	- 501728	- 447585	- 392019	- 322869	- 284173	1 90
2 00	- 524515	-.460475	- 398722	-.340461	- 286261	-.236315	2.00
2 10	-.484347	-.412908	-.348694	- 291231	- 239949	- 194255	2.10
2 20	-.435762	-.362581	- 299695	- 245447	- 198488	- 157714	2 20
2 30	- 382945	- 312187	-.253259	- 203769	- 161911	- 126298	2 30
2.40	-.329152	- 263683	- 210383	- 166512	-.130066	-.099556	2.40
2 50	- 276808	-.218407	- 171645	- 133735	- 102677	- 077009	2 50
2 60	- 227620	-.177197	- 137300	-.105316	-.079392	-.058182	2 60
2 70	- 182699	-.140495	-.107364	- 081012	- 059819	- 042614	2 70
2 80	-.142668	-.108440	-.081682	- 060501	- 043554	- 029872	2 80
2.90	-.107778	- 080951	-.059987	- 043420	- 030199	- 019560	2 90
3 00	-.077994	- 057791	-.041940	- 029389	- 019371	-.011318	3 00
3.10	- 053083	- 038624	- 027165	-.018032	- 010715	-.004823	3 10
3 20	- 032678	-.023053	- 015273	- 008988	- 003906	000208	3 20
3 30	-.016331	- 010658	- 005883	- 001918	001348	004024	3 30
3.40	- 003556	- 001017	001367	003485	005306	006839	3.40
3 50	.006139	.006275	006814	007498	.008195	008839	3.50
3 60	013228	011598	.010761	010367	.010212	010180	3 60
3.70	018156	015297	013480	012305	011527	010996	3 70
3 80	.021325	.017679	015207	013496	012283	.011399	3 80
3.90	.023094	.019012	016148	014099	.012602	.011482	3 90
4 00	023772	.019527	016476	014245	012583	011321	4 00
4 10	023625	019421	.016338	014045	.012312	010980	4.10
4 20	022874	018856	015856	.013591	.011857	.010510	4.20
4 30	021701	017966	015129	012956	011272	.009952	4.30
4.40	020251	016860	.014238	.012200	010603	009339	4 40
4 50	018642	015623	013246	011372	.009885	.008697	4.50
4 60	016962	014323	012204	010508	009145	008046	4 60
4.70	015279	.013010	011150	009636	008405	007401	4.70
4 80	013642	011722	010112	008779	.007679	.006773	4.80
4 90	012085	010486	.009111	.007951	006980	.006170	4.90

t	SKEWNESS 0	.1	2	3	.4	.5	t
5.00	.010942	.014005	014991	014581	013449	.012048	5 00
5.10	.007720	.010635	011950	012026	011373	.010381	5 10
5.20	005364	.007983	009436	009839	.009549	.008888	5.20
5 30	003672	.005926	.007384	007988	.007964	007564	5 30
5.40	002477	.004352	.005729	006439	.006601	006402	5 40
5.50	.001648	.003164	004409	005155	005439	005391	5 50
5 60	001081	002277	003367	004101	004457	004518	5.60
5 70	.000700	001623	002552	.003242	003634	003769	5 70
5 80	000447	001146	.001920	002543	.002948	003130	5.80
5.90	000282	000802	.001435	001992	.002380	002590	5 90
6.00	000175	.000556	001065	001549	001913	002134	6.00
6.10		.000382	000786	001198	001531	.001753	6.10
6.20		000261	.000576	000922	.001220	.001434	6.20
6.30		000176	000420	.000706	000969	001170	6.30
6 40		.000118	.000304	000538	.000766	000951	6.40
6.50		.000079	000219	000409	000604	.000771	6 50
6.60		.000052	000157	000309	000474	.000623	6.60
6.70			.000112	000232	.000371	000502	6 70
6.80			000079	000174	000290	000403	6.80
6.90			000056	000130	000225	.000323	6 90
7.00			000039	000097	000175	000259	7.00
7 10			000027	000072	000135	000206	7.10
7 20			000019	000053	000104	.000164	7.20
7.30			000013	000039	000080	.000130	7.30
7 40			000009	000029	000061	000103	7.40
7 50				000021	000047	000082	7 50
7 60				000015	000036	000064	7 60
7.70				000011	.000027	000051	7.70
7.80				.000008	000021	.000040	7.80
7.90				000006	.000016	000031	7 90
8.00				000004	.000012	000024	8 00
8.10					000009	.000019	8 10
8.20					.000007	.000015	8 20
8.30					000005	000012	8.30
8.40					000004	000009	8 40
8 50						000007	8 50
8.60						.000005	8 60
8 70						000004	8.70
8 80						.000003	8 80
8 90						000003	8 90
9.00						000002	9 00
9.10							9.10
9.20							9.20
9.30							9 30
9 40							9.40
9 50							9.50
9.60							9.60
9.70							9 70
9.80							9.80
9.90							9.90

t	SKEWNESS						t
	.6	.7	.8	.9	1.0	1.1	
5.00	.010631	.009320	.008162	.007164	.006315	.005598	5.00
5.10	.009292	.008236	.007273	.006424	.005690	.005061	5.10
5.20	.008075	.007240	.006451	.005737	.005107	.004550	5.20
5.30	.006980	.006334	.005697	.005103	.004568	.004096	5.30
5.40	.006004	.005517	.005010	.004523	.004074	.003670	5.40
5.50	.005140	.004785	.004391	.003996	.003622	.003279	5.50
5.60	.004382	.004135	.003835	.003520	.003212	.002923	5.60
5.70	.003721	.003560	.003339	.003091	.002841	.002600	5.70
5.80	.003148	.003055	.002898	.002708	.002507	.002308	5.80
5.90	.002654	.002614	.002509	.002366	.002207	.002045	5.90
6.00	.002230	.002230	.002166	.002063	.001940	.001809	6.00
6.10	.001867	.001897	.001865	.001795	.001701	.001597	6.10
6.20	.001559	.001609	.001603	.001558	.001490	.001408	6.20
6.30	.001298	.001362	.001374	.001350	.001302	.001239	6.30
6.40	.001078	.001150	.001176	.001168	.001136	.001090	6.40
6.50	.000893	.000968	.001004	.001008	.000920	.000957	6.50
6.60	.000737	.000814	.000856	.000869	.000861	.000839	6.60
6.70	.000608	.000683	.000728	.000748	.000749	.000735	6.70
6.80	.000500	.000572	.000618	.000643	.000650	.000643	6.80
6.90	.000410	.000478	.000524	.000552	.000563	.000562	6.90
7.00	.000336	.000398	.000444	.000473	.000487	.000490	7.00
7.10	.000274	.000332	.000375	.000405	.000421	.000428	7.10
7.20	.000224	.000276	.000317	.000346	.000364	.000373	7.20
7.30	.000182	.000229	.000267	.000295	.000314	.000324	7.30
7.40	.000148	.000190	.000225	.000252	.000271	.000282	7.40
7.50	.000120	.000157	.000189	.000215	.000233	.000245	7.50
7.60	.000097	.000130	.000159	.000183	.000201	.000213	7.60
7.70	.000078	.000107	.000133	.000155	.000172	.000185	7.70
7.80	.000063	.000088	.000112	.000132	.000148	.000160	7.80
7.90	.000051	.000073	.000093	.000112	.000127	.000139	7.90
8.00	.000041	.000060	.000078	.000095	.000109	.000120	8.00
8.10	.000033	.000049	.000065	.000080	.000093	.000104	8.10
8.20	.000026	.000040	.000054	.000068	.000080	.000090	8.20
8.30	.000021	.000033	.000045	.000057	.000068	.000078	8.30
8.40	.000017	.000027	.000038	.000049	.000059	.000067	8.40
8.50	.000014	.000022	.000031	.000041	.000050	.000058	8.50
8.60	.000011	.000018	.000026	.000035	.000043	.000050	8.60
8.70	.000009	.000015	.000022	.000029	.000036	.000043	8.70
8.80	.000007	.000012	.000018	.000025	.000031	.000037	8.80
8.90	.000005	.000010	.000015	.000021	.000027	.000032	8.90
9.00	.000004	.000008	.000012	.000017	.000023	.000028	9.00
9.10	.000003	.000006	.000010	.000015	.000019	.000024	9.10
9.20	.000003	.000005	.000008	.000012	.000016	.000020	9.20
9.30	.000002	.000004	.000007	.000010	.000014	.000018	9.30
9.40	.000002	.000003	.000006	.000009	.000012	.000015	9.40
9.50	.000001	.000003	.000005	.000007	.000010	.000013	9.50
9.60	.000001	.000002	.000004	.000006	.000009	.000011	9.60
9.70		.000002	.000003	.000005	.000007	.000010	9.70
9.80		.000001	.000003	.000004	.000006	.000008	9.80
9.90		.000001	.000002	.000004	.000005	.000007	9.90

	SKEWNESS						
t	.0	.1	.2	.3	.4	.5	t
10.00							10.00
10.10							10.10
10.20							10.20
10.30							10.30
10.40							10.40
10.50							10.50
10.60							10.60
10.70							10.70
10.80							10.80
10.90							10.90
11.00							11.00
11.10							11.10
11.20							11.20
11.30							11.30
11.40							11.40
11.50							11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

t	SKEWNESS 6	7	8	9	10	11	t
10.00		000001	000002	000003	000004	000006	10.00
10.10		000001	000001	000002	000004	000005	10.10
10.20		000001	000001	000002	000003	000004	10.20
10.30			000001	000002	000003	000004	10.30
10.40			000001	000001	000002	000003	10.40
10.50			000001	000001	000002	000003	10.50
10.60			000001	000001	000002	000002	10.60
10.70				000001	000001	000002	10.70
10.80				000001	000001	000002	10.80
10.90				000001	000001	000001	10.90
11.00					000001	000001	11.00
11.10					000001	000001	11.10
11.20					000001	000001	11.20
11.30					000001	000001	11.30
11.40						000001	11.40
11.50						000001	11.50
11.60							11.60
11.70							11.70
11.80							11.80
11.90							11.90
12.00							12.00
12.10							12.10
12.20							12.20
12.30							12.30
12.40							12.40
12.50							12.50
12.60							12.60
12.70							12.70
12.80							12.80
12.90							12.90
13.00							13.00
13.10							13.10
13.20							13.20
13.30							13.30
13.40							13.40
13.50							13.50
13.60							13.60
13.70							13.70
13.80							13.80
13.90							13.90
14.00							14.00
14.10							14.10
14.20							14.20
14.30							14.30
14.40							14.40
14.50							14.50
14.60							14.60
14.70							14.70
14.80							14.80
14.90							14.90

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THE SAMPLING VARIABILITY OF LINEAR AND CURVILINEAR REGRESSIONS

A FIRST APPROXIMATION TO THE RELIABILITY OF THE RESULTS SECURED BY THE GRAPHIC "SUCCESSIVE APPROXIMATION" METHOD

By

MORDECAI EZEKIEL[1]

Many statistical problems involve determining the change in one variable with changes in each of several others, all operating at the same time. Linear multiple correlation provides a method of making this determination, on the assumption that all the relations are linear. In many problems this assumption is not valid. To determine curvilinear relations without making assumptions as to the type of each curve except that it be a continuous function, a method of successive approximations by graphic fitting was presented six years ago; and it was demonstrated empirically that in cases of high correlation this method successfully determined the underlying curves.[2] It was also pointed out that multiple regression curves could be fitted by the least-squares method if specific parabolae or other first-degree equations were assumed for each variable, following methods previously sug-suggested by Yule.[3]

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[1] Formerly Senior Agricultural Economist, United States Department of Agriculture.

[2] Ezekiel, Mordecai. A Method of Handling Curvilinear Correlation for any Number of Variables. Quart. Pub, Amer Stat Assoc, XIX, No. 148, Dec., 1924.

[3] Yule, G. U. "On the Theory of Correlation," Jour. Roy Sta. Soc., Vol LX, p. 817 (1897). Apparently Wicksell had also suggested fitting regression curves to several variables simultaneously Wicksell, S D, Annals of Math. Stat., Vol I, No 1, pp. 3-15. Feb, 1930.

The advantage claimed for the successive approximation method was that it did not require assumptions as to the specific type of each curve, but instead permitted each regression to be indicated by the observations themselves

A new measure, the "index of multiple correlation," was suggested to measure correlation for curvilinear regressions in the same way that the coefficients of multiple correlation measured it for linear regressions.

No measure of the reliability of the net regression curves or of the index of correlation, was provided in the initial article. The usefulness of the results secured by this method has therefore been limited by the inability to state the confidence that could be placed in them even when based on a random sample, or to judge how large a sample would be necessary to infer, within any stated limits of precision and probability, the relations existing in the universe from which that sample was drawn.

This paper reports an attempt to determine the sampling error of multiple regression curves and indexes of correlation obtained by the successive approximation process, under conditions of simple sampling . The experimental method has been used to investigate the variability of results from 'successive samples drawn from the same universe under specified conditions and to establish error formulae inductively These experiments, representing the solution of over 150 multiple curvilinear correlation problems, indicate the possibility of establishing approximate expressions for the reliability of multiple regression curves and indexes of multiple correlation.[1] The results, however, are not fully consistent, and the error formulae are not completely satisfactory. The experimental results are therefore given in full, in the hope that the attention of mathematicians may be attracted to this problem, and that the tentative formulae may be modified to provide more rigorous and exact measures of the reliability of the curvilinear regressions and correlations.

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[1] The extensive computations involved in this investigation were carried through by Helen L. Lee and Della E. Merrick, and by others of the staff of the Division of Farm Management, U. S. Department of Agriculture Credit is due them for their intelligent and loyal assistance.

PART I—COEFFICIENTS AND INDEXES OF CORRELATION.

1 THE REDUCTION OF THE "DEGREES OF FREEDOM" BY FREE-HAND SMOOTHING.

When a line is fitted to a series of paired observations by the use of the formulae $Y=a+bX$, the assumption is made that the straight regression line is adequate to describe the relation. Two parameters one giving position to the line and the other slope, are required. For that reason, this equation will give a perfect fit to any two pairs of observations of X and Y. Furthermore, if the line is fitted to four pairs of observations, the determination of two parameters from four observations reduces the degree of freedom in obtaining the line from four to two, and the standard errors of the parameters must be determined with the number of degrees of freedom, N, equal to 2 instead of 4. Similarly, if a cubic parabola $Y=a+bX+cX^2+dX^3$ were fitted to ten observations, there would be only 6 degrees of freedom after determining the four parameters, and the standard errors would be based on $N=6$. In this case the four parameters determine position, slope, rate of change, and change in the rate of change.[1]

If instead of fitting a curve by the method of least squares or some other exact method, a free-hand curve is drawn by eye through the series of observations, it is necessary to make certain assumptions in drawing the curve, analogous to those represented in the parameters when more rigid methods are used. In addition to the basic assumption of continuity, these conditions may include:

(1) Whether the origin for $X=0$ will be at $Y=0$ or at some ordinate to be indicated by the data.

(2) Whether a straight line will be fitted (by ruler or thread) or whether a curve will be permitted.

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[1] The treatment of standard errors for small samples by "Student" and R. A. Fisher, as set forth in the latter's "Statistical Methods for Research Workers," give full recognition to these facts. Least square theory has always recognized that, for small samples, the number of parameters determined reduced the number of observations. See Wright, Thomas Wallace, and John Fillmore Hayford, "Adjustments of Observations," 1905, pp. 24-40, 132-133, and Merriman, Mansfield, "Method of Least Squares," 1911, pp. 80-82.

(3) If a curve, whether it will be limited (a) to a continuous arc of even curvature, (b) to a continuous parabola-like curve, (c) whether one or more inflections will be permitted, (d) whether the line will be so drawn as to minimize departures on the Y axis, the X-axis, or at right-angles to the line itself

It is evident that if a curve is drawn free-hand with its initial ordinate as indicated by the observations, with a continuous changing rate of curvature, and with no inflection, at least the three parameters of position, slope, and rate of change of curvature are represented, as shown by the corresponding equation for a parabola

$$Y = a + bX + cX^2$$

It is true that the free-hand curve may involve still more parameters, but three is the minimum. While the number of parameters represented in any free-hand curve cannot be exactly determined, it can be roughly estimated by a process of reasoning similar to that indicated above; and any measure of the sampling reliability of such free-hand curves would be more reliable if it allowed for the number of parameters assumed than if it ignored this reduction of the degrees of freedom.

It should be noted that while the process of fitting curves free-hand involves the "taste" of the investigator, represented in the conditions he places on himself as previously mentioned, and on his skill in drawing the line under those conditions, the process of fitting a curve by a mathematical formula also involves "taste" in deciding what formula to use If the conditions placed on the free-hand fitting are the same as those represented in the mathematical equation, the results may agree within the significant limits of error, and, therefore, either may be satisfactory for practical purposes.[1]

When coefficients of correlation or coefficients of multiple correlation are obtained from samples with a limited number of cases, the reduction in the number of degrees of freedom by the two or more parameters in the regression equation makes the observed correlation tend to exceed the true correlation in the universe from which the sample was obtained. Accordingly, even the usual linear correlation coefficients, if obtained from small samples, tends to exceed the true values. Adjustments to correct for this factor will be considered before going to the more complicated problem of adjustments in observed indexes of correlation.

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[1] Note the witty discussion of free-hand versus mathematical curves in the presidential address by E B Wilson, Proceedings: American Statistical Association, March, 1930.

2. BIAS IN COEFFICIENTS OF CORRELATION

Determining a coefficient of correlation from a finite sample reduces by 2 the number of degrees of freedom present As a consequence, there is a tendency for the computed correlation to exceed the true correlation in the universe, and a corresponding tendency for the computed standard error of estimate to fall below the true value. Exact measures of the "most likely" value of the correlation coefficient were given by Soper and others in 1917[1] and an elaborate method was provided for estimating it.[2]

Where a coefficient of multiple correlation for n_2 independent variables is determined from a finite sample of n' independent observations, the degrees of freedom are reduced by the $n_2 + 1$ parameters represented in the regression equation. If $n' = n_2 + 1$, the number of observations exactly equals the number of parameters to be obtained, the least square solution reduces to a simultaneous solution of the n' observation equations, and the coefficient of multiple correlation comes out 1.00 regardless of the presence or absence of correlation in the universe.

R. A Fisher called attention to this problem in 1924 and suggested an approximate adjustment of the observed correlation from limited samples by the equation

$$(1) \qquad 1 - \bar{R}^2 = \frac{n'-1}{n'-n_2-1}\left(1-R^2\right)$$

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[1] Soper, H. E., Young, A. W., Cave, B. M., Lee, A., and Pearson, K. On the Distribution of the Correlation Coefficient in Small Samples. A cooperative Study. Biometrika. Vol. XI, Part IV, May, 1917, pages 352-359.

[2] Locus, Cit., pp. 374-375.

where n' and n_2 have the same meanings as above, R is the correlation observed in the sample, and $\bar{R}$ is the most probable correlation in the universe.[1] This correction is very similar to that deduced independently by B. B. Smith in 1925, directly from the least square adjustment for number of constants.[2] In the same notation as above, Smith's adjustment:

$$\bar{R}^2 = 1 - \frac{1-R^2}{1-\frac{n_2}{n_1}}$$

may be stated

$$(2) \qquad 1-\bar{R}^2 \doteq \frac{n'}{n'-n_2}(1-R^2)$$

which differs from Fisher's formula only by the omission of the −1 from both numerator and denominator. In restating this formula a year ago[3] the present author modified it to the form

$$\bar{R}^2 = 1 - \frac{1-R^2}{1-\frac{n_2+1}{n'}}$$

or, stated in the same form as (1) and (2)

$$(3) \qquad 1-\bar{R}^2 = \frac{n'}{n'-n_2-1}(1-R^2)$$

This differs from both the previous equations in including the −1 term in the denominator but not in the numerator. The effect is thus to make the correction most severe; i. e., the corrected value departs still more from the uncorrected value than in either of the other forms.

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1. Fisher, R. A., The Influence of Rainfall on the Yield of Wheat at Rothamstead—Phil. Trans. B. ccxiii, 89-142; 1924.

2. Smith, B. B. Forecasting the Acreage of Cotton. Jour. Amer. Stat Assoc., March, 1925. Footnote on p. 41.

3. Ezekiel, Mordecai. Application of the Theory of Error to Multiple and Curvilinear correlation. Jour. Amer. Stat. Assoc, Supp, pp 99-104, Vol. XXIV, No 165-A. March, 1929.

The interpretation of correlation coefficients adjusted by any one of the three equations (1), (2), or (3) has been difficult because of lack of a definite explanation of the meaning of the adjusted coefficients. To determine their exact meaning, and to decide which one of the three forms of adjustment is most satisfactory, a study has been made of the relation of the adjusted values to the distribution of simple correlation coefficients when computed from random samples of various sizes drawn from universes with specified correlations. The "Cooperative Study" gives tables showing the exact theoretical frequency curves for zero order correlation coefficients, computed from samples of from 3 to 25, and 50, 100, and 400 observations, for true correlations ranging from 0 to .9, by tenths. Ordinates of the distributions of observed correlations are given for each value from $r = -1.00$ to 1.00 by .05 steps. With the frequency curve thus defined by as many as 41 ordinates, a rough integral of the curve was constructed by a cumulative summary of the ordinates. Then dividing by the total area, the proportion below any particular value was determined. When ρ (the true correlation in the universe) = 0, the summation was made from 0 in both directions to show the proportion of all samples showing correlations falling below the particular r, either plus or minus. When ρ exceeds 0, the summation was made from -1.00 to increasing values, to show the proportion of all the samples which show correlations falling below any particular value.

For each size sample investigated as described, more than 50 % of the theoretical observed correlations exceeded the true correlation. Thus for $\rho = .40$, with samples of 4, over 55 per cent of the samples showed r in excess of .40; 53 per cent with samples of 9, and about 51 per cent with samples of 50. But with $\rho = .80$, over 61 per cent of the samples showed r above .80 with samples of 4, 56 per cent with samples of 9, and 53 per cent with samples of 25. If we define the value which will be exceeded by exactly half the samples as the value which is most likely to be observed in any given sample, this "most likely" observed correlation is evidently in excess of the true value. The problem is to determine the adjustment equation, similar to eq. (1), (2), or (3), which will reduce the observed value to the correlation which exists in the universe from which it is most probable that that sample was drawn.

Frequency ogives (on a percentage basis) were constructed from the tables in the "Cooperative Study for $\rho = 0$, 0.2, 0.4, 0.6, 0.8, and 0.9, for $n' = 4$, 5, 9, 17, 25, 50, and 100. Equation (1) was then tested against these ogives, to determine what was the significance of the adjustment. For zero order correlations, equation (1) becomes

$$1-\bar{r}^2 = \left(\frac{n'-1}{n'-2}\right)(1-r^2)$$

Hence, with $\rho=0$ and $n'=9$, (r) would have to equal at least ±0.35 for $\bar{r}$ to be 0. Comparing this value, 0.35, with the frequency ogive for $\rho=0$, $n'=9$, it was found that only 35 per cent of the samples would give observed correlations larger than 0.35, or smaller than −0.35. Similarly for $\rho=0.6$ and $n'=17$, r would have to be 0.63 for $\bar{r}$ to be .60. For these conditions, 49 per cent of the samples would give observed correlation in excess of .63 Carrying out this same comparison for all of the ogives constructed gives results as shown in the following tabulation.

Size of sample (n')	When correlation in sample is					
	0.0	0.2	0.4	0.6	0.8	0.9
4	0.42	0.29	0.36	0.42	0.49	0.51
5	.39	.29	37	43	.48	.50
9	.35	.30	.38	44	48	49
17	.33	32	40	.45	.48	.49
25	.32	.34	42	46	48	49
50	.31	.37	44	.47	48	.50
100	31	.40	46	.48	49	.50

Proportion of samples, of specified sizes, drawn from universes of specified correlations, which show correlations in excess of the true value in the universe, even after adjusting the observed correlation by the formula

$$\bar{r}^2 = 1-\frac{n'-1}{n'-2}(1-r^2)$$

These values are determined from the graphs based on a rough integration by successive summations, and slight errors may have entered in making the graphic interpolations Hence the values cannot be regarded as precise. The error probably does not exceed 01 or .02 in any case. however, so the results are sufficiently exact to interpret the general effect of the correction formula.

It is evident from the table that when the true correlation is high, .80 or above, the probability of a value as large as that implied by the use of adjustment formula (1) is practically .50 Tests by the tables given in the "Cooperative Study" for the most probable value show that the probability becomes almost exactly .50 for larger samples and still higher correlations, the adjusted values by those tables and by the correction formula agreeing to the third or fourth decimal place.

Where the true correlation is low, however, the table indicates that the adjustment is too severe—that is, the probability of the true correlation in the universe being as high as the correlation shown after the adjustment is more than .50, and may be as high as .70 (for $n' = 4$ or 5 and $\rho = 0.2$). Even with this variation in the meaning of the adjusted value, however, equation (1) gives a valuable adjustment, since it indicates the probable correlation with almost exactly a .50 probability where the correlation is high, whereas it indicates the probable correlation with a higher probability—between .50 and .70—for those cases where the correlation is low and the standard error of the coefficient is correspondingly large

Comparison of equations (2) and (3) with the frequency ogives showed that where n' was small, the adjustment was more severe in the case of (3), and less severe in the case of (2), and did not in either case tend to approximate the 50 per cent probability, except where n' was very large. In some cases equation (2) gives corrected values so low that such cases are likely to occur more than 50 per cent of the time, and accordingly the probability would be even less than .50 that the correlation is really as high as shown by the adjusted coefficient.

It may be concluded that equation (1) gives the most satisfactory simple method for adjusting coefficients of simple or multiple correlation to remove the positive bias. *The adjusted value* thus obtained may be defined as *the value that most probably exists in the true universe, in the case of a high correlation, or a value slightly below the probable true value, in the case of a low correlation.*

The adjustment of the standard error of estimate may next be considered. When a standard deviation, σ_s, is calculated from the items in a sample of n' cases, the probable standard deviation of the items in the universe, σ_x, may be computed (following Fisher) as

$$\sigma_x^2 = \frac{n'\sigma_s^2}{n'-1}$$

So if the standard error of estimate is calculated by the usual formula

$$S_e^2 = \sigma_x (1 - R^2)$$

but the adjusted correlation, $\bar{R}$, is substituted for R, and the value just shown is used for σ_x, the equation becomes

$$S_e^2 = \frac{n'\sigma_x^2}{n'-1}\left[\frac{n'-1}{n'-n_r-1}(1-R^2)\right] \tag{4}$$

$$S_e^2 = \frac{n'\sigma_x^2}{n'-n_r-1}(1-R^2)$$

This is identical with the equation given by Fisher[1], though in different form.

3. CORRECTING FOR BIAS WITH INDEXES OF (CURVILINEAR) CORRELATION

Where correlation is measured with respect to curvilinear regressions, the greater number of parameters represented in the regression curve increases the tendency for the observed correlation to exceed the actual and requires a more drastic correction of the observed values. Where the regression curve is determined by a definite equation, the number of parameters is known, and the observed correlation may be adjusted to the most probable true correlation by the use of equation (1), as before Since the number of parameters, rather than the number of independent variables, now becomes of moment, the equation may be restated for curvilinear correlation

$$1-\bar{\rho}^2 = \frac{n'-1}{n'-m}(1-\rho^2)$$

using m to designate the number of parameters, and ρ and $\bar{\rho}$ to designate the observed and the adjusted index of correlation. This formula may be used either for simple or for multiple curvilinear correlation. Thus if the regression equation

$$X_1 = a + b_2 X_2 + b_2'(X_2^2) + b_3 X_3 + b_3'(X_3^2)$$

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[1] Fisher, R. A., Statistical Methods for Research Workers. 1928. P. 117, first equations; page 135, 2nd equation.

had been fitted, m would equal 5. For a sample of 20 observations and an observed multiple correlation of 0.80, the most probable true correlation would be but 0.74.

Where the regression curve or curves have been fitted free-hand, the observed correlation may be even more in need of adjustment than where a definite equation has been employed[1]

It is true that the number of parameters which it would take to duplicate the free-hand curve by a definite mathematical function cannot be exactly determined without finding some equation which will exactly represent the curve. On the other hand, even an approximate estimate of the number of parameters which would be required provides a better basis for judging the probable true correlation than does the observed correlation taken alone. Such an approximate estimate may be made by considering how many degrees of position, change, or movement are represented in the graphic curve. The following list suggests some of these:

(a) Position
(b) Direction
(c) Change of direction
(d) Change in the change of direction

Where several different free-hand regression curves have been obtained by the method of successive approximation, the number of parameters represented by each one must be estimated separately. Only a single "position" parameter is required, since the origin of each regression is purely arbitrary, depending upon the constant in the regression equation, and the origin assumed for each of the other curves. That is, in the curvilinear regression equation

$$X_1 = a + f(X_2) + f(X_3) + f(X_4)$$

the value of a depends upon the origin used in graphing each of the functions.

Once the number of parameters represented in the regression equation has been estimated, equation (4) may be used to adjust the observed correlation Until more exact information is available, the explanation of the precise meaning of the adjusted value which has just been developed for the coefficient of linear correlation, may be assumed (by analogy) to apply to the adjusted index of (curvilinear) correlation as well.[1]

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[1] Ezekiel, Mordecai. Application of the Theory of Error to Multiple and Curvilinear Correlations. Jour. Amer. Stat Assoc., March, 1929, Supp., pp. 99-104, Vol. XXIV, No. 165-A.

4 SAMPLING ACCURACY IN COEFFICIENTS OF CORRELATION

Although equations (1) and (4) may be used to find the *most probable* correlation in the universe from which a given sample has been drawn, they do not give any measure of the *range within which* the true value probably lies, for any specified degree of probability

It has long been recognized that coefficients of correlation, computed from small samples drawn from a universe in which some correlation exists, show a very skew distribution Even for samples of a size most used in actual research—up to $n = 100$ or larger—the distribution is so skewed that the computed standard error of the correlation coefficient is of relatively little value. Even with fairly large samples the chances of the observed value departing from the true value by four or five times its standard error are very much greater than any interpretation based upon the normal curve would indicate.[2]

Recent investigations by "Student" and by R. A. Fisher have developed means of determining the reliability of correlation coefficients while allowing for the skewness of their distribution That phase of the subject will not be developed in this article; it is referred to here merely to call attention to the fact that even after the most probable value for the true correlation has been determined, it may still be necessary to take account of how much confidence can be placed in that value—of how far the correlation obtained from the sample, even after adjusting as suggested, is likely to vary from the true correlation of the universe for any stated odds of probability[1]

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[1] The adjusted correlation corresponding to a given observed correlation, for any size of sample and value of m, may be more readily determined from a graphic chart, instead of eq. (1) or (4). Such a chart is shown in the appendix to "Methods of Correlation Analysis," by the present author, page 404. (John Wiley and Sons, 1930.)

[2] "Student," On the Probable Error of a Correlation Coefficient. Biometrika, Vol. VI., p 302, 1908
Soper, H. E., On the Probable Error of the Correlation Coefficient to a Second Approximation Biometrika, Vol. IX, p. 91, 1913.
Fisher, R. A., Distribution of the Correlation Coefficients of Samples, Biometrika, 10, p. 507, 1915.
Soper, H. E., A. W. Young, B. M. Cave, A. Lee, K. Pearson. Distribution of Correlation Coefficients in Small Samples. Appendix 11, to the papers of "Student" and R. A. Fisher. Biometrika, XI, p. 328-413.

It must be recognized that the interpretation of the reliability of a correlation merely serves to indicate the significance that may be attached to the observed correlation, in view of the possibility of variation of the observed value from the true value in the universe due solely to random variation in sampling. If the conditions under which the sample is obtained do not fulfill the assumptions of simple sampling, then obviously Fisher's methods cannot be used unless the necessary reservations or modifications are added.

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1. Fisher, R A On the "Probable Error" of a Coefficient of Correlation Deduced from a Small Sample. Metron, 1, No 4, p 3, 1921—Statistical Methods for Research Workers, pp. 159-175, 2nd edition, 1928—The General Sampling Distribution of the Multiple Correlation Coefficient. Proc. Roy. Soc., A Vol. 121, pp 654-673 1928

The methods developed by Fisher in the last of these articles have been made more readily available by the construction of graphic charts, both for simple and multiple correlations, which are given in the present author's "Methods of Correlation Analysis," pp. 400-403.

PART II—LINEAR AND CURVILINEAR REGRESSIONS

1 SAMPLING VARIABILITY OF LINEAR REGRESSIONS

Relatively little attention has been given in practical research work to the reliability of the regressions determined. Many correlation studies especially where multiple correlation has been employed, have been misinterpreted because proper attention has not been given to the standard errors of the regression coefficients As was pointed out recently,[1] this sampling variation may readily be so great in practical work as to invalidate the conclusions as to the effect of various variables, even when samples of considerable size are employed.

Fortunately, regression coefficients, derived from finite samples selected by random sampling, tend to be distributed in a normal distribution in the same way as does the arithmetic mean, so that elaborate devices necessary to allow for skewed distribution are not necessary If the necessary corrections are made for the failure of the distribution to be normal when the number of degrees of freedom falls below 30, the standard error of a linear coefficient of gross regression or of partial regression may be employed with only the same restrictions as apply in the case of the arithmetic mean. More recently the formula for regression errors has been extended by Working, Hotelling, and Schultz to develop the standard errors of each constant for curves fitted by least-square methods.[2]

Where the regression is represented only by a plotted curve instead of by a definite equation, the reliability of the curve has been unknown. Obviously, it cannot be estimated from the constants represented in the curve, for they are unknown, and only their number may be roughly estimated Some knowledge of the variability of such regression curves may, however, be obtained experimentally

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[1] Ezekiel, Mordecai The Application of the Theory of Error to Multiple and Curvilinear Correlations Jour, Amer. Stat. Assoc. Proceedings, 19th annual meeting, Vol XXIV, No. 165-A', pp. 99-104, March, 1929

[2] Working, Holbrook, and Hotelling, Harold Applications of the Theory of Error to the Interpretation of Trends Jour. Amer. Stat. Assoc. Proc., Vol. XXIV, 165-A, pp. 73-85, March, 1929.
Schultz, Henry. Discussion of above paper. pp 86-88
Schultz, Henry The Standard Error of a Forecast from a Curve Jour Amer. Stat. Assoc., June, 1930

2 Outline and Summary of Experimental Study of Sampling Variability of Multiple Curvilinear Correlation Results

The study was conducted by first constructing a set of data in which a dependent variable, X_1, was related to several independent variables according to known curvilinear regressions, and in which a certain known portion of the variance of X_1 was not related to any of the independent variables A second universe was then constructed with the same underlying functions, but with a different proportion of random variation in the dependent variable. Successive samples of various sizes were drawn at random from both "universes" and net (partial) regression curves and indexes of multiple correlation were computed separately for each sample The net regression curves obtained in successive samples of the same size were compared with the true curves and with each other to see how far the results determined from the samples differed from the true values, and how much variance there was among them The variability of the curves, for samples of different size, different true correlations, and different points along the curves, was then studied, and it was found possible to construct an error formula to estimate the standard error of the regression curves from the values obtained in the individual samples Checking this formula by applying it to each of the samples previously determined, the actual errors were found to be in fair agreement with the estimated errors.

For a more rigorous test of the new error formula for regression curves, two new synthetic universes were constructed Samples of various sizes were drawn from them, net regression curves computed separately for each sample, and the actual departures of the computed curves from the true curves checked against the error indicated by the new formula The agreement in this test was not so good as in the previous case, although 66.5 per cent of the ordinates of the curves showed errors no greater than their computed standard errors, only 20.3 per cent fell between 1 and 2 times the computed values, while 7.5 per cent fell between 2 and 3 times, as compared to 68.3, 27 2 and 4.3, the proportions to be expected if the distribution were normal.

On the other hand, 5.8 per cent of the ordinates had errors exceeding 3 times the computed standard error, and some departures in excess of 5 times the computed standard error were obtained. It is evident from these results that either (a) the tentative formula is not adequate to estimate the standard errors of regression curves determined by the free-hand method, or (b) that net regression curves obtained by the successive approximation process are so unstable that their errors cannot be represented by a normal curve, and possibly may be impossible of estimation by any mathematical process. In the hope that the attention of others may be drawn to this problem, and a more satisfactory error formula be obtained, the experimental study is given subsequently in as full detail as possible.

The indexes of multiple correlation obtained from successive samples of the same size, were studied with respect to (1) bias and (2) variability. As has been previously reported[1], the indexes of multiply correlation show an average positive bias even larger than that of coefficients of multiple correlation. Indexes of multiple correlation apparently require a correction which takes into account both the number of observations and the estimated number of constants represented in the regression curves, according to equation (4) already discussed. Further study of the variability of the correlations showed that as far as could be judged from the relatively small number of replications of each size sample (5 to 16) they tend to have a standard error of the order of

$$\sigma_\rho = \frac{(1-\rho^2)}{n'-m} \tag{5}$$

where n' and m have the same meaning as for equation (4), and where ρ represents the observed index of multiple correlation. If this very rough approximation for their sampling errors is found adequate, it would seem logical to expect Fisher's determination for the sampling error of multiple correlation coefficients to apply equally well to indexes of multiple correlation.

In concluding this summary, it must be reiterated that these conclusions are only tentativhe. They provide at least some indication of the reliability of curvilinear correlation results, for which previously nothing had been known The error formulae are only first approximations, however, and in the case of the error of net regression curves, are such a poor approximation that much more work remains to be done before the results of such analyses can be used with anything like the degree of confidence that can be felt in older and more well-established statistical procedures.

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[1] *Loc. Cit.*, Proc Amer. Stat. Assoc., March 1929, p. 100

Details of Experimental Study

3. Construction of Synthetic Universes

The set of data used in the initial sampling was constructed as follows:

1. Values for X_2 were obtained by taking the sum of values from two dice. The throws were repeated 500 times, giving 500 values.

2. To insure some curvilinear correlation between X_2 and X_3, values of X_3' were computed for each value of X_2, according to the following function.

Value of X_2	Value of X_3'	Value of X_2	Value of X_3'
2	3	8	6
3	4	9	6
4	5	10	7
5	5	11	8
6	6	12	9
7	6		

One die was then thrown, and the value for X_3 computed as the die reading $+X_3'$ $[\Sigma = \text{die reading} + f(X_2)]$.

3. Values for X_4' were then computed for each value of X_3, according to the following function:

Value of X_3	Value of X_4'	Value of X_3	Value of X_4'
3	4	10	0
4	3	11	0
5	2	12	0
6	1	13	0
7	1	14	0
8	1	15	0
9	1		

Again, one die was thrown, and the reading of the die added to the X_4' value to get X_4. This gave a set of 500 values of X_2, X_3, and X_4, fairly normally distributed, with positive correlation between X_2 and X_3 ($r = +.534$); with a negative correlation between X_4 and X_2 ($r = -.489$); and between X_3 and X_4 ($r = -.234$); and with all of the inter-correlations more or less curvilinear.

4. Values for a dependent variable, X_1, were then calculated according to the relation

$$X_1 = f(X_2) + f(X_3) + f(X_4) + e \quad ,$$

where the values for each of the functions were read from the assumed regression curves tabled below, and where e was obtained by throwing two dice, and taking the sum of the readings.

VALUES FOR ASSUMED REGRESSION CURVES

X_2	$f(X_2)$	X_3	$f(X_3)$	X_4	$f(X_4)$
2	2.6	4	2.0	1	0.0
3	3.4	5	1.5	2	0.2
4	4.0	6	1.3	3	0.7
5	4.4	7	1.0	4	1.7
6	4.7	8	1.0	5	3.0
7	5.0	9	1.3	6	4.1
8	5.0	10	1.7	7	5.0
9	5.0	11	2.1	8	5.0
10	5.0	12	2.8	9	4.5
11	5.0	13	3.6	10	3.3
12	5.0	14	4.4	11	2.5
		15	5.2		

Values for a second dependent variable, Y, were obtained by using the same assumed regressions, but obtaining the value for e by throwing a single die, rather than two dice. This gave two sets of 500 observations, both identical as to the independent variables, but with different dependent variables, and with the true correlation higher in one universe than in the other, since the dependent variable included a smaller proportion of random variation in one case than in the other. The complete set of 500 paired observations are shown in Table A.

4. Drawing Random Samples

Thirty-one separate samples were drawn from each of the 2 "universes"; 5 samples of 100 observations each; 10 samples of 50 observations; and 16 samples of 30 observations. In making the drawings, slips numbered from 1 to 500 were mixed in a box, and drawn at random. They were stirred afresh between each drawing. In making the drawings for the X_1 universe, the slips were not returned to the box until each sample was completed; so that the same set of data would appear only once in each sample. In making the drawings for the Y universe, each slip was returned to the box as soon as its number was noted. In a few cases this resulted in the same observations appearing twice in the same sample. While 500 is not an "infinite" universe as compared to a sample of 100, the difference in the method of drawing appeared to make no practical difference in the variability in the two sets of samples. However, the fact that the samples made an appreciable proportion of the "universe" would mean that the variability in the observed results would not be quite as large as if drawn from an infinite universe. Using Bowley's statement of this the maximum effect[1], however, which would be for the samples of 100, would make the of the observed deviations about one-tenth smaller than it would have been if determined by drawings from an infinite universe of similar characteristics.

For, following Bowley,

$$\sigma_{s'} = \sigma_s \sqrt{1 - n_s / n_u}$$

Where, $\sigma_{s'} = \sigma$ of actual sample, from a finite universe

$\sigma_s = \sigma$ of a similar sample, from an infinite universe

n_s = number of cases in sample

n_u = number of cases in the finite universe

Hence where $n_u = 500$, $n_s = 100$, then $\sigma_{s'} = .894\,\sigma_s$

Since the effect of the limited universe on the variation in the results can thus be estimated, the results can be transformed to what they would probably have been had a much larger universe been available for study.

5. CURVILINEAR REGRESSIONS DETERMINED FROM THE SAMPLES

Net regression curves were determined for each sample by the method of successive graphic approximations, and indexes of multiple correlation were computed for each set of curves. Each sample was carried through successive approximations until no further significant increase in correlation was found by further modifications of the curves. From 2 to 4 approximations were necessary, in various cases. The multiple correlation found for each sample at the first (linear) solution, and for each successive set of curves, are shown in Table B. For the Y universe, a multiple correlation was run to adjust, by least squares, the slope of each regression curve according to the formula.[1]

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[1] Bulletin Int. Institute Statistics, Proceedings, Rome, 1925. Annex by A. L. Bowley, Cambridge Univ. Press.

$$Y = a + b'_2\left[f(X_2)\right] + b'_3\left[f(X_3)\right] + b'_4\left[f(X_4)\right]$$

The indexes of multiple correlation (necessarily higher than the previous indexes) as found by this process are also shown in Table B. The further study of the sampling variability of the regression curves was based on the set of regression curves for each individual sample which showed the highest correlation for that sample

6. ERRORS IN REGRESSION CURVES FROM THE SAMPLES

The net regression curves determined from each successive sample were all put on a comparable basis by adding a constant to each so that the central ordinate of each would equal the central ordinate of the corresponding true regression curve The differences between the adjusted ordinates at other points along the curves and the true ordinates would then show the errors in the curves. That is, the difference between ordinates at the central value and the ordinates at other points along the curve, as shown for the curves determined from the samples, were compared with the same differences for the true curves.

This procedure centered attention on the reliability of the slope and shape of the curves, rather than on the accuracy of their position. It is true that in linear correlation, the a as well as the b of the formula $Y = a + bx$, is subject to sampling errors, and formulae have been devised to compute its standard error. In the present case, however, it seemed desirable to first solve the problem of the shape and slope of the curve, before attacking the further problem of its position.

The departures of the curves found in the several samples from the true values for each curve are shown in Table C, for selected ordinates The central point of reference (and therefore the point of 0 error) was taken at approximately the mean value of each independent variable.

The individual samples were studied to see if there was any relation between the correlation observed in individual samples and the errors in the regression curves. No relation whatever was found between the size of the correlation in the individual sample and the size of the errors for the sample so long as samples of the same size and drawn from the same universe were compared.

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[1] See pages 445-447, Dec 1924, Jour Amer Stat. Assoc., for the original discussion of this process.

Standard errors for the linear partial regression coefficients were computed for each sample by the standard formula given by Yule, and, modified, by R. A. Fisher:

$$\sigma^2_{b_{12.34}} = \frac{\sigma_1^2 (1 - R^2_{1.234})}{n'\sigma_2^2 (1 - R^2_{2.34})}$$

When the actual errors in the regression curves for individual samples were compared with these standard errors, again no relation was found for samples of the same size and drawn from the same universe For that reason it was decided to abandon further study of the characteristics of individual samples, and instead study the characteristics of each entire set of samples of the same size and from the same universe.

7. Derivation of Tentative Error Formula

Study of the errors showed that, so far as could be judged from the limited number of observations, they had a marked tendency to a normal distribution. However, to prevent undue weighting of single extreme cases, the average deviation was used instead of the standard deviation as a basis for summarizing the results shown by different samples of the several sizes. These average deviations are shown in Table 1 (page 298).

Each of these results would be expected. The true standard error of estimate for Universe X is 2.39, and for Universe Y is 1.80, or 75.3 per cent as large. It would therefore be reasonable to expect that, other things being the same, the errors in the ordinates of the regressions for Universe Y would average only three-quarters as large as the corresponding errors for Universe X. Stating each mean error (Table 1) in Universe Y as a percentage of the corresponding mean error in Universe X, and taking the geometric mean of these percentages, it appears that on the average the errors in Universe Y are 78.5 per cent as large as in Universe X, or in fair agreement with the proportion expected The extent to which average error shown in Table 1 for the selected ordinates in Universe X are correlated with the average error for the corresponding ordinate in Universe Y are shown graphically in Figure 1[1] It is evident that the individual group averages agree fairly well with the expected relation Accordingly, it was concluded that any formula for the standard error of net regression curves would have, for one component, $\bar{S}_{1.234}$, the standard error of estimate for the dependent variable, just as does the formula for the probable error of a linear net regression coefficient, which is

$$\sigma^2_{b_{12.34}} = \frac{\bar{S}^2_{1.234}}{n'\sigma^2_2\left(1-R^2_{2.34}\right)}$$

TABLE 1.

Average deviation of errors in net regression curves, at selected ordinates for various sizes of sample

		Universe X			Universe Y		
X_2	$f(X_2)$	16 samples of 30	10 samples of 50	5 samples of 100	16 samples of 30	10 samples of 50	5 samples of 100
3	11.4	1.66	1.19	0.34	0.90	0.82	0.50
5	12.4	0.93	0.63	0.24	0.48	0.50	0.26
7	12.9	0.00	0.00	0.00	0.00	0.00	0.00
9	13.0	0.72	0.56	0.34	0.38	0.32	0.24
11	13.0	1.48	1.09	0.77	0.71	0.58	0.50
X_3	$f(X_3)$						
5	12.5	1.65	1.25	1.04	1.38	0.52	1.10
7	12.0	0.84	0.44	0.38	0.38	0.18	0.16
9	12.3	0.00	0.00	0.00	0.00	0.00	0.00
11	13.1	0.61	0.47	0.24	0.41	0.56	0.52
13	15.6	1.35	0.93	0.82	1.56	1.24	0.88
X_4	$f(X_4)$						
2	10.2	0.69	0.80	0.38	0.58	0.80	0.50
3	10.7	0.52	0.54	0.48	0.39	0.36	0.34
4	11.7	0.35	0.36	0.40	0.30	0.16	0.22
5	13.0	0.00	0.00	0.00	0.00	0.00	0.00
6	14.1	0.28	0.29	0.12	0.34	0.54	0.22
7	15.0	0.72	0.67	0.40	0.68	0.68	0.54
8	15.0	1.50	1.09	0.76	1.14	0.92	0.66
9	14.5	2.26	1.60	1.00	1.34	1.25	0.74

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[1] This and subsequent figures will be found at the conclusion of the paper.

It is evident from Table 1 and from Figure A, which shows the data graphically for Universe X, (a) that in general the larger the sample the smaller the average error; (b) that the further from the center ordinate, the larger the error; and (c) since the errors in Universe X were usually larger than in Universe Y, that the lower the true correlation, the larger the error.

The influence of sample size may next be considered The number of observations is involved in two ways in the results shown in Table 1. In the first place, the average error tends to vary somewhat inversely with the size of sample. But in addition, it tends to vary with the distance from the central ordinate. Since the independent variables were composed of elements derived from dice readings, their distribution was roughly normal As a result, the number of observations upon which the regression curves were based was largest toward the center portions, and thinned out toward the extremes. In the graphic approximation method of determining the curves, each portion of the curve is determined from the cases falling within that portion, rather than from all the cases as a whole. Accordingly, it seems logical to try to relate the observed differences in the average deviation of the errors to differences in the number of cases from which they were determined, rather than to the total size of sample.

There is no precise range within which the observations can be said to be considered in free-hand fitting Instead of trying to measure the exact *number* of cases within any specified range, therefore, it seemed desirable to establish a measure of the *concentration* of observations at any point along the curve Thus, for example, if within a given interval of X_1, with a group interval of u units, there are n_u observations, we can express the concentration of observations at the mid-point of that group by the relation

$$n_k = n_u \left(\frac{\sigma_x}{u_x} \right)$$

If the group-interval is taken equal to the standard-deviation of the variable, n_k will be simply the number of cases falling within that group. If, however, the group-interval is made either larger or smaller than the standard-deviation, this equation will measure the *concentration* of observations *in terms of the number per standard-deviation range*. In a rectangular distribution, changing the value of u_x

would change the size of n_u to a corresponding extent, so the value of n_k would be independent of the group-interval selected. In a normal distribution, however, n_k would be only an approximation of the true value which would be secured from the theoretical distribution when the total number of cases was made very large and u_x was made infinitely small.

On the basis of the foregoing reasoning, it was thought that the differences in the average deviations within each universe as shown in Table 1, might be explained by differences in the number of cases which each *portion* of each curve was based upon. In sampling theory the dispersion of values of a constant determined from successive samples ordinarily varies with $\frac{1}{\sqrt{n}}$, rather than with $\frac{1}{n}$, hence, in this case, it was tentatively assumed that the value $\frac{1}{\sqrt{n_k}}$ would be a component of the formula for the error of ordinates of regression curves. This hypothesis was tested by adjusting the average shown in Table 1 by multiplying each of these by the factor $\frac{1}{\sqrt{n_k}}$, determining the n_k in each case from the true distribution of that variable in the whole universe, and from the total number of cases in the samples. These average differences would presumably reflect the true distribution of each independent variable in the original universe, since the variations in distribution in different samples would tend to cancel out. We may therefore use the distribution of the entire universe to indicate the average distribution within samples of specified sizes drawn from that universe. The calculation of n_k for each ordinate in accordance with this method is shown in Table 2.

TABLE 2

Calculation of n_k values for selected ordinates and various sizes of samples.

Group	Number of cases (n_v)				Value of n_k[1]		
	In entire Universe	30	50	100	30	50	100
X_2							
3	32	1.92	3.2	6.4	2.171	2.803	3.964
5	55	3.30	5.5	11.0	2.846	3.674	5.197
9	53	3.18	5.3	10.6	2.794	3.607	5.102
11	38	2.28	3.8	7.6	2.366	3.055	4.370
X_3							
5	9	0.54	0.9	1.8	1.081	1.396	1.974
7	70	4.20	7.0	14.0	3.015	3.892	5.505
11	91	5.46	9.1	18.2	3.437	4.437	6 276
13	18	1.08	1.8	3.6	1.529	1.974	2.792
X_4							
2	62	3.66	6.2	12.2	2.771	3.577	5.060
3	76	4.56	7.6	15.2	3.093	3.993	5.648
4	79	4.74	7.9	15.8	3.153	4.071	5.757
6	91	5.46	9.1	18.2	3.385	4.370	6.181
7	55	3.30	5.5	11.0	2.631	3.397	4.804
8	26	1.56	2.6	5.2	1.809	2.335	3.303
9	16	0.96	1.6	3.2	1.419	1.832	2.591

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[1] Computed from formula $n_k = n_v \left(\frac{\sigma_x}{v_x}\right)$, with $v_x = 1$, $\sigma_2 = 2.455$; $\sigma_3 = 2\ 164$, $\sigma_4 = 2\ 098$, $v_x = 1$, since the frequencies for 3 include 2.5 to 3.5; for 5, 4.5 to 5.5, etc.

TABLE 3

Average deviation of errors in net regression curves, at selected ordinates, adjusted to error per unit observation per standard-deviation range

Group	Universe X			Universe Y		
X_2	[1] 30	50	100	30	50	100
3	3.60	3.34	1.35	1.95	2.30	1.98
5	2.65	2.31	1.25	1.37	1.84	1.35
7	0.00	0.00	0.00	0.00	0.00	0.00
9	2.01	2.02	1.73	1.06	1.15	1.22
11	3.50	3.33	3.33	1.68	1.77	2.16
X						
5	1.78	1.75	2.05	1.49	0.73	2.17
7	2.53	1.71	2.09	1.15	0.70	0.88
9	0.00	0.00	0.00	0.00	0.00	0.00
11	2.10	2.09	1.51	1.41	2.48	3.26
13	2.06	1.84	2.29	2.39	2.45	2.46
X						
2	1.91	2.86	1.92	1.61	2.86	2.53
3	1.61	2.16	2.71	1.21	1.44	1.92
4	1.10	1.47	2.30	0.95	0.65	1.27
5	0.95	1.27	0.74	1.15	2.36	1.36
6	0.00	0.00	0.00	0.00	0.00	0.00
7	1.89	2.28	1.92	1.79	2.31	2.59
8	2.71	2.55	2.51	2.06	2.15	2.18
9	3.21	2.93	2.59	1.90	2.29	1.92

When the values in Table 1 are multiplied by the corresponding n_k values, from Table 2, the adjusted values shown in Table 3 are obtained. Averaging together all the values in Table 3, average adjusted errors of 1.89 are secured for samples of 300 cases, 2.04 for samples of 50, and 1.98 for samples of 100 cases. It is evident that most of the difference due to different sizes of samples has been eliminated. However, even after this adjustment, the errors tend to increase as the ordinate departs from the assumed point of origin at the center. This same relation holds for linear regression lines. The standard error of any point on a regression line (in relation to the origin at $M_y = 0$) is $\sigma_b x$, and hence increases directly as x increases. A line continues out with the slope given it by b, and any error in b has a progressive influence on the accuracy of the line. The free-hand curve, on the contrary, is more flexible, and does not continue in any determinate direction. Hence it would hardly be supposed that the errors in the ordinates of the curve would increase with increasing values of X so rapidly as does the standard error of the straight line. The errors shown in Table 3 may be tested with respect to this hypothesis by averaging, for each universe, the errors shown by the three sizes of samples for the several selected ordinates and relating the resulting averages to the departures from the assumed means. To put these departures in comparable terms for the three variables, they may be stated in terms of standard deviation units. Carrying these operations through, the data appear as shown in Table 4.

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[1] Number of observations in each of the successive samples.

TABLE 4

Average adjusted deviation of errors at selected ordinates, contrasted with departure from origin

Group	Departure from origin	$\frac{\text{Departure}}{\sigma}$	$\sqrt{\frac{d}{\sigma}}$	Average adjusted errors Universe X	Average adjusted errors Universe Y
X_2					
3	4	1.63	1.06	2.76	2.08
5	2	0.81	0.90	2.07	1.52
7	0				
9	2	0.81	0.90	1.92	1.14
11	4	1.63	1.06	3.39	1.87
X_3					
5	4	1.85	1.36	1.86	1.46
7	2	.92	0.96	2.11	91
9	0				
11	2	.92	0.96	1.90	2.38
13	4	1.85	1.36	2.06	2.43
X_4					
2	3	1.41	1.20	2.23	2.33
3	2	.95	0.97	2.16	1.52
4	1	.48	0.69	1.62	.96
5	0				
6	1	.48	0.69	.99	1.62
7	2	.95	0.97	2.03	2.23
8	3	1.43	1.20	2.59	2.13
9	4	1.91	1.38	2.91	2.04

It is evident from Table 4 that the average error, adjusted for size of sample, increased as the departure from the origin increased. This is shown more clearly in Figure 2, where the average error is plotted against the departures from the origin. This figure, however, indicates that the relation is not linear, as the errors do not increase in proportion. When the average errors are plotted against the departures on semi-log paper, however, as shown in Figure 3, the relation is substantially linear, and is of such an order as to suggest that the errors vary with the square-root of the departures, rather than the departures themselves The line drawn in on each chart, with such a slope as to coincide with the square roots, parellels the relation fairly well, so from this it may be concluded that another constituent of the error formula will be

$$\sqrt{\frac{\text{Units departure from origin}}{\sigma_x}}$$

If the origin is made at the mean of X , the independent factor, X_2, X_3, etc., this segment of the error formula may be stated (using $x = X - M_x$)

$$\sqrt{\frac{x}{\sigma_x}}$$

Each of the adjusted errors shown in Table 4 may be further adjusted by dividing each one by $\sqrt{\frac{dep}{\sigma_x}}$, the value shown in the third column. They may also be adjusted to allow for the difference in the original standard errors of estimate in the two universes, as noted earlier. The standard error in Universe Y was 1.80 and Universe X, 2.39, so the errors may be made comparable by dividing those from each universe by the corresponding standard error of estimate. Performing these two operations, the average deviations of the errors appear as shown in Table 5 These average deviations are now so adjusted as to eliminate differences due to (1) number of observations in each portion of the distribution, (2) departure from origin, and (3) standard error of estimate in the universe As stated in Table 5, the average deviations are in per cent of the deviations that would have been estimated from an equation representing the three elements discussed.

TABLE 5

Average deviation of errors at selected ordinates adjusted for n_k, $\sqrt{\frac{\bar{x}}{\sigma_x}}$ and s_e

Group	Universe X	Universe Y	Average
Average X_2 – 3	1.09	1.09	
5	0.96	0.94	
9	0.89	0.70	
11	1.33	0.98	
Average X_2	1.07	0.93	1.00
X_3 – 5	0.57	0.60	
7	0.92	0.53	
11	0.83	1.38	
13	0.63	0.99	
Average X_3	0.74	0.88	0.81
X_4 – 2	0.78	1.08	
3	0.93	0.97	
4	0.98	0.77	
6	0.60	1.30	
7	0.83	1.28	
8	0.90	0.99	
9	0.88	0.82	
Average X_4	0.84	1.03	0.94

Averaging all values for each variable, as shown in Table 5, there still remains some difference in the average errors. The errors for $f\ (X_3)$ are smaller on the average than the errors for either of the other variables, while those for $f\ (X_2)$ are larger. This suggests that some element other than those already considered influences the errors, and that it differs with individual independent variables.

The formula for the standard error of a linear net regression coefficient contains the term

$$\frac{1}{\sqrt{1 - R^2_{2.34}}}$$

which allows for the intercorrelation between the independent variables. The more closely an independent variable may be estimated from the other independent variables, the less accurately its net regression line can be determined. The same relation might be expected to hold true of multiple regression curves We can test this by comparing the average adjusted errors, just computed, withh the intercorrelation, as follows:

Regression	Mean Adju ted Error	$\sqrt{1-R^2}$	$\frac{\text{Mean Error}}{1-R^2}$
$f(X_2)$	1.00	$\sqrt{1-R^2_{2.34}} = 0.757$	0 76
$f(X_3)$	0 81	$\sqrt{1-R^2_{3.24}} = 0.844$	0.68
$f(X_4)$	0.94	$\sqrt{1-R^2_{4.23}} = 0.874$	0.82

It is evident that the means vary somewhat inversely with the $\sqrt{1-R^2}$ values. They may therefore each be multiplied by the corresponding $\sqrt{1-R^2}$ value to secure the final adjusted values, as shown in the last column[1]. This column now shows the average deviation of the errors actually observed stated in per cent of an estimated error computed from a theoretical equation composed of the four elements developed separately.

The average deviations of the observed errors varies from 68 to 82 per cent of the estimated error in each case, as contrasted to the value of 80 per cent to be expected if the equation gave the standard error. This is consistent with the fact that the standard error of estimate is included as the initial value in the equation. Furthermore, since the samples were drawn from a limited universe, the variation observed would tend to be slightly less than if they were drawn from an infinite universe with the same characteristics, which is consistent with the observed values falling mostly a little below the expected value of 0.80. The elements considered in estimating the error may therefore be said to give the standard error of the regression curves.

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[1] This demonstration is by no means convincing proof of the need of including this adjustment. After this final adjustment, the discrepancy between the smallest and largest average errors, 0.68 and 0.82, is still as great as it was between the smallest and largest before, 0 81 and 1 00. On logical grounds, however, some such adjustment for the closeness of inter-relation between the independent variables is necessary, and by analogy, this method seems a possibility. It may be, however, that the index of (curvilinear) multiple correlation, $P_{2.34}$, should be used in the adjustment, rather than the coefficient of multiple correlation,

By a combination of induction and deduction, of which the foregoing is a condensed re-statement, a tentative formula for the standard error of the ordinates of a net regression curve was constructed from the four elements developed separately. They may be combined as follows[1]:

I. $$e_{f(X_2)} = \left(\bar{S}_{1.234}\right)\left(\frac{1}{\sqrt{n_k}}\right)\sqrt{\frac{x}{\sigma_x}}\left(\frac{1}{\sqrt{1-R^2_{2.34}}}\right)$$

or writing n_k out in full,

II. $$= \left(\bar{S}_{1.234}\right)\left(\frac{u}{\sigma_x\ n_u}\right)\sqrt{\frac{x}{\sigma_x}}\left(\frac{1}{\sqrt{1-R^2_{2.34}}}\right)$$

Hence

III. $$e^2_{f(X_2)} = \frac{\bar{S}^2_{1.234}\ u_x\ x}{n_u\,\sigma^2_x\,\left(1-R^2_{2.34}\right)}$$

8. TESTING TENTATIVE FORMULA BY SAMPLES DRAWN FROM THE ORIGINAL UNIVERSE

The formula which has just been shown was derived from the *average* errors shown by all the samples, using the known facts about each universe—the standard error of estimate, the frequency distributions and the standard deviations of each independent variable, and the inter-correlations among the independent factors in working out the estimated errors. But for practical use in estimating the reliability of regressions determined from a single sample all that would be known about the universe would be what could be inferred from that sample, and the standard errors of the regression curves would have to be computed from the values so obtained. The next step of the experiment, therefore, was to calculate the standard error separately for each sample in turn, using only the values obtained from each one. These computations were made for each independent variable for each abscissa listed in Table C. The actual error of the regression curve at that point was then compared to the calculated standard error, and the ratio

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[1] Equation (III) may be restated in a simpler form for practical computation, and the operations of working out the standard error for selected ordinates along the net regression curves may be organized in a systematic manner, as shown in the author's "Methods of Correlation Analysis," pages 384 to 389.

$$\frac{\text{Observed error}}{\text{Calculated standard error}} = T$$

computed for each selected abscissa. If the computed error was the true standard error of the regression curve, these ratios should then be distributed according to the normal curve, and should have a standard deviation of 1.00.

The test was first applied to all the samples from both universes without including the term $1 - R^2_{2.34}$ in the error formula.

The standard deviation of the ratios σ_T was calculated separately for each selected abscissa of each independent variable with results as follows:

TABLE 6

Standard Deviation of Ratios of Actual Errors to Calculated Errors, as shown by 62 separate samples

Value of independent variable	Errors in $f(X_2)$	Errors in $f(X_3)$	Errors in $f(X_4)$
2			0.96
3	1.13		1.06
4			0.98
5	1.10	0.81	
6			1.19
7		0.82	1.21
8			1.23
9	0.89		1.11
11		1.13	
13	1.34	0.87	
All values	1.19	0.94	1.10

It is evident (1) that σ_T does not tend to increase appreciably as the abscissa departs from the mean of the independent variable; and (2) that the results based on the errors computed from individual samples are on the average quite consistent with those based on the facts from the universe. This is shown more fully in the following comparison:

Regression	Errors from individual samples		Errors from entire universe; mean adjusted error
	σ_T	$0.80\ \sigma_T$	
$f(X_2)$	1.19	0.95	1.00
$f(X_3)$	0.94	0.75	0.81
$f(X_4)$	1.10	0.88	0.94

Taking 0.80 of the σ_T gives an approximate measure of the average deviations of the T values, to compare with the average deviation of the adjusted errors as calculated in Table 5. The average deviation of the T values ranges from 93 to 95 per cent of the average adjusted errors, showing the same average differences from variable to variable as were shown in Table 5 and suggesting the need of some element in the error formula to allow for the inter-correlation among the independent variables.

For the next step in the test, the term $1 - R^2_{2.34}$ was included in the error formula for $f(X_2)$ and the corresponding terms were included in the other formulas, using, in each case, the R values shown by each individual sample. Calculating the T values by comparing the actual errors with these revised estimates, and calculating their standard deviations, results were secured as follows.

Regression	σ_T, using full error formula
$f(X_2)$	0.77
$f(X_3)$	0.76
$f(X_4)$	0.90

The σ_T is calculated from 0 as origin, disregarding differences in the average error from zero. It is evident that in these sample results the errors, on the average, are somewhat less than would be expected from the formula, as σ_T falls below the unity. The distribution of the errors is also important. Figure 4 shows the distributions of the T values and compares it with the corresponding normal distribution. The extent of the agreement with the normal distribution may be judged from the following comparison:

Value of	Per cent of total frequencies in range			
	$f(X_2)$	$f(X_3)$	$f(X_4)$	Normal distribution
Over 3.00			0.5	0.14
2.00 to 2.99	0.9		2.6	2.14
1.00 to 1.99	6.5	4.4	11.9	13.59
0.00 to 0.99	46.1	40.1	31.3	34.13
0.00 to −0.99	38.3	46.2	43.2	34.13
−1.00 to −1.99	6.5	8.4	10.5	13.59
−2.00 to −2.99	1.7	0.9		2.14
−3.00 and larger				0.14

Although the distributions are not exactly normal, they agree fairly well. The different variables give slightly different distributions, however. For $f(X_3)$, in farticular, the distribution of the errors appears to be skewed, with more negative errors than positive ones. This may be due to a slight bias in the free-hand method of fitting the curve, which in this instance, for a very peculiarly-shaped regression curve, led to a slight but persistent error in the fitted curve. This possible individual bias in fitting the curve free-hand will be taken up again subsequently.

The test of the error formula described above was not a complete proof of the adequacy of the formula, since it used the same samples as those from which the original formula was constructed. For a more rigorous test the formula would have to be tried out on completely new samples secured from a different universe. Such a test was made in the next phase of the investigation.

9. Testing Tentative Formula by Samples Drawn from a New Universe

A new "universe" was constructed for testing purposes, by methods parallel to those described before. In this case only two independent variables were used. There were 328 observations in the universe and 45 samples were selected at random—15 of 10 observations, 15 of 20 and 15 of 40. (The number of observations was taken as small as 10 so as to make an extreme test of the value of the sampling formula.) Multiple curvilinear regressions were determined for $f(X_2)$

and $f(X_3)$, and the standard error of selected ordinates was computed by equation (III) The value of T was then computed by dividing the actual errors by the expected. The distribution of these errors is shown in Figure 5, as contrasted with the normal curve.

When the standard deviations of T are computed separately for each size of sample, the results are as follows.

	Size of Sample		
	10	20	40
$f(X_2)$	1 29	1.30	1.23
$f(X_3)$	1.46	1.80	1 79

Combining the distribution for both $f(X_2)$ and $f(X_3)$, the distributions of the errors for each size of sample are as follows·

Value of T	Size of sample 10	20	40	Normal Distribution
	Per cent of total	Per cent of total	Per cent of total	Per cent of total
Over 3.00	2.0	2.4	2.9	0.14
2.00 to 2.99	3 3	4.5	3 2	2.14
1.00 to 1 99	10 7	9 2	12.8	13.59
0.00 to 0.99	34.2	30 8	31.1	34 13
0.00 to –0.99	35.8	34.4	33.3	34.13
–1.00 to –1.99	8.3	11.5	7.8	13.59
–2.00 to –2.99	2 4	3 6	5.5	2.14
–3 00 and larger	3 3	3.6	3 4	0 14

There were many more wide departures—of 3.00 or larger—than would be expected if the errors had a normal distribution, with σ = the estimated standard error. Instead of only 5 per cent of the errors exceeding twice the estimated standard errors, from 11 to 15 per cent were this large Yet the general distribution of the errors (Figure 5) was in fair agreement with a normal distribution.

Two elements may contribute to the greater variation in the actual errors than in the estimated With samples of the size involved—10 to

40 cases—the shape of various portions of the curve is determined by much less than 30 observations, and in some cases by 10 or less. With such small samples, Student and Fisher have shown that for arithmetic means and other constants, the distribution of actual error ÷ estimated error does not follow the normal curve and has a σ in excess of unity. It may be that some modification needs to be introduced into equation (III) to take account of this tendency before it can be correctly applied to small samples. From Student's table for small samples[1], 15 per cent of the errors would be expected to exceed twice the standard error if there were 3 degrees of freedom in the sample, and 10 per cent if there were 5. This indicates a reasonable number of cases, as compared with the size of the samples used in these tests. But whether $\frac{n_v \sigma}{U}$, or some other fraction of the total number of observations, would give the proper number of cases to use in entering the table, has not been determined, and more work needs to be done on this phase of the problem.

A second element of error appears to lie in using $\frac{1}{1-R^2_{2.34}}$ as one element of the error formula, instead of using the index of correlation. $\frac{1}{1-P^2_{2.34}}$. Substituting the index of correlation for the coefficient in the error formula was tried in two of the samples where the T values were the highest, and in both cases it much improved the accuracy of the estimated error—reducing values of T from 5.0 to 3.0, from 8 3 to 4 7, from 6.7 to 3.8, etc. It would appear that wherever the inter-correlation between the independent factors is markedly curvilinear, the accuracy of the estimate of the error could be much improved by measuring that curvilinear inter-correlation, and using it in computing the standard error of the function.

In view of the two sources of variation mentioned above, the fact that the variation of the actual errors ranges from 23 per cent to 79 per cent in excess of the variation of the estimated errors does not necessarily mean that the suggested formula (eq. III) is entirely inadequate, but may mean only that the necessary reservations in the use of the formula have not been applied. On the other hand, the fact that the actual results do vary as widely as this from the expected suggests that the formula can be used only as a very tentative approximation to the standard error of the regression curves until its possibilities and limitations have been more definitely determined.

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[1] This table is reproduced, in abridged form, in the author's "Methods of Correlation Analysis," on pages 19 and 392.

10 Free-hand versus Mathematical Net Regression Curves

It was noted earlier that there appeared to be some tendency toward bias in fitting the first set of curves. The errors from the second universe, as shown in the last set of results showed a little of the same tendency, with the average error not falling exactly at 0. To test whether determination of the regression curves mathematically would eliminate this bias, mathematical partial regression curves were fitted by least squares to one set of samples from the second universe. The 15 samples of 20 observations were used, and two types of curves were fitted—the parabola and the cubic parabola. The regression equations were therefore:

$$(1)\quad X_1 = a + b_2 X_2 + b_2' X_2^2 + b_3 X_3 + b' X_3^2$$

$$(2)\quad X = a + b_2 X_2 + b_2' X_2^2 + b_2'' X_2^3 + b_3 X_3 + b' X_3^2 + b_3'' X_3^3$$

The estimated error was calculated for selected ordinates, using the same equation (III) as derived for free-hand methods, and T and σ_T computed. The values of σ_T were as follows:

	Simple parabola	Cubic parabola
$f(X_2)$	0 77	0.95
$f(X_3)$	0.90	1.13

It would appear, therefore, that equation (III) gives about as good results in estimating the reliability of net regression curves mathematically determined as it does in estimating the reliability of those secured by free-hand fitting.

Even with the curves fitted by least squares, however, there was some tendency to bias, as is illustrated in Figures 6 and 7. It is evident from these figures that neither the free-hand curve nor the mathematical curve exactly reproduced the true curve, even on the average of the fifteen samples. The average amount of bias is shown in the following statement.

AVERAGE BIAS IN FITTING REGRESSION CURVES

Value of independent variable	Average error[1] in $f(X_2)$			Average error[1] in $f(X_3)$		
	Parabola	Cubic parabola	Free-hand curve	parabola	Cubic parabola	Free-hand curve
2	0.16	0.09	0.62	-0.35	-0.31	-0.69
3	-0.06	-0.03	0.21	-0.10	-0.20	-0.31
4	-0.05	0.00	0.05	-0.06	-0.09	-0.15
5				-0.04	0.00	-0.02
6	0.11	0.02	-0.05			
7	0.11	-0.04	- .07	0.00	0.05	-0.05
8	0.13	-0.07	-0.10	-0.05	0.05	-0.07
9				-0.06	0.07	-0.18
10	0.23	0.43	-0.16	-0.09	0.09	-0.25
12	0.42	0.46	-0.24	-0.13	-0.09	-0.50
14				-0.27	-0.35	-0.79

In this particular, where the true curve is of such a slope as to be fairly well represented by a parabola or cubic parabola, the mathematical curves appear to give a slightly more accurate fit, on the average, than do the free-hand curves. The standard deviation of the errors, however, is only slightly greater for the free-hand curves than for those fitted by the cubic parabolae, as shown by the following tabulation.[2]

---

[1] Taken with regard to sign

[2] At first glance it seems strange that the regressions fitted by the cubic parabola should have, on the average, larger errors than those fitted by the simple parabola. The explanation may be that the extra constant allowed the cubic parabola to follow more closely the individual characteristics of each sample; but that in fitting those (partly random) relations more closely, the regressions were distorted from the true underlying relation.

Standard deviation of errors (absolute values).

	Free-hand	Parabolic	Cubic
$f(X_2)$	0.98	0.70	0.84
$f(X_3)$	0.91	0.65	0.89

Where the true regression is of such shape that it could not be represented by any simple equation, it seems likely that the free-hand method would give a more accurate fit than would a mathematical equation which was not capable of representing the particular relation involved. Since, in practical investigations, the shape of the net regression curve is usually unknown to start with, the most satisfactory procedure would seem to be to use the free-hand method to determine the approximate shape of the curves, and then, if their shape appeared to follow any definite types by least-squares as a final check on the shape of the curves.

CONCLUSION

This article is only a progress report The experiments reported here suggest that it may be possible to develop a formula for the standard error of net regression curves fitted free-hand. The problem has not been completely solved; the tentative formula which is developed has given only fair results in experimental tests; and several points are in need of further study. I hope at some future time to carry this investigation further, but my present plans make it necessary to lay it aside for a year or more. I am, therefore, publishing this preliminary report now, in the hope that others may be led to attack the same problem.

Mordecai Ezekiel

FIGURE A

AVERAGE ERRORS OF REGRESSION CURVES

Universe X

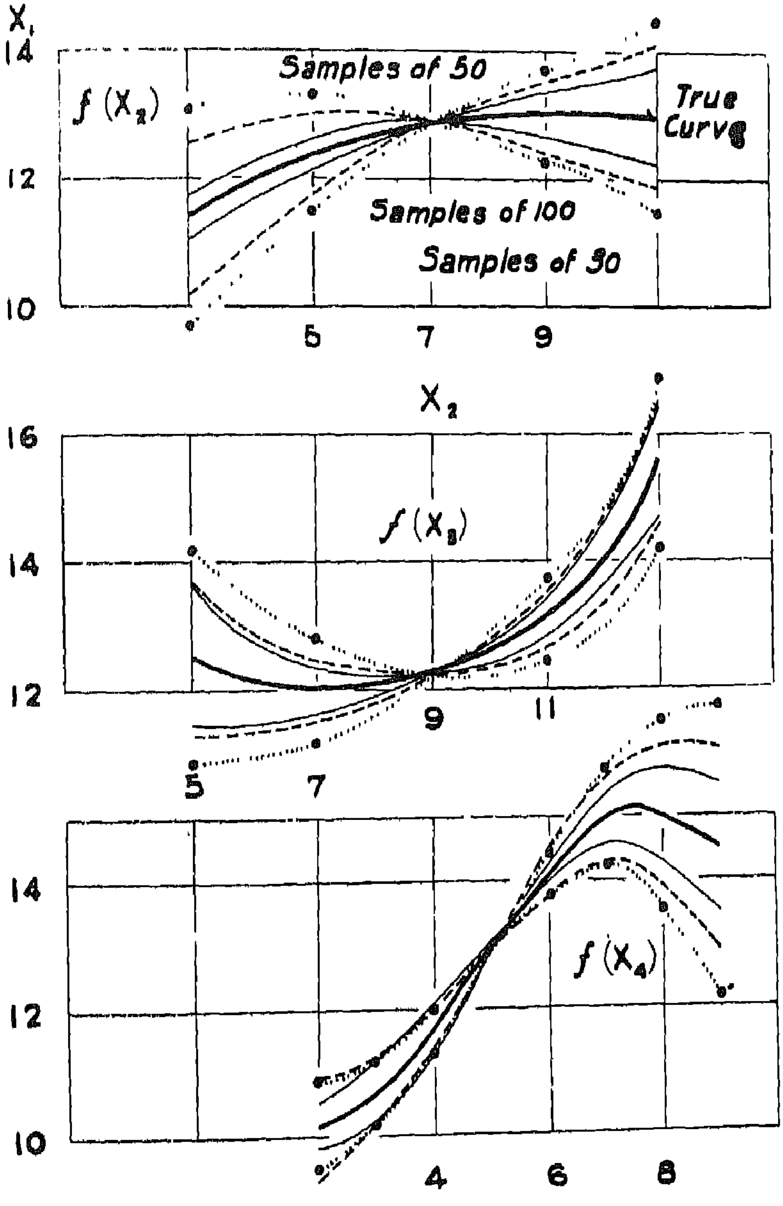

FIGURE 1

CORRELATION BETWEEN CORRESPONDING AVERAGE ERRORS IN UNIVERSES WITH DIFFERENT STANDARD ERRORS OF ESTIMATE

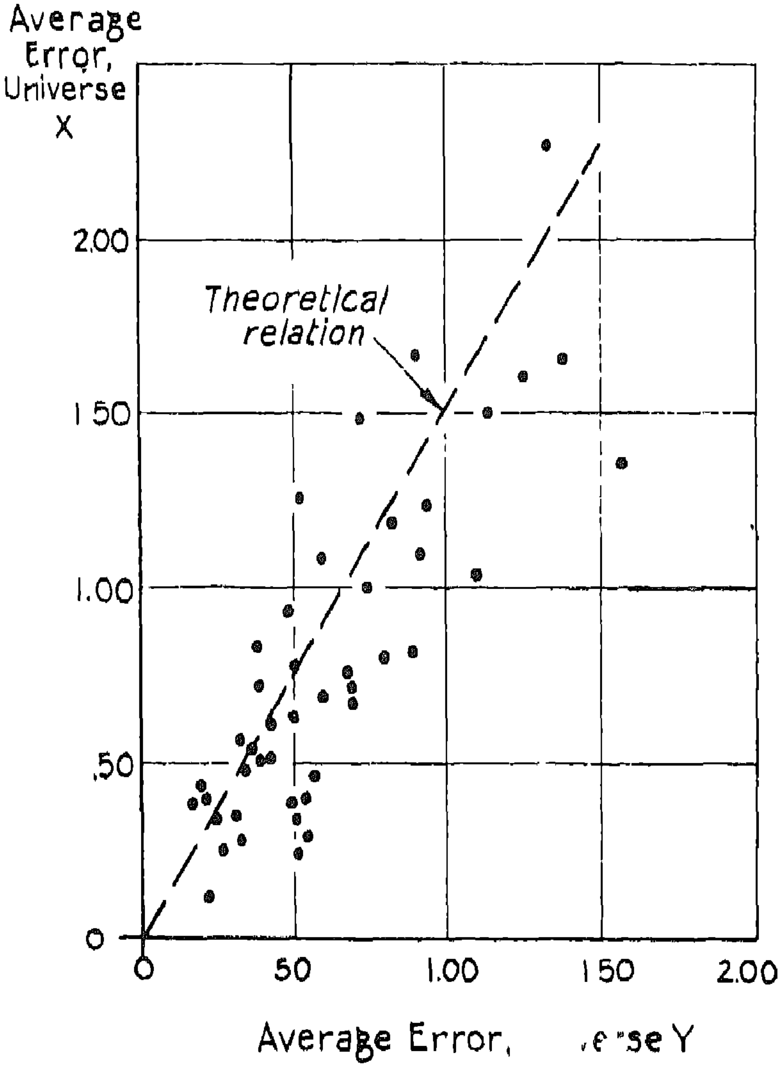

FIGURE 2

RELATION OF AVERAGE ERRORS, ADJUSTED FOR N_{κ}, TO DEPARTURE FROM CENTER

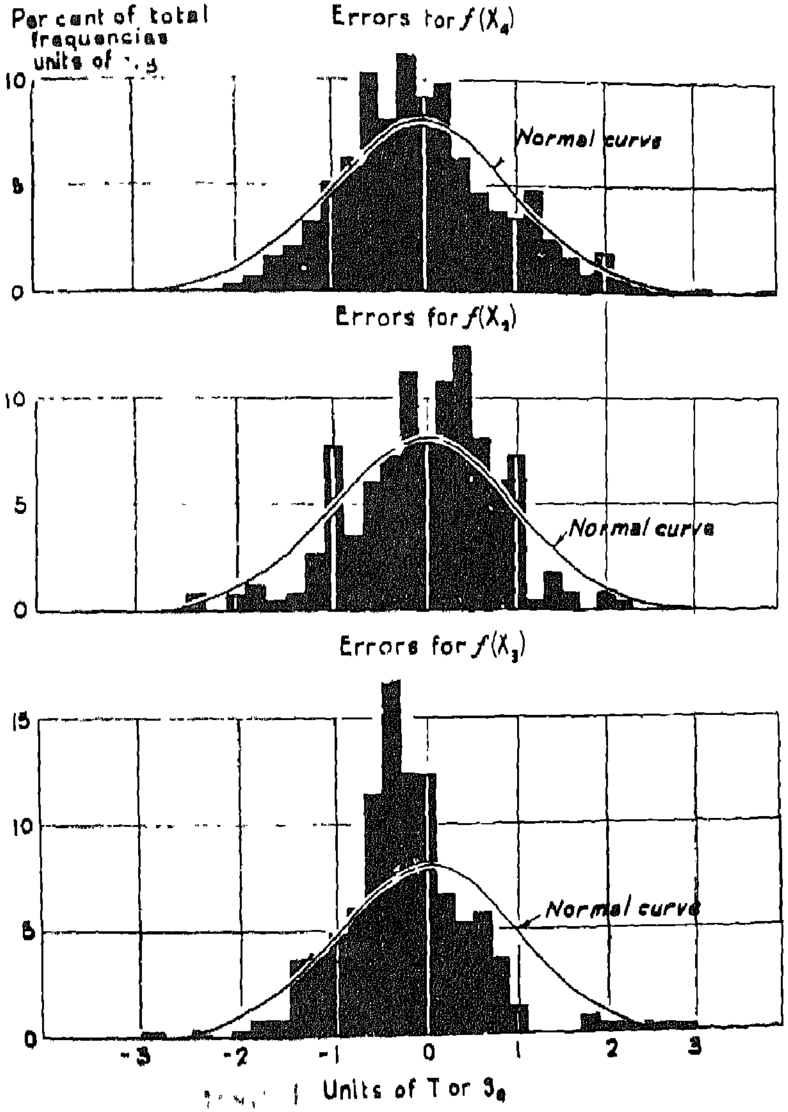

FIGURE 3

RELATION OF AVERAGE ERRORS ADJUSTED FOR N_K TO DEPARTURE FROM CENTER

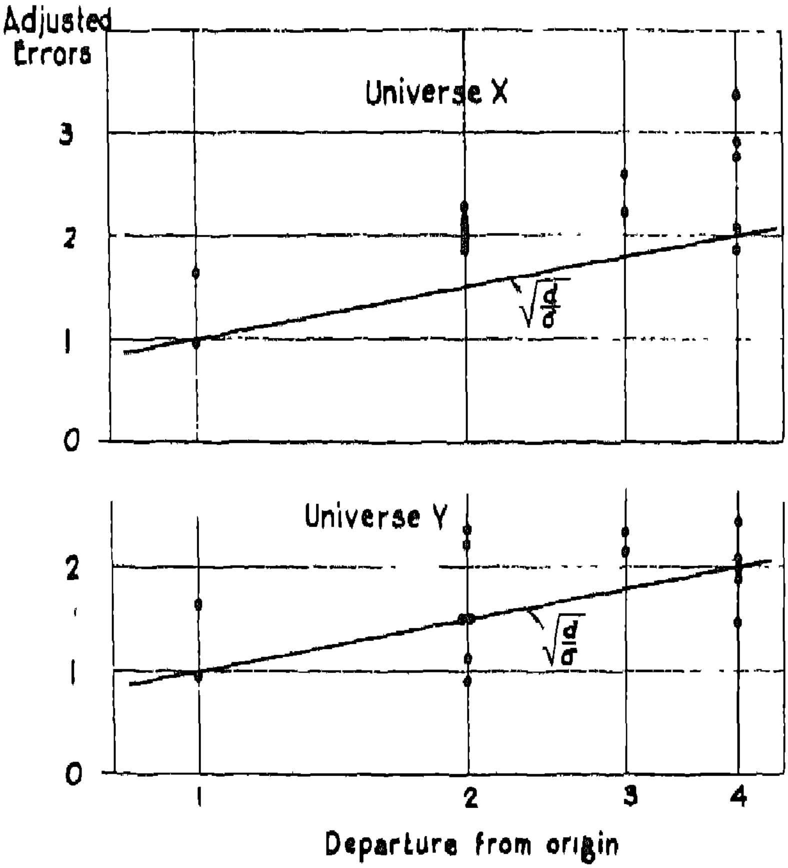

FIGURE 4

FREQUENCY DISTRIBUTIONS OF ERRORS

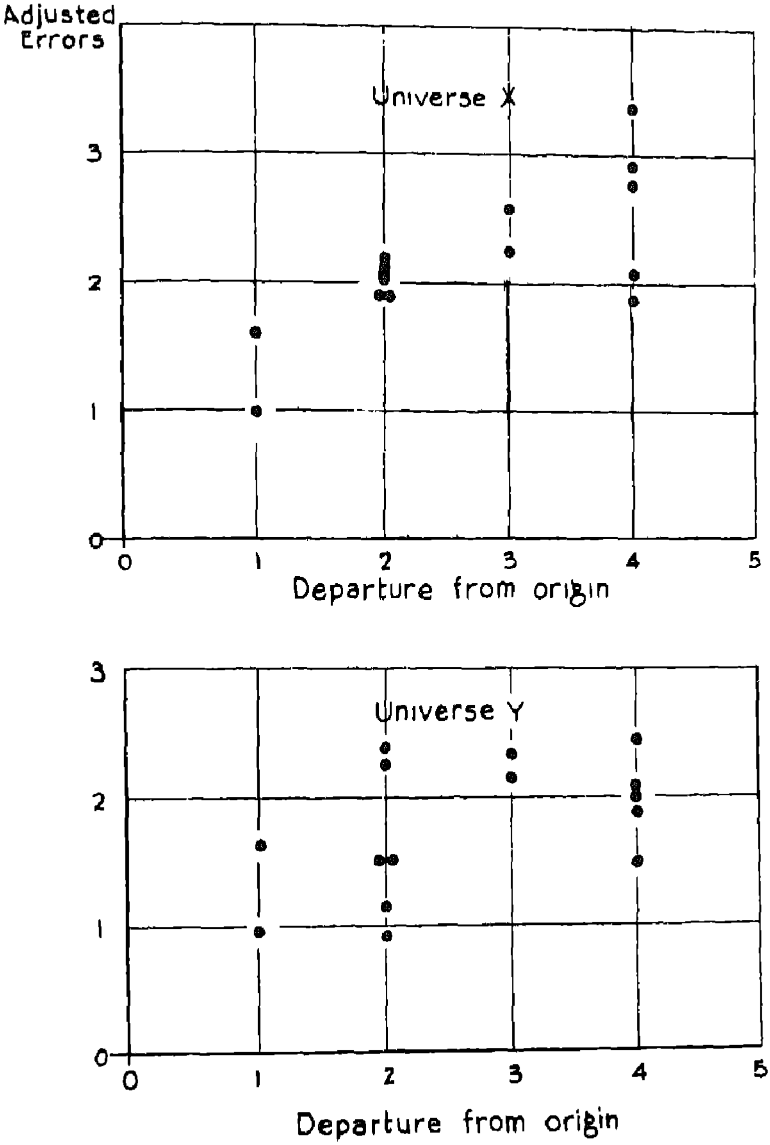

FIGURE 5

FREQUENCY DISTRIBUTIONS OF ERRORS

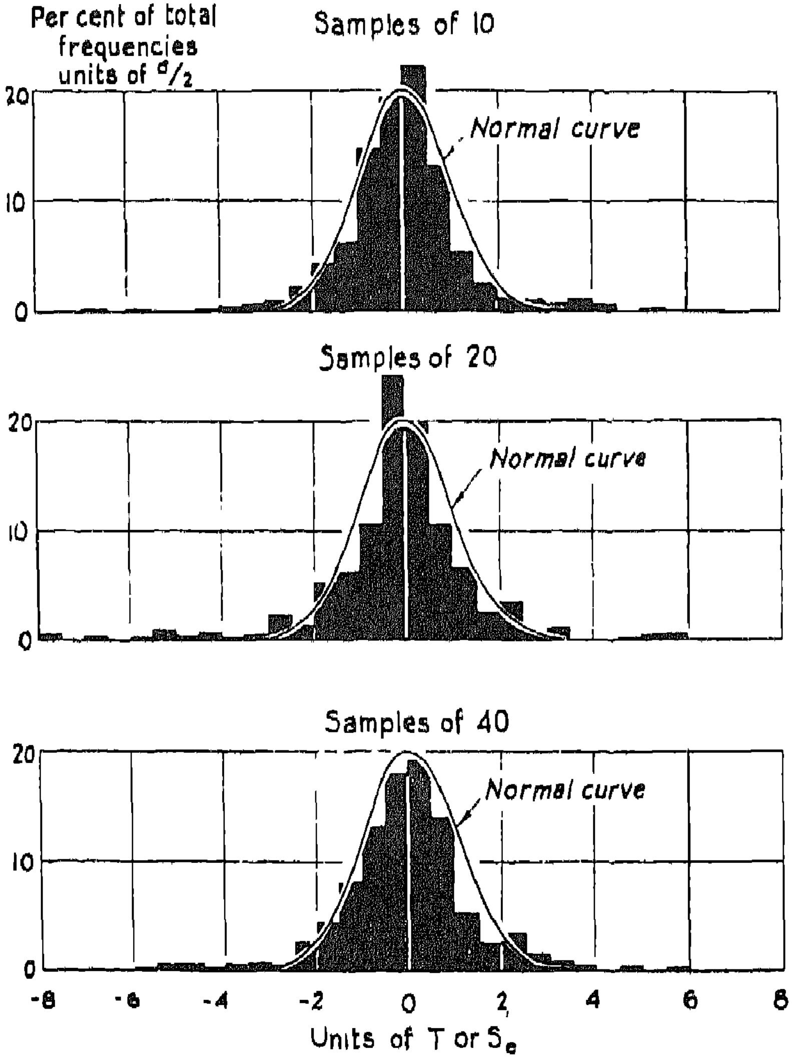

FIGURE 6

AVERAGE CURVES FITTED BY THREE METHODS $f(X_2)$

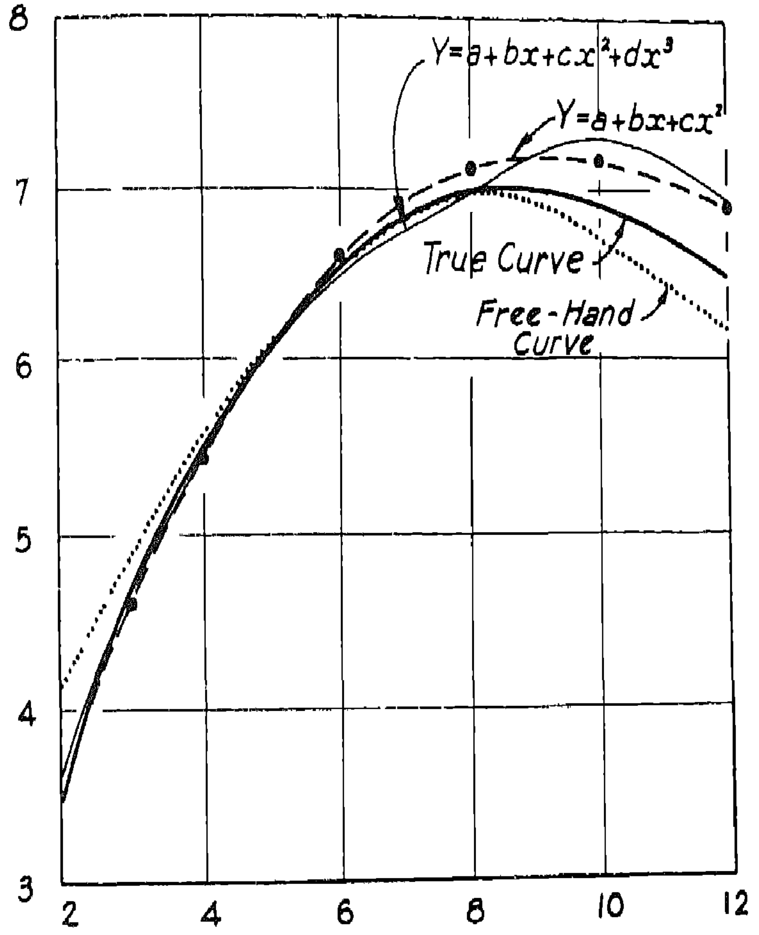

FIGURE 7

AVERAGE CURVES FITTED BY THREE METHODS $f(X_3)$

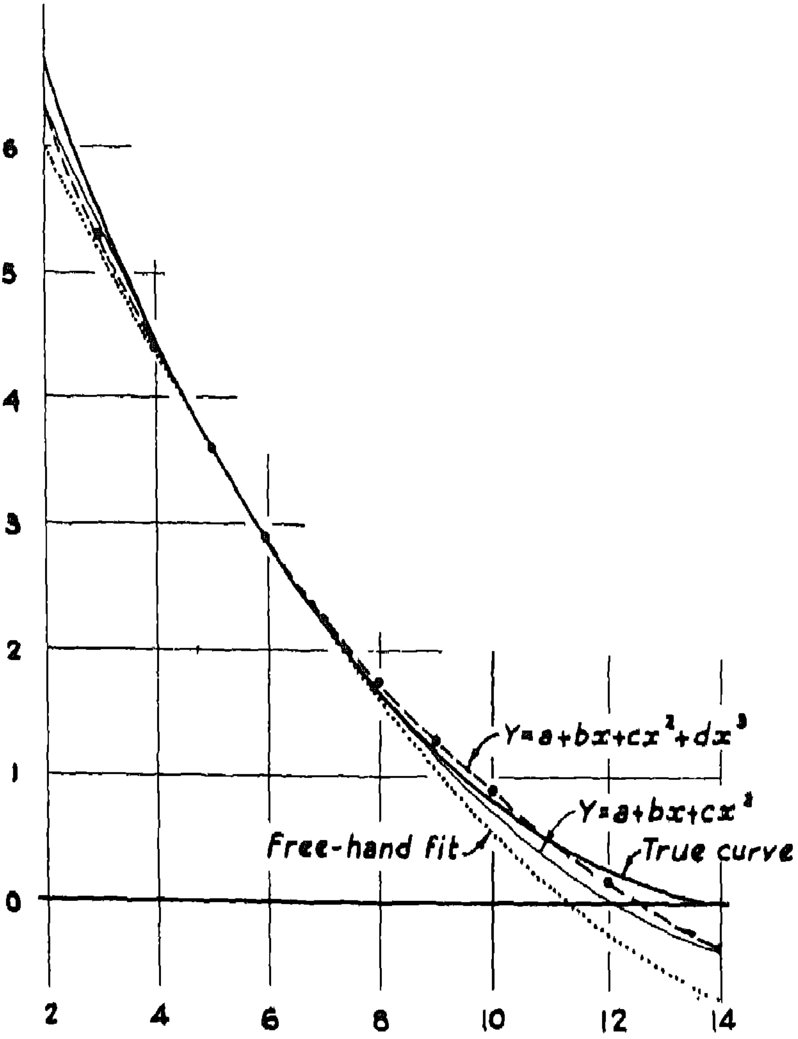

TABLE A—SYNTHETIC DATA FOR SAMPLING STUDY

No	X_2	X_3	X_4	X_1	Y
1	5	6	3	9.4	10.4
2	6	9	2	14.2	9.2
3	8	9	4	16.0	11.0
4	7	12	5	16.8	12.8
5	10	13	2	14.8	14.8
6	6	10	6	18.5	12.5
7	8	11	4	13.8	14.8
8	2	4	10	9.9	9.9
9	10	13	2	10.8	11.8
10	11	11	3	18.8	9.8
11	4	10	8	17.7	15.7
12	4	9	7	18.3	12.3
13	7	9	3	11.0	8.0
14	7	8	4	16.7	10.7
15	10	11	2	14.3	10.3
16	11	9	5	15.3	12.3
17	11	9	6	13.4	16.4
18	10	11	2	15.3	13.3
19	10	13	3	15.3	15.3
20	6	8	3	14.4	9.4
21	8	12	4	17.5	14.5
22	6	7	3	13.4	10.4
23	10	11	5	21.1	12.1
24	11	9	4	18.0	11.0
25	7	7	3	15.7	8.7
26	7	9	2	11.5	8.5
27	6	10	2	12.6	7.6
28	9	10	3	12.4	9.4
29	6	9	5	13.0	10.0
30	12	10	2	15.9	11.9
31	6	8	3	11.4	7.4
32	11	12	4	16.5	15.5
33	4	8	9	11.5	13.5
34	9	10	6	13.8	11.8
35	7	10	7	19.7	13.7
36	11	13	1	12.6	13.6
37	6	11	7	20.8	17.8
38	6	8	3	8.4	7.4
39	5	11	4	16.2	9.2
40	7	9	6	16.4	14.4
41	5	8	4	15.1	9.1
42	6	7	2	16.9	6.9
43	7	12	6	18.9	16.9
44	10	10	2	14.9	10.9
45	4	8	4	16.7	10.7
46	3	5	5	13.9	8.9
47	11	11	3	15.8	9.8
48	7	11	7	22.1	18.1
49	10	13	4	17.3	12.3
50	5	7	4	12.1	8.1
51	8	7	4	12.7	13.7
52	9	11	3	14.8	10.8
53	7	12	3	19.5	10.5
54	9	11	2	15.3	10.3
55	8	9	4	12.0	9.0
56	9	7	3	13.7	11.7
57	10	8	1	12.0	11.0
58	11	11	5	20.1	12.1
59	4	11	8	18.1	12.1
60	7	11	7	21.1	18.1
61	9	8	2	15.2	12.2
62	8	11	2	14.3	8.3
63	3	7	7	12.4	10.4
64	8	8	7	17.0	15.0
65	3	8	9	17.9	11.9
66	9	12	2	12.0	11.0
67	4	11	9	17.6	16.6
68	6	7	2	14.9	10.9
69	7	12	6	16.9	16.9
70	2	8	10	15.9	8.9
71	5	10	4	18.8	8.8
72	8	11	5	17.1	12.1
73	9	10	2	10.9	8.9
74	11	13	5	14.6	17.6
75	5	8	8	17.4	11.4
76	5	7	7	21.4	15.4
77	7	8	2	15.2	11.2
78	6	9	2	14.2	9.2
79	6	12	4	16.2	10.2
80	3	8	6	14.5	9.5
81	7	7	3	16.7	7.7
82	8	8	3	12.7	10.7
83	6	8	4	15.4	9.4
84	8	12	7	22.8	15.8
85	12	14	3	16.1	16.1
86	7	11	4	15.8	10.8
87	5	7	6	16.5	15.5
88	11	13	3	17.3	11.3
89	7	11	2	12.3	10.3
90	7	11	4	12.8	9.8
91	7	7	6	12.1	13.1
92	7	9	6	15.4	16.4
93	7	8	5	14.0	10.0
94	6	10	7	20.4	13.4
95	11	13	5	22.6	14.6
96	8	7	6	14.1	16.1
97	6	7	6	14.8	11.8
98	2	4	6	11.7	13.7
99	7	8	4	11.7	11.7
100	10	13	6	20.7	16.7

TABLE A—SYNTHETIC DATA FOR SAMPLING STUDY (Continued)

No.	X_2	X_3	X_4	X_1	Y
101	6	12	4	17.2	15.2
102	3	5	8	17.9	15.9
103	7	10	2	13.9	11.9
104	8	7	3	14.7	7.7
105	6	7	6	19.8	11.8
106	4	10	5	12.7	11.7
107	11	14	6	20.5	16.5
108	3	9	9	16.2	11.2
109	9	9	6	18.4	15.4
110	3	5	8	17.9	12.9
111	12	14	6	20.5	14.5
112	9	8	6	12.1	16.1
113	7	7	3	10.7	10.7
114	7	8	2	14.2	7.2
115	6	9	4	13.7	10.7
116	11	9	5	19.3	13.3
117	5	8	6	17.5	10.5
118	7	11	4	14.8	14.8
119	6	9	4	11.7	13.7
120	7	8	2	15.2	11.2
121	7	7	2	14.2	7.2
122	3	7	10	13.7	9.7
123	5	6	5	15.7	9.7
124	8	8	6	18.1	16.1
125	10	8	3	14.7	9.7
126	6	8	7	19.7	13.7
127	7	12	3	13.5	12.5
128	6	10	4	16.1	9.1
129	4	6	4	16.0	10.0
130	6	12	2	9.7	9.7
131	11	9	3	13.0	13.0
132	5	11	7	15.5	12.5
133	3	8	5	15.4	8.4
134	10	12	6	21.0	16.9
135	12	10	4	18.4	11.4
136	8	11	6	19.2	15.2
137	7	11	6	19.2	13.2
138	3	8	9	17.9	12.9
139	10	9	5	17.3	13.3
140	6	8	3	15.4	8.4
141	8	9	2	10.5	12.5
142	5	9	4	16.4	10.4
143	8	11	6	23.2	13.2
144	8	11	5	17.1	12.1
145	9	12	6	23.9	14.9
146	6	11	3	17.5	13.5
147	5	9	8	17.7	13.7
148	11	14	4	18.1	17.1
149	8	11	6	21.2	13.2
150	10	12	1	16.8	9.8
151	7	9	5	17.3	11.3
152	9	8	5	12.0	11.0
153	7	12	3	13.5	14.5
154	7	10	6	14.8	11.8
155	8	8	7	17.0	14.0
156	3	9	6	12.8	13.8
157	12	12	1	14.8	12.8
158	12	12	3	20.5	9.5
159	7	7	3	16.7	10.7
160	2	6	9	17.4	14.4
161	8	12	3	13.5	11.5
162	9	11	7	21.1	17.1
163	6	12	4	18.2	10.2
164	7	9	3	13.0	9.0
165	6	10	5	21.4	15.4
166	9	10	5	15.7	11.7
167	6	12	7	15.5	18.5
168	9	11	2	15.3	11.3
169	7	11	6	21.2	12.2
170	5	8	8	17.4	12.4
171	4	11	9	20.6	15.6
172	8	7	6	17.1	15.1
173	6	9	3	12.7	7.7
174	12	14	6	22.5	14.5
175	7	7	2	12.2	7.2
176	7	7	2	14.2	8.2
177	4	7	6	16.1	15.1
178	7	11	4	17.8	13.8
179	7	10	6	16.8	12.8
180	10	11	5	16.1	16.1
181	11	12	5	14.8	14.8
182	9	12	6	20.9	12.9
183	10	9	1	13.3	10.3
184	7	11	4	16.8	9.8
185	9	12	3	18.5	9.5
186	8	12	4	17.5	14.5
187	11	11	4	12.8	14.8
188	5	9	4	12.4	12.4
189	6	7	4	11.4	12.4
190	2	7	11	12.1	10.1
191	5	7	6	17.5	10.5
192	5	11	5	12.5	11.5
193	7	8	7	18.0	15.0
194	9	11	5	13.1	15.1
195	11	13	6	23.7	13.7
196	9	10	4	13.4	13.4
197	7	12	4	15.5	15.5
198	6	8	4	12.4	11.4
199	4	6	4	9.0	12.0
200	3	10	10	18.4	9.4

TABLE A—SYNTHETIC DATA FOR SAMPLING STUDY (Continued)

No	X_2	X_3	X_4	X_1	Y
201	4	10	7	17.7	14.7
202	2	6	11	9.4	12.4
203	6	11	6	17.9	11.9
204	7	9	4	16.0	14.0
205	3	10	6	16.2	14.2
206	8	8	5	17.0	12.0
207	5	10	8	16.1	12.1
208	4	11	8	17.1	13.1
209	8	11	5	16.1	16.1
210	7	8	6	13.1	11.1
211	11	11	1	14.1	13.1
212	8	12	6	17.9	15.9
213	7	10	6	19.8	15.8
214	6	11	2	17.0	12.0
215	8	12	2	17.0	14.0
216	10	10	1	13.7	7.7
217	10	13	3	19.3	15.3
218	6	9	4	17.7	10.7
219	8	10	6	17.8	12.8
220	6	12	5	14.5	15.5
221	8	9	5	18.3	12.3
222	8	7	6	14.1	13.1
223	7	7	7	17.0	13.0
224	10	8	2	14.2	11.2
225	7	9	7	18.3	12.3
226	10	8	1	9.0	7.0
227	6	7	5	19.7	12.7
228	5	11	4	16.2	12.2
229	10	10	3	13.4	13.4
230	7	12	6	17.9	13.9
231	6	7	4	16.4	8.4
232	2	6	8	16.9	14.9
233	3	10	6	13.2	11.2
234	9	7	5	16.0	10.0
235	10	8	6	18.1	12.1
236	3	5	5	12.9	9.9
237	8	7	7	20.0	13.0
238	9	11	6	16.2	13.2
239	7	10	3	11.4	12.4
240	3	10	7	15.1	13.1
241	4	11	5	18.1	15.1
242	7	11	2	13.3	8.3
243	9	8	3	11.7	11.7
244	4	10	5	12.7	14.7
245	5	7	3	15.1	12.1
246	7	8	5	18.0	13.0
247	4	11	6	13.2	11.2
248	3	10	9	16.6	14.6
249	8	7	2	14.2	9.2
250	5	10	8	18.1	17.1

No.	X_2	X_3	X_4	X_1	Y
251	11	14	5	18.4	13.4
252	5	6	8	20.7	13.7
253	4	9	5	15.3	11.3
254	3	5	7	15.9	15.9
255	9	11	4	14.8	9.8
256	11	11	1	15.1	8.1
257	9	7	6	13.1	11.1
258	7	9	3	14.0	13.0
259	6	10	7	18.4	16.4
260	6	8	4	15.4	8.4
261	5	8	4	14.1	9.1
262	9	8	3	11.7	8.7
263	7	11	4	18.8	10.8
264	2	4	6	15.7	14.7
265	9	11	3	17.8	13.8
266	7	8	4	15.7	11.7
267	11	11	5	14.1	15.1
268	5	11	8	20.5	17.5
269	6	12	2	15.7	9.7
270	6	8	2	13.9	7.9
271	8	11	4	13.8	10.8
272	8	7	6	13.1	15.1
273	5	7	3	9.1	7.1
274	5	10	5	20.1	10.1
275	11	12	1	14.8	11.8
276	7	9	7	22.3	17.3
277	5	7	8	18.4	11.4
278	6	7	6	13.8	10.8
279	5	6	6	15.8	15.8
280	8	12	3	14.5	13.5
281	5	10	3	14.8	7.8
282	10	11	4	17.8	14.8
283	7	12	6	21.9	12.9
284	5	8	3	18.1	9.1
285	12	15	5	21.2	15.2
286	4	11	4	10.8	9.8
287	3	10	5	19.1	14.1
288	8	10	6	20.8	16.8
289	11	13	5	20.6	16.6
290	10	11	2	15.3	8.3
291	4	10	9	20.2	13.2
292	7	7	5	19.0	15.0
293	8	7	5	16.0	14.0
294	3	7	8	17.4	10.4
295	6	10	6	17.5	15.5
296	4	11	4	10.8	10.8
297	5	6	3	17.4	12.4
298	8	7	7	22.0	17.0
299	12	10	2	13.9	11.9
300	4	8	9	13.5	11.5

TABLE A—SYNTHETIC DATA FOR SAMPLING STUDY (Continued)

No.	X_2	X_3	X_4	X_1	Y
301	6	11	6	20 9	16.9
302	6	11	2	13 0	9 0
303	8	7	6	19.1	16 1
304	12	15	6	23 3	19.3
305	7	9	3	14.0	11.0
306	3	5	5	10 9	11.9
307	8	11	7	18 1	17.1
308	9	8	7	19.0	17.0
309	5	10	4	13 8	12.8
310	7	9	2	12 5	9.5
311	7	7	5	12 0	11.0
312	3	8	8	18 4	15.4
313	4	6	6	16.4	13 4
314	7	9	4	19 0	9.0
315	3	5	7	17.9	15.9
316	6	8	7	16 7	11.7
317	2	8	7	12 6	10.6
318	9	11	3	14 8	12.8
319	11	10	3	18.4	11.4
320	7	9	7	22.3	12.3
321	7	10	7	20.7	14 7
322	5	7	4	14.1	9.1
323	7	8	2	14.2	8.2
324	9	11	7	22 1	17.1
325	5	6	4	17.4	8 4
326	8	10	7	19.7	16.7
327	12	11	1	12.1	10.1
328	5	8	6	16.5	14.5
329	8	8	5	13 0	13.0
330	6	7	2	14 9	7.9
331	8	7	5	13.0	10 0
332	3	8	6	13.5	11.5
333	8	9	6	18.4	12.4
334	9	7	3	12 7	7.7
335	7	8	7	15.0	17.0
336	7	9	3	14.0	8.0
337	7	10	3	12.4	12.4
338	11	14	2	17 6	14.6
339	4	11	9	18.6	16.6
340	7	7	3	17.7	7.7
341	4	6	5	12.3	14.3
342	2	4	8	16.6	14.6
343	3	5	6	16.0	12.0
344	7	8	6	17.1	13.1
345	11	11	6	18.2	14.2
346	8	12	5	16.8	11.8
347	9	10	2	16.9	8.9
348	7	8	4	15.7	12.7
349	11	10	4	17.4	14.4
350	7	7	3	13 7	9 7
351	4	9	7	12.3	12.3
352	7	7	4	15.7	9.7
353	7	12	3	16 5	9.5
354	9	11	3	17 8	12.8
355	6	8	5	14.7	11.7
356	9	10	5	19.7	12 7
357	9	8	4	19.7	10.7
358	7	9	2	11.5	10.5
259	5	6	3	13.4	9.4
360	5	8	6	15 5	12 5
361	12	15	2	21 4	15 4
362	6	12	2	17.7	13 7
363	8	7	7	20.0	17.0
364	6	12	3	11.2	13.2
365	6	9	6	12.1	14.1
366	8	8	6	21.1	13.1
367	10	8	3	9.7	8.7
368	7	10	7	18.7	14 7
369	5	9	3	10.4	11.4
370	12	14	4	19.1	14.1
371	7	8	6	15.1	15.1
372	4	7	8	16.0	11.0
373	3	6	7	18 7	14.7
374	7	7	3	12.7	9.7
375	9	8	6	17.1	12.1
376	6	9	4	11.7	13.7
377	7	7	3	15.7	9.7
378	8	8	4	11.7	8.7
379	5	10	5	13.1	15.1
380	5	11	3	19.2	12.2
381	11	13	2	20.8	9.8
382	9	11	5	18.1	11.1
383	8	9	7	18.3	12.3
384	3	7	8	16.4	12.4
385	4	7	9	11.5	12.5
386	5	8	7	15.4	13.4
387	5	7	3	15.1	8.1
388	9	11	6	16.2	13.2
389	10	8	6	14.1	11.1
390	8	8	4	12.7	9.7
391	8	7	6	15.1	16.1
392	8	9	3	12.0	11.0
393	7	10	7	21.7	17.7
394	10	8	2	14.2	8.2
395	9	12	2	14.0	9.0
396	4	6	4	16.0	9.0
397	4	8	5	11.0	13.0
398	10	9	3	13.0	8.0
399	7	11	2	13.3	13.3
400	9	10	5	16.7	11.7

TABLE A—SYNTHETIC DATA FOR SAMPLING STUDY (Continued)

No.	X_2	X_3	X_4	X_1	Y
401	7	12	6	14.9	13.9
402	2	6	8	18.9	13.9
403	5	10	4	15.8	13.8
404	8	12	2	19.0	9.0
405	7	8	7	21.0	12.0
406	6	9	6	16.1	11.1
407	6	11	4	14.5	13.5
408	6	12	6	21.6	14.6
409	9	12	7	22.8	18.8
410	8	7	3	15.7	11.7
411	4	11	8	17.1	12.1
412	10	9	3	19.0	9.0
413	7	12	6	17.9	13.9
414	11	10	1	13.7	11.7
415	8	11	7	19.1	18.1
416	4	11	9	13.6	11.6
417	6	8	5	14.7	12.7
418	3	10	5	13.1	9.1
419	8	7	5	16.0	11.0
420	5	6	5	13.7	12.7
421	4	11	8	18.1	12.1
422	7	10	3	11.4	13.4
423	5	6	6	13.8	13.8
424	3	6	5	17.7	12.7
425	11	11	1	13.1	10.1
426	7	11	6	22.2	12.2
427	4	11	8	18.1	13.1
428	7	9	4	16.0	14.0
429	6	9	3	13.7	11.7
430	8	9	2	9.5	10.5
431	11	14	3	15.1	12.1
432	6	12	5	18.5	11.5
433	5	6	6	16.8	14.8
434	9	9	7	21.3	13.3
435	9	11	3	12.8	13.8
436	7	10	4	16.4	10.4
437	5	6	7	13.7	12.7
438	5	8	8	20.4	16.4
439	7	11	2	14.3	13.3
440	3	8	10	14.7	12.7
441	12	14	3	16.1	15.1
442	8	10	2	14.9	12.9
443	7	11	6	23.2	17.2
444	8	11	3	13.8	11.8
445	4	6	9	19.8	14.8
446	9	10	7	17.7	14.7
447	5	9	7	17.7	14.7
448	5	7	6	18.5	14.5
449	7	9	2	14.5	9.5
450	11	13	5	21.6	15.6
451	6	12	7	20.5	18.5
452	7	8	5	17.0	13.0
453	5	7	8	14.4	14.4
454	9	9	6	20.4	15.4
455	3	7	10	12.7	10.7
456	9	7	7	16.0	16.0
457	12	11	1	9.1	9.1
458	8	8	3	14.7	9.7
459	11	13	1	16.6	11.6
460	6	9	2	8.2	9.2
461	6	8	5	15.7	14.7
462	3	5	9	15.4	13.4
463	6	12	2	18.7	9.7
464	7	10	2	13.9	12.9
465	11	11	2	11.3	13.3
466	9	11	5	16.1	13.1
467	8	12	7	17.8	17.8
468	9	10	7	20.7	14.7
469	7	8	5	18.0	12.0
470	9	10	3	14.4	10.4
471	10	13	3	15.3	13.3
472	7	10	6	14.8	15.8
473	11	11	2	10.3	13.3
474	7	12	5	12.8	15.8
475	5	8	7	20.4	16.4
476	12	13	6	17.7	15.7
477	4	10	5	16.7	9.7
478	9	10	4	19.4	14.4
479	9	7	6	18.1	12.1
480	9	10	7	16.7	15.7
481	6	8	6	18.8	15.8
482	5	8	4	9.1	12.1
483	11	10	4	16.4	12.4
484	8	7	4	16.7	9.7
485	6	8	6	16.8	10.8
486	12	15	5	16.2	15.2
487	4	7	9	16.5	14.5
488	5	11	6	18.6	15.6
489	4	6	5	15.3	12.3
490	7	11	5	16.1	13.1
491	12	13	1	16.6	13.6
492	7	10	2	16.9	9.9
493	11	14	4	15.1	14.1
494	6	12	4	18.2	10.2
495	7	11	4	16.8	13.8
496	7	12	4	13.5	12.5
497	10	10	4	19.4	14.4
498	9	10	2	16.9	8.9
499	5	9	8	17.7	11.7
500	9	9	4	13.0	11.0

TABLE B—COEFFICIENTS AND INDEXES OF MULTIPLE COR

(uncorrected for num

Sample No	UNIVERSE X				
	R	ρ			
		1st curves	2nd curves	3rd curves	4th curves
Samples of 30					
1	.620	.689	.714	.751	
2	.705	.737	.754	.756	
3	510	.545	.703	.775	
4	.463	487	.516	.508	548
5	.741	.614	679	736	
6	.486	.688	.720	731	
7	.681	.777	.787	.801	
8	.532	.589	659	.696	
9	.608	.649	.598	659	
10	.469	539	597	.743	
11	.745	.792	.813	.818	
12	.614	590	.628	752	
13	.771	.741	.815	.790	
14	.586	.742	.794	.798	
15	.551	.569	.574	.578	
16	.634	759	809	.826	
Samples of 50					
17	.529	.473	.545	.543	.510
18	.507	.536	.621	.622	655
19	.418	.493	.541	.534	.536
20	.512	.686	.693	.702	.706
21	.526	686	.721	.733	.745
22	.704	.721	730	.727	.726
23	.666	.724	747	.756	.753
24	.517	.650	.659	.679	.684
25	.703	.723	.721	.727	729
26	.609	.646	646	677	699
Samples of 100					
27	543	.629	.671	.684	
28	.679	.685	.699	.699	
29	.557	.649	.673	673	.673
30	.565	.590	.656	.675	.678
31	.576	.650	.682	.682	.686

RELATION FOUND AT EACH SUCCESSIVE APPROXIMATION

ber of variables)

Sample No	UNIVERSE Y					
	R	P				
		1st curves	2nd curves	3rd curves	4th curves	
			Samples of 30			
47	.697	.698	.705			715
48	588	781	.787			.794
49	.639	.800	836			.858
50	.659	782	.797			.801
51	.643	.812	829			.851
52	.668	.745	.722			746
53	.837	.877	.895			.898
54	677	.736	697			.737
55	.505	.679	.702			720
56	.707	767	.778			.782
57	.594	639	.665			.676
58	.580	.660	.661			.669
59	.684	.762	779			785
60	.825	880	.876			.881
61	.461	.621	.619			.642
62	590	.756	.803			.819
			Samples of 50			
37	.721	.786	.803	793	804	.804
38	.649	.686	.731	.713	.703	
39	.705	.730	.736	.738		
40	.679	.786	797	.796		
41	.764	.773	.804			.805
42	.725	.764	.759	.723		
43	.721	.772	.798	.799	800	.800
44	.688	.749	.781	.777		
45	.647	676	.691	.695	.699	699
46	564	.672	.731	.733	.736	.736
			Samples of 100			
32	710	.764	.769	769		
33	482	.644	.656	644	.650	
34	.668	.755	760	.762		
35	.760	.794	.799	.802		
36	555	.663	.673	.687	.684	

TABLE C—FOR UNIVERSE X. SAMPLING ERRORS IN NET

(The errors are observed or-

Ordinate	True Regression	SAMPLES OF 30															
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
X_2	$f(X_2)$																
3	11.4	.8		1.3	1.5	.8	3.0	1.1	1.5	1.8	2 1	2.4	2.1		3.0	.1	2.8
5	12.4	.6	1.4	.9	.4	.0	1.5	1.3	.2	.7	1.1	1.1	.7	3 4	.7	.1	.8
7	12.9																
9	13.0	1.0	.4	1 3	.4	.4	4	1.3	1.5	.7	9	.5	.1	1.0	.7	.2	.8
11	13 0	2.2	1.3	1.3	8	1.1	.6	3.1	3.6	1.1	2.1	1.9	.4	1.0	.9	.4	1.8
X_3	$f(X_3)$																
5	12.5	2.1		1.8	.5	1 2	2.4		2.7	2.6	1.2	1.1			.2		2.3
7	12.0	.5	.2	1.8	.6	3	1.8	1.0	1.5	1.7	.2	.7	4	.6	.7	1.0	1.5
9	12.3																
11	13.1	.9	.2	.8	.4	.0	.0	2.1	.1	.5	.2	1.8	1.5	.0	1.1	.4	.3
13	15.6		2 2	1.8	1 6	1.4	.8	2.6		1.9	.3	2.0	1.8	.4	.4	.8	.7
X_4	$f(X_4)$																
2	10 2	.7	.4	.1	1.0	.6	1.1	.8	5	1 4	.6	.5	.1	1.4	1.3	.1	.4
3	10.7	.0	6	.3	1	.7	1.2	.7	.7	1.1	3	.5	.2	1.2	.2	.5	.1
4	11.7	.2	.3	.3	.2	.4	.7	.8	.5	.5	0	.7	.1	.7	.1	.5	.1
5	13.0																
6	14 1	.1	0	.1	1	.2	1	.3	3	.7	.8	.4	.4	.4	.0	.4	.2
7	15.0	.8	.5	1.2	.0	.1	1.0	.5	.3	1.9	7	.8	1.2	1.3	.1	.6	.5
8	15.0	2.8	1.9	1.8	.2	1 0	2.7	.7	.8	2.7	1.6	1.0	2 7		.9	.1	1.6
9	14.5	2.0		.6	1 3	2.9		.9	2.9	3.7	3.7	2.1	3.6		1.4	1.1	3.0

TABLE C (Continued)—FOR UNIVERSE Y

Ordinate	True Regression	SAMPLES OF 30															
		47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62
X_2	$f(X_2)$																
3	11.4	2.1	5	.9	.2	2.6	1.0	.2	1.0	1.2	.7	.0	.5	.3	1.8	1.0	.4
5	12.4	1.0	.2	3	.1	.8	.6	.3	.2	.4	.7	.8	.4	.1	1.1	.6	5
7	12.9																
9	13.0	7	.0	0	.3	.4	.9	.3	.0	.3	.2	.1	.4	3	1.5	.1	.6
11	13 0	1.3	.0	.1	.9	1.3	9	.2	.0	.5		.4	.8	.1	3.1	.1	1.0
X_3	$f(X_3)$																
5	12 5				1.5		1.8				.6					1.6	1.9
7	12 0	.2	.3	.1	.0	.2	.3	.8	.5	.2	.4	.7	.6	.9	.1	6	.6
9	12 3																
11	13 1	.6	.1	1 3	.9	1.2	.3	.4	.0	.0	.1	.3	.2	.2	.4	.3	.3
13	15 6	3.4	1.7	1.4		1.6	2.2	.4	1.7	2.0	.3	.9	2.0		2.5	.9	.9
X_4	$f(X_4)$																
2	10.2	.5	1.1	.5	.1	.0	.6	.3	.4	1.2	.0	1.2	.7	.6	.8	.4	.9
3	10.7	4	.2	8	.2	.1	1	.0	.7	1.2	.4	.1	.0	.6	.6	.7	.1
4	11.7	.5	.1	.6	.1	.1	.2	.0	.5	.7	.3	.3	.1	.2	.5	.5	.1
5	13.0																
6	14.1	.2	.3	1	.1	.5	.1	5	.4	.0	.0	4	.0	.9	1.3	.5	.1
7	15.0	.5	8	.0	.0	7	1	1 2	4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

REGRESSION CURVES, FOR SELECTED ORDINATES

dinates minus true ordinates)

SAMPLES OF 50										SAMPLES OF 100				
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1 4	5	0	2.6	.8	1.3	1 5	2.7	.9	.2	.6	5	.2	.4	0
.1	.8	0	1 3	1.1	.4	1 2	.9	.4	1	.2	3	.1	.1	.5
9	1.1	.2	3	.4	.6	.7	.8	5	.1	2	1	.1	.4	9
1 7	1.8	4	6	1.9	1.2	.1	1.5	1.2	1.1	.1	.8	.1	1.8	.9
2	4	1			5.4	.6	1.0	1	2 2	1 4	2	1.1	2 3	.2
5	.2	.3	0	.3	1.0	.6	.3	2	1	5	.2	.5	.5	.2
.2	2	1.1	.4	.1	.4	.2	.4	6	1 1	.7	.2	.2	.1	.0
1 8	1 2	7	1 8	0	1.0	7	1 0	.7	.4	.1	.8	1.6	.4	1.2
1.2	1 3	1 0	1	8	.3	5	.7	8	.4	.2	.5	4	.6	2
1 2	1.1	1 6	2	1	.0	.2	.0	3	1	.4	7	.3	.9	.1
.7	.5	9	2	.1	.4	.2	.1	.2	.3	.3	.5	4	6	.2
1	7	.3	.4	.0	.1	.1	.6	.4	.2	0	.1	2	1	.2
.4	1.0	4	.7	.3	.7	.7	1.3	.7	.5	7	2	.1	.7	.3
9	3	3	1.7	.4	1 9	1.2	1.8	1 6	.8	.4	1.3	.3	.9	.9
.3	1 3	1 6		1 4	2.7	1 7	1.9	2 5	1 1	.1	2.7	.2	2	1.8

SAMPLES OF 50										SAMPLES OF 100				
37	38	39	40	41	42	43	44	45	46	32	33	34	35	36
1 2	1 4	1 1	6	.9	.1		2.2	.5	7	9	.5	.0	1 1	.0
1	1.3	.7	5	1.0	.2	.7	.6	.3	.4	.3	.1	.2	.7	0
6	.3	.3	.4	4	1	.1	.4	.0	5	.3	1	.3	.2	.3
1.6	7	.5	2	.1	.2	8	.7	2	.7	6	.3	8	.3	.5
	1 0	5	4		.7			.0		4		.1	1 1	2.8
.2	.5	.1	.8	.1	8	1	.1	1	.4	.2	.1	1	.1	.8
.4	.1	.0	.2	.5	.6	1.6	.1	.0	.3	8	.1	.9	.4	.4
1.0	2.1	1.2		.6	.5	1.2	1.5	1.1	2.3	.7	1.8	.1	.8	1.0
.5	.8	.2	1 3	1.1	1.8	1.0	.9	.1	1.3	.2	1 0	8	.6	.4
3	.7	.3	.6	.1	1.1	.2	5	2	1.0	.3	.7	.0	.1	.6
.0	.4	.3	.1	.4	.6	.0	.0	.1	3	.3	.3	1	1	.3
.9	.1	.0	.2	.6	.3	.1	1.4	2	.9	3	1	.3	.2	.2
.8	.0	.0	.0	1.1	8	.3	1.0	.0	1.2	.9	.3	.2	1.2	.1
1.2	1 0	.6	.3	.8	1.9	.7	1 2	4	1.5	.8	1 2	6	.6	.1
2.1	2 5	1.1	4	.2	3.2		1.7	1.0	1.7	2	1 8	9	.4	.4

TRANSFORMATIONS OF BIMODAL DISTRIBUTIONS

By

G. A. Baker

I. INTRODUCTION

Several men have concerned themselves extensively with the transformation of frequency distributions, for instance, Edgeworth, Kapteyn, Arne Fisher, and H L. Reitz (see 1, bibliography) The first three of these men have been concerned with transformations as a means of extending the scope of the normal distribution and Gram-Charlier system as a method of description. Rietz has been more interested in the properties of the transformed distributions.

There are three types of transformations that are of particular importance:

(1) $u = x^n$ because it has a physical interpretation.

(2) $u = \log x$ because Arne Fisher and others find it useful.

(3) $u = e^x$ because it is the inverse of (2).

These three transformations will be discussed in some detail for bimodal frequency distributions. It is interesting to note that it is possible to transform a bimodal distribution into a unimodal distribution and vice versa by means of these transformations The general scheme of the first part of the following is that of H. L. Heitz (see 1, bibliography).

The latter part of this paper consists of a few remarks on transformations in general.

II. THE TRANSFORMATION $u = x^n$

In the following theorems it will be understood that *one* means *at least one* and that a frequency function is to have a total area of unity.

The transformation $u = x^n$ has a very clear physical interpretation, for if the diameters of oranges are distributed as $f(x)$ then the distribution of the volumes of these oranges would be obtained by making the transformation $u = kx^3$.

Theorem I.

Given a continuous bimodal frequency function of positive variates $y = f(x)$ with a range $0 < a \leq x \leq e$ with modes at $x = b$, $x = d$ and antimode at $x = c$, $(a < b < c < d < e)$ $f(a) = f(e) = 0$ and with a continuous derivative, then the frequency distribution $v = \phi(u)$, $\left[\phi(u) \equiv \frac{1}{n} u^{\frac{1-n}{n}} f\left(u^{\frac{1}{n}}\right)\right]$ of positive variates, $u = x^n$ has modes as follows:

Case I. $n > 1$

(1) one mode $a^n < u \leq b^n$ always, and (2) one mode and one antimode $c^n < u \leq d^n$ if $\left|(1-n) f(u^{\frac{1}{n}})\right| < u^{\frac{1}{n}} f'(u^{\frac{1}{n}})$ somewhere in this interval.

Case II: $0 < n < 1$

(1) always one mode $b^n \leq u < c^n$, (2) a mode and antimode $d^n \leq u < e^n$ if $\left| u^{\frac{1}{n}} f'(u^{\frac{1}{n}}) \right| > (1-n) f(u^{\frac{1}{n}})$ somewhere in this interval.

Case III. $n < 0$

(1) One mode $d^n \leq u < e^n$ (2) one mode and one antimode $b^n \leq u < c^n$ if $\left| u^{\frac{1}{n}} f'(u^{\frac{1}{n}}) \right| > (1-n) f(u^{\frac{1}{n}})$ somewhere in this interval.

Proof:

Since $u^{\frac{1}{n}}$ is taken to be positive, then if $\frac{dv}{du}$ is to be zero we must have

$$(1) \qquad (1-n) f(u^{\frac{1}{n}}) + u^{\frac{1}{n}} f'(u^{\frac{1}{n}}) = 0$$

Also we have by hypothesis

$$(2) \qquad f(a) = f(e) = 0$$

$$(3) \qquad f'(b) = f'(c) = f'(d) = 0$$

and that $f'(x)$ is continuous.

From these considerations the proof of the theorem follows quite simply; for instance

Case I $\quad n > 1$

In the interval $e^n > u \geq d^n$ (1) is negative. At $u = c^n$ (1) is negative. In the interval $c^n < u < d^n$, $u^{1/n} f'(u^{1/n})$ is positive and hence from continuity there is a maximum and minimum or not according as $u^{\frac{1}{n}} f'(u^{\frac{1}{n}}) < |(1-n) f(u^{\frac{1}{n}})$ or not for every u in this interval At a^n (1) is $u^{\frac{1}{n}} f'(a)$ which is zero or positive, while at b^n (1) is negative If $f'(a)$ is positive there is clearly a maximum at the point where the sign of the continuous derivative changes from positive to negative in the interval $a^n < u \leq b^n$ If $f'(a)$ is zero it follows that there is also a maximum, since $v = 0$ at $u = a^n$ and then increases before decreasing at $u = b^n$

The other cases follow from exactly similar reasoning

Theorem II

In case the bimodal continuous frequency function $y = f(x)$ (of Theorem I) is symmetrical about the antimodal line $x = c$, then the mean value of u in the frequency distribution $v = \phi(u)$ of $u = x^n (n \neq 0 \text{ nor } 1)$ is less or greater than its median value according as the value of n lies between 0 and 1 or outside of these bounds.

The first moment of the transformed distribution is given by $u = \frac{1}{n}\int_{a^n}^{e^n} u^{\frac{1}{n}} f(u^{\frac{1}{n}})\, du = \int_{a}^{e} x^n f(x)\, dx$ i. e., we have $\bar{\mu} = \mu'_n$ where μ'_n is the n th moment about the origin of the original frequency distribution $y = f(x)$. Denoting the mean value of x by $\bar{x}$, it is known[1] for every set of positive values that $\mu'_n < \bar{x}^n$, when u lies between 0 and 1, and that $\mu'_n > \bar{x}^n$ when u lies outside this interval.

Since $\bar{x} = c$ when $y = f(x)$ is symmetrical about this line, the theorem follows. This follows Rietz exactly.

Theorem III.

In case the continuous bimodal frequency function $y = f(x)$ (of Theorem I) is symmetrical about the antimodal line $x = c$, the frequency distribution $v = \phi(u)$ of $u = x^n$ $(n \neq 0 \text{ nor } 1)$ has the following relations between its modes and its median.

Case I. $n > 1$

One mode $\leq$ median, in any case, and one greater if $|(1-n)f(u^{\frac{1}{n}})| < u^{\frac{1}{n}} f'(u^{\frac{1}{n}})$ somewhere in the interval $c^n < u \leq d^n$

Case II. $0 < n < 1$

One mode $\leq$ median, in any case, and one greater if

$$|u^{\frac{1}{n}} f'(u^{\frac{1}{n}}) > (1-n)\, f(u^{\frac{1}{n}})$$

at some point $d^n \leq u \leq e^n$

Case III $n < 0$

One mode $\geq$ median, in any case, and one less if

$$|u^{\frac{1}{n}} f'(u^{\frac{1}{n}})| > (1-n)\, f(u^{\frac{1}{n}})$$

at some point $b^n \leq u \leq c^n$.

As an example of a transformation $u = x^n$ which transforms a bimodal frequency function satisfying the conditions of the previous theorems into a unimodal distribution consider the following.

Take $n = 37$ and $f(x) = -x^4 + 12x^3 - 50x^2 + 84x - 44$,
$0 < \alpha \leq x \leq \beta < 6$
If $\frac{dv}{du} = 0$, then

$$(1-n)\, f(u^{\frac{1}{n}}) + u^{\frac{1}{n}} f'(u^{\frac{1}{n}}) = 0 \qquad (1)$$

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[1] See J. L. W. V. Jensen, Acta Mathematica, Vol 30 (1906), pp 180-187.

Instead of the variable $u^{\frac{1}{n}}$ we may just as well write an x. Hence (1) becomes

$$(2) \quad F(x) = 32x^4 - 396x^3 + 1700x^2 - 2940x + 1984$$

Calculating Sturm's functions for (2), it is easily seen that the transformed distribution has only one mode and that in the interval $0 < u < 2^{1/7}$

III. TRANSFORMATION $u = \log x$

Theorem IV.

Given a continuous bimodal frequency function (of Theorem I) with a range $1 < a \leq x \leq e$, then the frequency distribution $v = \phi(u)$ $[\phi(u) = e^u f(e^u)]$ of positive variates $u = \log x$ has one mode, in any case, $\log d \leq u < \log e$ and has a mode and antimode in the interval $\log b \leq u < \log c$ if $|e^u f'(e^u)| > f(e^u)$ somewhere in this interval.

This follows very simply from considering $\frac{dv}{du}$, which is

$$e^{2u} f'(e^u) + e^u f(e^u)$$

Theorems similar to those stated under the transformation $u = x^n$ concerning the relative position of the modes and median of the transformed distribution may be stated here.

As an example of a bimodal frequency distribution satisfying our hypothesis and which is transformed into a unimodal distribution by the transformation $u = \log x$ consider

$$f(x) = -x^4 + 16x^3 - 92x^2 + 224x - 148, \quad 1 < \alpha \leq x \leq \beta < 7$$

The condition for the vanishing of the derivative of the transformed distribution takes the form

$$F(x) = -5x^4 + 64x^3 - 276x^2 + 448x - 148$$

By calculating Sturm's functions for $F(x)$ it is easily seen that the transformed distribution has but one mode and that in the interval $\log 6 < u < \log 7$.

IV. TRANSFORMATION $u = e^x$

Theorem V.

Given a continuous bimodal frequency function (of Theorem 1), then the frequency distribution $v = \phi(u)$, $\left[\phi(u) = \frac{1}{u} f(\log u)\right]$ of positive variates $u = e^x$ has one mode $e^a < u \le e^b$, and has a mode and antimode in the interval $e^c < u \le e^d$ if $f'(\log u) = f(\log u)$ at some point in this interval.

For $$\frac{dv}{du} = \frac{1}{u^2}\left[f'(\log u) - f(\log u)\right]$$

from this the theorem follows.

Theorems similar to those stated concerning the relative positions of the median and modes of the transformed distribution in the case of the transformation $u = x^n$ may be stated here also.

As an example of a bimodal frequency distribution that satisfies our hypothesis and is transformed into a unimodal distribution by the transformation $u = e^x$ consider

$$f(x) = -x^4 + 16x^3 - 92x^2 + 224x - 148, \quad 1 < \alpha \le x \le \beta < 7$$

The condition that the derivative of the transformed distribution vanish takes the form

$$F(x) = x^4 - 20x^3 + 140x^2 - 408x + 372$$

By calculating Sturm's functions for $F(x)$ it is easily seen that the transformed distribution has only one mode and that in the interval $e^1 < u < e^3$.

V. TRANSFORMATIONS IN GENERAL

Suppose that we have a frequency distribution the distribution of whose parameters due to random sampling we know. If we transform this distribution what will happen to the distributions of the estimates of the parameters? It appears that, in view of the fact that bimodal and possibly multimodal distributions may be transformed by fairly simple transformations into unimodal distributions, there will be no simple relation between the change in the frequency distribution and the corresponding changes in the distributions of the estimates of the parameters by means of random samples. As a specific example of these general remarks consider the following

If the normal curve

(1) $$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

is transformed by the transformation

(2) $$u = x^2$$

giving

(3) $$F(u) = \frac{1}{2\sqrt{2u}} \quad \frac{e^{-\frac{1}{2}u}}{u}$$

Then, applying a general method for finding the distribution of the means of samples, first developed by J. O. Irwin (2, bibliography), the mean values of the u's are found to be distributed as proportional to

(4) $$\frac{\left(\frac{n}{2}\right)^{\frac{n}{2}} x^{\frac{n-2}{2}} e^{-\frac{1}{2}nx}}{\Gamma\left(\frac{n}{2}\right)}$$

Then

(5) $$u'_m = \frac{(n+2) - - - - (n+2m-2)}{n^{m-1}}$$

If $m \geq 1$

Whence $\mu_0 = 1$ $\qquad \beta_1 = \frac{8}{n}$

$\mu_1 = 1$ $\qquad \beta_2 = 3 + \frac{12}{n}$

$\mu_2 = \frac{2}{n}$ $\qquad \underset{n\to\infty}{L} \beta_1 = 0$

$\mu_3 = \frac{8}{n^2}$ $\qquad \underset{n\to\infty}{L} \beta_2 = 3$

$\mu_4 = \frac{12n+48}{n^3}$

Thus we see that, although the sampled population is J-shaped, the distribution of the estimates of the means ultimately approaches the normal distribution but that this approach is rather slow.

It has been shown (see 3) that (4) is also the distribution of the estimates of the second moment by means of samples of n drawn from (1), the second moment of the sample being taken about the mean of (1). This is a special example of a general consideration that is of considerable interest in this connection.

It has been shown (see 3) that, formally, the distribution of the estimates by means of samples of n of the mth moment of a population represented by $f(x), a \leq x \leq b$, the m th moment of the sample being taken about the mean of $f(x)$ is given by the solution of the integral equation

$$(6) \qquad F(s) = \int_{\alpha}^{\beta} \psi(x)\, e^{sx}\, dx$$

where $\psi(x)$ is the unknown distribution of n times the estimates of the m th moment of the population about the mean of the population and

$$F(s) = \left(\int_{a}^{b} f(x)\, e^{sx^{m}}\, dx\right)^{n}$$

and if m is even

$$\alpha = 0$$

$$\beta = \text{larger of } na^{m},\ nb^{m}$$

if m is odd

$$\alpha = na^{m}$$
$$\beta = nb^{m}$$

Now, the formal development for finding the distribution of the means of samples of n drawn from a population represented by $f(x)$ transformed by the transformation $u = x^{m}$ leads to a relation equivalent to (6) (see 2). This result may be stated as

Theorem VI.[1]

If the distribution of the estimates of n times the m th moment of a population represented by $f(x)$, $a < x < b$ about the mean of $f(x)$ exists as a solution of (6) it is identical with the distribution of the estimates by means of samples of n of n times the mean, measured from the mean of $f(x)$, of the population represented by $f(x)$ transformed by the transformation $u = x^m$.

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[1] This theorem permits of an obvious generalization to the case of the k th moment of the transformed distribution.

This enables us to formally identify these two problems so that anything that is true of one distribution is also true of the other.

With other transformations the relation between the distribution of the means of random samples from the transformed distribution and the distribution of the estimates of the parameters of the original distribution become much more complicated.

Further, we might say a few words with regard to the possibility of transforming various types of distributions into various other types.

Suppose that $f(x)$ is a continuous frequency function of positive variates, $a \leq x \leq b$, and that $f'(x)$ is continuous in this closed interval. Now make the transformation

$$(7) \qquad u = \phi(x)$$

and suppose that $\phi(x)$ is such that (7) can be solved explicitly for x, i. e.,

$$(8) \qquad x = \psi(u)$$

Then $f(x)\,dx$ becomes, assuming $\psi'(u)$ is continuous,

$$f\left[\psi(u)\right]\psi'(u)\,du$$

$$(9) . \qquad U(u) = f\left[\psi(u)\right]\psi'(u)$$

Supposing that f is known, what can we do towards fixing the form of U by a suitable choice of ψ ?

Now, the simplest of all possible frequency distributions, from the standpoint of description by means of a continuous function, is one in which the probabilities of all values of the variate are equal. Hence we will suppose for illustration that

$$(10) \qquad U(u) = f[\psi(u)]\psi'(u) = k$$

whence, putting $\psi(u) = y$

we have

$$(11) \qquad \int f(y)\,dy = ku + c$$

Suppose that

$$(12) \qquad f(x) = \alpha x + \beta$$

Then

$$(13) \qquad \psi(u) = \frac{-\beta \pm \sqrt{\beta^2 + 2\alpha(ku+c)}}{\alpha}$$

From this it is apparent that if $f(x)$ is any polynomial whose degree is less than four and which is positive $a \leq x \leq b$ may, conceivably, be transformed into a rectangular distribution.

If in place of k we were to put a specified function, say the normal function, we would run into considerable difficulty.

In (9) we may regard ψ as known and then ask what forms of f may be transformed into certain specified forms For instance, let us take

$$u = \log x$$
$$x = e^u$$

Then $U(u) = f(e^u)\,e^u$

$$(14) \qquad U'(u) = e^u\left[f(e^u) + e^u f'(e^u)\right]$$

Now, since $u>0$, it is apparent that if $f(x)=c$ that (14) has no zero.

Let us put, for illustration, $U'(u) \quad 0$ or $U(u) = k$

Then $f(x) \quad \frac{k}{x}$

However, if we were to suppose that (14) vanished at only one point at exactly two points, etc., instead of identically it would be very difficult to express this in terms of the form of

VI. SUMMARY

It has been shown that unimodal distributions may be transformed into bimodal distributions by means of rather simple transformations. This suggests that bimodal distributions are not necessarily the result of heterogeneity

The fact that a badly misshapen distribution may be transformed into something that is approximately normal does not seem to be of much aid in determining the distribution of the estimates of the constants of the original distribution.

The problem of transforming a specified distribution into another specified distribution is very difficult in gneral but could, perhaps, be handled to an adequate degree of approximation in special cases.

G. A. Baker

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ERROR AND UNRELIABILITY IN SEASONALS

By

EDGAR Z PALMER

An article in the *Annals of Mathematical Statistics* for February, 1930, entitled 'A Mathematical Theory of Seasonals," by the Statistical Department of the Detroit Edison Company, has three objects. It presents a mathematical version of the time series analysis, suggests the "interpolation" method of computing seasonals, and constructs a theoretical time series as a test of the new method. The mathematical analysis and the theoretical series are based upon the assumption that the trend, cycle, and seasonal are proportional to each other, while the "errors" or residuals are additive in nature. The reasoning is not necessarily valid for series where the cycle or the seasonal is additive rather than proportional to the trend.

The interpolation method as proposed consists in (1) finding the total of the items for each of the twelve months, and (2) dividing each total by a function which theoretically contains the trend and the cycle insofar as they influence the particular month In practice this twelve-month function turns out to be a smooth trend curve, and the method of its calculation inspires little confidence that it can reflect much cyclical influence. The function for each month is simply a weighted sum of the annual totals, the weights varying for different months. The early years are weighted more heavily in finding the values of the function which apply to the first half of the year, while the later years are given a greater weight in the second half of the year. The function is influenced almost solely by trend, or rather, by the difference between the first year and the last year of the data, since these two years are the only ones whose weights vary considerably from month to month. It is certain that no cyclical movement, however violent, can have the proper effect upon this function unless it affects the two extreme years.

Since the interpolation method involves dividing the monthly totals (for which may be substituted the monthly means) by a function which is mainly composed of trend, we are justified in considering it a variation of the well-known monthly-means method.[1] In the theoretical series of the Detroit Edison article, the means of each month, corrected for trend by the Davies method, yield a seasonal index almost identical with that obtained by the interpolation method (see Table I). It should be noted that we used the very easily computed semi-means line to correct the monthly means for trend. The semi-means trend in this series is not as steep as the theoretical trend used in the con-

TABLE I

SEASONALS OF THE THEORETICAL SERIES

As Obtaind by Five Methods

			1	2	3	4	5
						Ratio to trend-cycle	
Month	Theoretical index*	True index	Monthly means	Interpolation*	Link relative*	Moving mean	With free-hand correction
January	.990	.978	.931	.938	.975	.978	989
February	.930	.908	880	.885	.890	918	905
March	1.050	1 020	984	.988	.988	1 003	1.037
April	1.020	1.028	1.018	1.021	1 007	1.051	1.035
May	1.040	1 063	1.062	1.065	1 030	1 069	1.064
June	980	.969	986	.986	.962	982	.957
July	.980	.978	994	993	972	.986	976
August	1.000	1.007	1 019	1.017	1 013	1.024	1 016
September	.980	1.009	1 031	1.028	1.033	1.004	.995
October	1.040	1 056	1 084	1.079	1 099	1.049	1.055
November	.990	.974	.990	985	1.004	.955	.958
December	1.000	1.013	1 016	1 009	1.027	.982	1.014

* From the Detroit Edison article

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[1] Davies, *Economic Statistics* (1922), p. 117.

struction of the series. For the purposes of this quick and easy seasonals method, however, the semi-means trend is accurate enough.

Any one series used in the comparison of methods should, of course, be viewed merely as an illustration, or at most as a sample, of their results. A thousand such series, constructed upon a thousand variations in assumptions, is necessary for determinative comparisons. Long and short series, large and small seasonals, cycles, trends, and irregulars, regular and irregular seasonals, curved and straight trends, additive and proportional combination of the factors: each of these attributes introduces some elements of error into the computation of sasonals, and the errors are not necessarily constant as between different methods.

An instance of the danger of using any single theoretical series occurs in the Detroit Edison article. If we test the residual factor, we find that it also contains some seasonal variation. The true seasonal index of the series, then, is the theoretical seasonal modified by whatever seasonal is to be found in the residuals. There is, as might be expected, some seasonal inequality in the cyclical factor as well, but since it is the task of the method used to eliminate this cyclical influence, we do not consider it a part of the true seasonal. No method, however, can be expected to distinguish between a seasonal arbitrarily designated as the theoretical, and one which is added as part of the residual factor. In Table I, the first column gives the theoretical, and the second column the true seasonal, of the series.

The authors compare their results with those by the link-relative method, and find that the interpolation method gives an index slightly closer to the theoretical seasonal. For further test, we have computed the seasonal by the somewhat more logical ratio-to-trend-cycle method. We used the twelve-month moving mean, centered on the sixth month, to represent the combined trend and cyclical factors. Then we found the ratios of each monthly item in the series to the corresponding moving mean figure, and, after arraying the ratios for each of the twelve months, we found the modified median of each array. •This is approximately the method suggested by the Federal Reserve Board,[1] and gives a seasonal with very much less error than the previously mentioned methods. (See Table II)

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[1] Joy and Thomas, The Use of Moving Averages in the Measurement of Seasonal Variations, *Journal of the American Statistical Association*, vol. 23, p 241.

The ratios to the moving mean may be expected to be free from both trend and cyclical influence. However, the moving mean has its faults, especially in its tendency to cut corners when the true cycle makes a sharp change of direction, and in its failure to extend to the ends of the series. For this reason we made another computation, using a corrected moving mean. The data and the moving mean were graphed together, and a free-hand curve was drawn (without reference to the theoretical trend-cycle curve given in the article), correcting the moving mean in three places and extending it to the limits of the series. The seasonal index computed from the ratios to the new trend-cycle curve had an error about half that of the interpolation method. (See Table II)

TABLE II

ERROR OF SEASONALS

Method	(1) Mean deviation from theoretical seasonal	(2) Mean deviation from true seasonal	(3) Mean deviation from 1.0000	(4) Ratio of (2) to (3)
Theoretical seasonal	——	.0158	0258	.612
True seasonal .	.0158	——	0324	——
Monthly means	.0291	.0197	.0388	.508
Interpolation	.0269*	.0170	.0378	.450
Link relative	.0277*	.0198	.0355	.558
Ratio to trend-cycle:				
Moving mean .	.0208	.0132	.0344	.384
With free-hand correction	.0164	.0078	.0328	.239

* From the Detroit Edison article.

When W. I. King[1] first proposed the use of a free-hand curve in this connection, objection was raised that this introduced the personal equation into what should be mechanically determinable In many series, however, any experienced statistician would draw a curve which would fit the data better than the moving mean. There is not as much discretion involved in drawing the curve as there is in choosing between two mechanical methods. The possible error due to the personal factor is very much less than the error made certain by the use of any more mechanical method.

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[1] King, An Improved Method for Measuring the Seasonal Factor, *Journal of the American Statistical Association*, vol. 19, p. 301.

An important test of the reliability of the methods of finding seasonals, which may be applied to actual series where the true seasonal is unknown, consists in an examination of the monthly arrays. The monthly-means and the interpolation methods depend for their reliability upon the distribution of the arrays of the original data from the means of each month. Similarly, the link-relative method depends upon the scatter of the link relatives about their medians, and the ratio-to-trend-cycle methods upon the arrays of ratios. We measured the dispersion for each month for any method by the mean deviation of its array about its central tendency. This should be divided by the central tendency itself to obtain a relative dispersion measure for that month Then the mean of all twelve dispersion measures was taken,

TABLE III

UNRELIABILITY OF SEASONALS

Method	Relative mean deviation of monthly arrays
Monthly means	1838
Interpolation	1838
Link relative	0734
Ratio to trend-cycle.	
Moving mean	.0621
With free-hand correction	.0492

to give an indication of the unreliability of the method as a whole. The great unreliability of any method based on the monthly means is apparent from Table III, as well as the superiority of the ratio-to-trend-cycle method with free-hand correction.

Some question may arise concerning the propriety of submitting the link relatives to this test of reliability, because the manipulations to which the medians are subjected before they emerge as a seasonal index may decrease the error inherent in the spread of the monthly arrays. We have not been able to derive the algebraic relationship between the mean deviation of the link relatives and the corresponding unreliability of the final seasonal indexes based upon them. In erratic series, the process of computing link relatives tends to heighten the spreading effects of rapid changes in direction; conversely, cumulative multiplication of the median link relatives possibly decreases the error. If so, the link-relative method is not as unreliable as Table III would seem to show it.

The penalty which the computer pays for accuracy and reliability is, of course, a longer time of computation. The time required for the application of each method to the given series is shown in Table IV. The time allowed is for each operation to be performed twice, and the results checked against each other. In addition to the five methods used throughout this article, the time is given for a short cut to the best method, involving much more of the personal equation The short cut consists in not computing the moving mean at all, but drawing the trend-cycle curve altogether free-hand.

TABLE IV

COMPUTING TIME OF SEASONALS

Method	Time in minutes
Monthly means	60
Interpolation	110
Link relative	160
Ratio to trend-cycle:	
Moving mean	285
With free-hand correction	495
All free-hand curve	371

This timing, of course. assumes that the seasonal index is the whole objest of the computation. If we were finding the seasonal only as a part of a general statistical analysis of the series, the seasonal should not be charged with the full time for the steps which are useful for other parts of the analysis. The trend-cycle curve. for instance, has other uses than in computing seasonals. In considering the question of speed, it should also be recognized that the various methods do not have the same relative time for series of different length, nor when more elaborate calculating equipment is available than we used.

High speed of computation is not as necessary in seasonals as it is, shall we say, in index numbers The calculation of a seasonal is a task that does not have to be repeated often for any one series. It is more in the nature of a capital expenditure than a current routine. For student theses, and for investigations where the computation of seasonals is merely incidental or can be roughly done, the monthly-means method is adequate. But for a positive study of actual seasonal influences, and for the elimination of the seasonal factor from indexes published currently by research bureaux, the best method should be used regardless of the longer time needed.

Edgar Z Palmer

MODIFICATIONS OF THE LINK RELATIVE AND INTERPOLATION METHODS OF DETERMINING SEASONAL VARIATION

By

Richard A. Robb

In a recent paper[1] the statistical department of the Detroit Edison Company have introduced a new method of calculating seasonal variation in a time series. Briefly, the time series ${}_i u_x$ is represented by the function ${}_i u_x = f(x)\ c(x)\ s(x) + \varepsilon_x$ where $f(x)$ represents secular trend, $c(x)$ cycle, $s(x)$ seasonal, and ε_x residual errors, and by the Method of Least Squares the seasonal variation for any one month will be given by

$$(A) \qquad s_i = \frac{\Sigma\, {}_i u_x\, f(x) \cdot c(x)}{\Sigma \left[f(x)\, c(x) \right]^2} \qquad i = 1, 2, 3, \ldots 12.$$

where $s(i)$ represents the seasonal variation in the i th month and the summations in the right hand member of the equation are taken over the years covered by the time series.

If the Method of Moments be used

$$(B) \qquad s_i = \frac{\Sigma\, {}_i u_x}{\Sigma \left[f(x)\ c(x) \right]}$$

The trouble lies in the determination of the denominator $\Sigma \left[f(x)\ c(x) \right]^2$ or $\Sigma \left[f(x) \cdot c(x) \right]$. The Detroit Edison have overcome this difficulty by smoothing the observed time series with a sixth degree parabola, keeping the total population for each year unchanged over a period of seven years. In this way seasonal

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1. A Mathematical Theory of Seasonals, *Annals of Math. Stat.*, I, p. 57.

variation is obtained from (A) or (B), (B) being much easier to handle than (A).

There appears to be an objection in fitting a curve over a period of seven years and thus for successive seven year intervals obtaining smoothed values for a time series of any given length The ordinates of the smoothed curve are not equally weighted as, for example, in fitting curves over a ten-year period, the first smoothed ordinate for a year is given by *one* curve, the second by *two* curves, the third by *three* curves, the fourth, fifth, sixth and seventh by *four* curves, the eighth by *three,* the ninth by *two* and the tenth by *one.* To overcome this I decided that my smoothed curve should have the same zero, first and second moments as the observed curve over a period of twelve months. This simply means that a parabola of 2nd degree was fitted to the successive twelve month intervals, and as above a smoothed curve will be obtained for any length of time.

If the observed values u_x are plotted against the corresponding values of x and a parabola of second degree fitted to the points

$$u_{-n},\ u_{-n+1},\ \cdots\cdots,\ u_0,\ \cdots\cdots,\ u_n$$

determining the constants by the Method of Least Squares, the ordinate of the curve at $x = 0$ is taken as the graduated value of u_0. If $n = 6$ this would involve thirteen observed ordinates, whereas I desire twelve. This difficulty, however, is easily removed by finding a first approximation to my graduated value by using thirteen ordinates; having found the corresponding seasonal variation by (A) or (B), the thirteenth ordinate is divided by this seasonal factor. The parabola which is to represent the smoothed curve given by trend × cycle is then found from the twelve ordinates subject to seasonal, trend and cycle influences, and a thirteenth from which seasonal has been eliminated.

The graduated ordinate at $x = 0$ corresponding to u_0 is (first approximation)

$$\text{(C)} \quad u'_0 = \frac{1}{143}\Big[25u_0 + 24(u_1 + u_{-1}) + 21(u_2 + u_{-2}) + 16(u_3 + u_{-3}) + 9(u_4 + u_{-4}) - 11(u_{-6} + u_6)\Big] - [c]$$

For example, if we take thirteen ordinates, commencing at January, 1904, and finishing at January, 1905, the first approximation for July is

$$\text{July} = \frac{1}{143}\left[\begin{array}{l} -11\ (\text{Jan.})\ 1904 + 9\ (\text{Mar.}) + 16\ (\text{Apr.}) + 21\ (\text{May}) \\ +24\ (\text{June}) + 25\ \ (\text{July}) + 24\ (\text{Aug.}) + 21\ (\text{Sept}) \\ +16\ (\text{Oct.}) + 9\ (\text{Nov.})\ -11\ (\text{Jan})\ 1905 \end{array}\right]$$

where (1) I have designated the production for any one month by the corresponding name of the month, and (2) the formula is rearranged in a form suitable for the calculating machine.

If formula (B) is used it is readily seen that to obtain the seasonal variation for any month we must

(1) Sum together all the Januaries, then all the Februaries, etc. It should be noted that, as the first six months and the last six months of a time series are not weighted equally with the others, no graduated points were found for these periods. In consequence, as will be seen in practice, two sets of summations of the different months are required, the first including every year except the last, and the second excluding the first year. Then apply formula (C) This gives

$$\Sigma\left[f(x)\ \ c(x)\right]$$

(2) Divide $\Sigma\,.\,u_x$ by $\Sigma\left[f(x)\ \ c(x)\right]$

Having obtained a first approximation to the seasonal factors, $\Sigma\left[f(x)\ \ c(x)\right]$ is recomputed as explained above. In practice this is quickly executed, as will be seen in an example completely worked out below.

To illustrate this method I have taken the theoretical time series given by the Detroit Edison. Summing the productions for the various months, we have Table I.

To find the seasonal for *July*, for example, we have to find the value of

$$\Sigma[f(x)\cdot c(x)] = \frac{1}{143}\left[\begin{array}{l} -11(20434) + 9(21621) + 16(22615) + 21(23035) \\ +24(21129) + 25(21508) + 24(22118) + 21(22212) \\ +16(23186) + 9(21215) - 11(21215) \end{array}\right]$$

Using formula (B) the seasonal for July is $s_7 = \frac{21508}{22280}\ 0.965$

TABLE I

Month	1904-1914	1905-1915	1st approx.	2nd approx
January	20,434	21,215	971	.973
February	19,425	20,143	918	.919
March	21,621	22,389	1.015	1 014
April	22,615	23,196	1 045	1.039
May	23,035	24,231	1.061	1 062
June	21,129	22,567	974	.974
July	21,508	22,820	965	.967
August	22,118	23,077	987	.993
September	22,212	23,707	1.011	1 010
October	23,186	24,964	1 071	1 067
November	21,215	22,712	987	.982
December	21,836	23,182	1 002	1 005
			1200 07	1200 05

In this way the seasonals in Column 4 of Table I were obtained

The second approximation is obtained with little extra trouble; for July, on account of the thirteenth ordinate, in this case the January of the following year, which has a seasonal of .971, we have to replace the last term in $143\ \Sigma\left[f(x)\quad c(x)\right]$ given above by $\frac{-11(21215)}{.971}$, i. e., the recomputed $\Sigma\left[f(x)\quad c(x)\right]$ is now $\frac{3186016 - 11(21215)(0.0299)}{143}$, the reciprocal of 0.971 being 1.0299.

The seasonals obtained with these corrections are given in Column 5 of Table I.

Comparing the seasonals with actual values, we have the following table.

	Jan	Feb.	March	April	May	June
Actual Seasonal	990	.930	1.050	1 020	1.040	980
Computed Seasonal	.973	919	1.014	1.039	1.062	.974
Error	- 017	-.011	-.036	+ 019	+.022	-.006
	July	**Aug.**	**Sept**	**Oct.**	**Nov.**	**Dec.**
Actual Seasonal	980	1.000	.980	1.040	.990	1.000
Computed Seasonal	.967	993	1.010	1.067	.982	1.005
Error	- 013	-.007	+.030	+ 027	-.008	+.005

The mean and standard deviations are compared with the Interpolation Method of the Detroit Edison.

	Mean Deviation of Errors	Standard Deviation of Errors
	.0168	.0194
Interpolation Method	.0269	.0337

It will be noticed that the new method of smoothing yields a standard deviation which is roughly a little greater than half that obtained by the Interpolation Method.

To test whether any actual difference in Seasonals would be obtained by using formula (A), the ordinates of the smoothed curve were found by formula (C) and are given in Table II. The seasonals were as follows:

Jan.	Feb.	March	April	May	June	
.967	.988	1.009	1.072	.986	1 005	
July	**Aug.**	**Sept.**	**Oct.**	**Nov.**	**Dec.**	**Total**
.971	.906	1.007	1.042	1.062	.980	11.995

figures practically identical with those previously obtained.

In Figure I, I have plotted against the various months (1) the actual Seasonal Indices, (2) those given by the Detroit Edison Inter-

TABLE II

	1904	1905	1906	1907	1908	1909
January		1724	1916	2105	1306	1656
February		1765	1968	2092	1180	1765
March		1824	1955	2041	1121	1889
April		1855	1956	2049	1141	2007
May		1885	1956	2093	1187	2129
June		1854	1954	2148	1218	2216
July	1405	1880	2019	2127	1223	2310
August	1490	1882	2063	2064	1253	2370
September	1522	1859	2078	1917	1307	2458
October	1569	1867	2099	1772	1376	2462
November	1634	1890	2122	1618	1465	2502
December	1709	1939	2139	1454	1566	2538

	1910	1911	1912	1913	1914	1915
January	2565	1919	2157	2521	2233	1699
February	2574	1872	2222	2562	2123	1803
March	2571	1861	2293	2590	2022	1920
April	2544	1852	2370	2592	1900	2057
May	2529	1869	2441	2609	1863	2248
June	2461	1900	2514	2645	1821	2449
July	2382	1935	2540	2640	1774	
August	2275	1972	2554	2624	1735	
September	2167	1993	2536	2511	1645	
October	2036	2008	2488	2405	1634	
November	1950	2048	2477	2314	1586	
December	1931	2113	2464	2271	1613	

polation Method, and (3) those given by the method of this paper.

As it will be interesting to note how the smoothed values of the ordinates of the time series agree with the actual, I have given below the Mean Deviation of errors from actual for the various months.

Month	Jan.	Feb.	March	April	May	June
M. D.	58	41	50	43	32	34
Month	**July**	**Aug.**	**Sept.**	**Oct.**	**Nov.**	**Dec.**
M. D.	23	24	23	37	42	67

For the whole period the Mean Deviation is 39.5.

The Deviations are small, the January Mean Deviation being roughly 3 per cent of the mean production for January; for December it is 3.4 per cent.

On the assumption that the Seasonal Index for any one month is constant for a given time series, it will be seen that Least Squares can be used in several ways to yield Seasonals. I give one example of its use, obtaining Seasonals by a method closely allied to the Link Relative method.

In the Link Relative method link relatives are formed for all the different months. This involves the greater part of the calculation, and it seemed feasible that instead of calculating link relatives and finding median values one could assume that the production for any one month with reference to that for the previous month is given by

$$
\begin{array}{ll}
\text{February} & = a_1 \text{ January} \\
\text{March} & = a_2 \text{ February} \\
\cdot \;\; \cdot \;\; \cdot & \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\
\cdot \;\; \cdot \;\; \cdot \;\; \cdot & \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\
\text{December} & = a_{11} \text{ November} \\
\text{January} & = a_{12} \text{ December}
\end{array}
$$

where, as before, the name of the month stands for the production for that month, and $a_1, a_2, \ldots, a_{12}$ are constants which can be determined by the Method of Least Squares. For February $= a_1$ Janu-are, we have

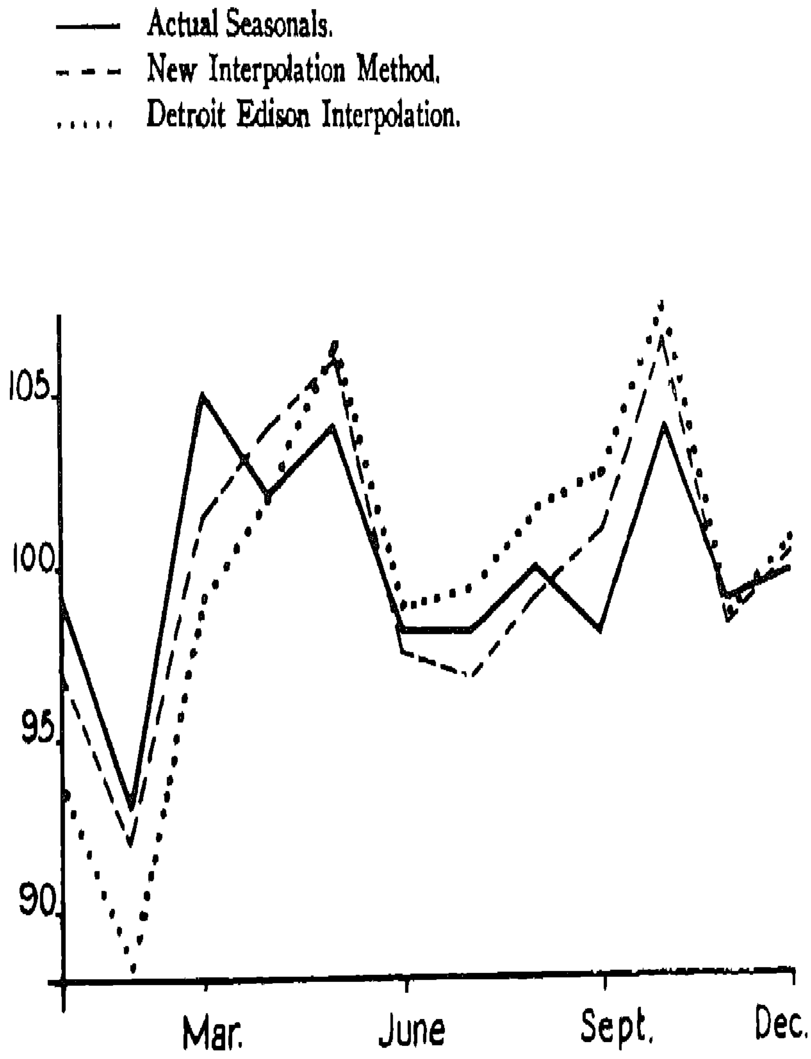
$$a_1 = \frac{\Sigma\,(\text{February})\,(\text{January})}{\Sigma\,(\text{January})^2}$$

the summations extending over the years of the time series.

Considering our time series to be $u_1, u_2, u_3, \cdots u_{12}, u_{13}, \cdots$

$$a_1 = \frac{u_1 u_2 + u_{13} u_{14} + u_{25} u_{26} + \cdots}{u_1^2 + u_{13}^2 + u_{25}^2}$$

FIGURE I


Actual Seasonals.
New Interpolation Method.
Detroit Edison Interpolation.
105
100
95
90
Mar.
June
Sept.
Dec.

i e, the observed productions for two successive months are multiplied together and summed and the whole divided by the sum of the squares of the production of the first of the months.

These coefficients a_1, a_2, . . . correspond to the median link relatives, and the procedure is then similar to that used in that method, i. e., January is assumed to be 100.0, etc. We thus get the following table

TABLE III

	(1)	(2)	(3)	(4)	(5)
Month	a_r	Chain Relative	(2) Adjusted	Seasonal Indices	Error from Actual divided by 100
January	.951	100.0	100 0	95.5	–.035
February	1.106	95 1	94.8	90.5	–.025
March	1.043	105 2	104.3	99.6	–.054
April	1.037	109.6	108 3	103.4	+.014
May	.934	113 7	111.9	106.8	+.028
June	.997	106.2	104.1	99.4	+.014
July	1.018	106.0	103.5	98 8	+.008
August	1.019	107.9	105.0	100.3	+ 003
September	1.056	110 0	106.6	101.8	+.038
October	.915	116.2	112.2	107.1	+.031
November	1.020	106.3	102.2	97.6	–.014
December	.967	108 4	103 9	99.2	–.008
January		104 8	100.0		

The Standard Deviation of the Errors of column (5) is found to be ± 0.0269, which is considerably less than that of the Link Relative Method.

If we assume that $_{o}u_{x}$ can be represented by the points on a theoretical curve $_{o}u_{x} = s(x)\; f(x)\; c(x) + \varepsilon_{x}$ as given by the Detroit Edison Statistical Department, it will be seen that February a_1 (January) gives

$$a_1 = \frac{s(2)}{s(1)}\;\frac{\Sigma\left[f(x)\; c(x)\right]\left[f(x+1)\; c(x+)\right]}{\Sigma\left[f(x)\; c(x)\right]^2}$$

where, if $x = 1$ corresponds to the first January of time series, $x = 1$, 13, 25, etc.

$\frac{S(2)}{S(1)}$ can therefore be found as soon as a value can be obtained for the adjustment factor $\frac{\Sigma \psi(x)\, \psi(x+1)}{\Sigma [\psi(x)]^2}$ where $\psi(x) = f(x)\, c(x)$ If the time series is smoothed by the method already discussed, satisfactory values of $\psi(x)$ are obtained and the adjustment factors easily computed. The smoothed values of the ordinates of the theoretical time series are given in Table II. As logarithmic correction, which has already been employed, assumes a constant adjustment factor for any pair of consecutive months, it will be interesting to find whether the assumption of a theoretical curve for the time series yields better adjustment factors than the constant one used in logarithmic correction.

TABLE IV

Month	Adjustment (1)	Chain Relative (2)	(2) adjusted (3)	Seasonal (4)	Error from actual/100
January		100.0	100.0	97.2	-.018
February	.994	94.5	94.4	91.8	-.012
March	.993	103.8	103.6	100.7	-.043
April	.991	107.3	107.1	104.1	+.021
May	.980	109.0	108.7	105.7	+.017
June	.984	100.2	98.8	96.0	-.020
July	.996	99.5	99.0	96.2	-.018
August	1.000	101.3	100.8	98.0	-.020
September	1.015	104.7	104.1	101.2	+.032
October	1.016	112.3	111.6	108.5	+.045
November	1.006	103.3	102.5	99.6	+.006
December	.995	104.8	104.0	101.1	+.011
January	.996	100.9	100.0		

The adjustment factors are given in column (1) of Table IV, and the corresponding seasonal indices in column (4). The standard deviation of errors, ± .0248, is less than that obtained with the adjustment as used in the link relative, but in this particular case the adjustment factor, using logarithmic correction, would be .996 for each month, differing little from the factors using smoothed ordinates. It will be noted that, owing to accidental errors, the chain relative for January is 100.9, not 100, and an arithmetical correction has to be applied.

From this one sees that, taking accidental errors into account, logarithmic correction is well adapted for reduction purposes

Finally, I found the Seasonal Indices by the Variate Difference Method. In this method the trend is removed and second differences taken, which are treated by Fourier Analysis. For the second differences I obtained

$$\begin{aligned}\Delta'' u = &+0.02 + 0.822\cos(\theta - 339^\circ 52') + 3.958\cos(2\theta - 314^\circ 53')\\ &+3.902\cos(3\theta - 39^\circ 34') + 4.374\cos(4\theta - 25^\circ 58')\\ &+9.942\cos(5\theta - 293^\circ 39') + 0\,775\cos 6\theta.\end{aligned}$$

yielding the seasonal indices:

Jan.	96.9	May	105 6	Sept.	101.9
Feb.	89 2	June	99.4	Oct.	106 3
Mar.	100.8	July	98.3	Nov.	97.8
Apr.	103.4	Aug.	102.2	Dec.	98.3

Dividing the seasonals by 100 and comparing with Actual values, Mean and Standard Deviation of errors from the actual seasonals are

TABLE V

SEASONAL INDICES

Month	Actual Values	Interpolation		Link Relative	Modified Link Relative		Variate Difference
		Detroit Edison	Robb		Log Corec-tion	Theoretical Correction	
January	990	.938	973	.975	955	.972	.969
February	.930	.885	919	.890	905	918	.892
March	1 050	.988	1.014	.988	.996	1.007	1.008
April	1.020	1 021	1 039	1.007	1.034	1.041	1.034
May	1 040	1.065	1 062	1.030	1.068	1 057	1 056
June	.980	.986	.974	962	.994	.960	.994
July	980	.993	967	.972	.988	.962	.983
August	1,000	1 017	.993	1.013	1 003	.980	1.022
September	.980	1.028	1.010	1.033	1.018	1.012	1.019
October	1.040	1.079	1.067	1 099	1.071	1.085	1.063
November	.990	.985	.982	1.004	.976	.996	.978
December	1 000	1.009	1.005	1.027	992	1 011	.983
S. D		.0337	.0194	.0338	0269	.0248	.0246

given by ± 0.022 and ± 0.0246, which are still considerably less than those obtained by Link Relative or Detroit Edison Interpolation. For reference I have put in Table V the results obtained from all the methods mentiond in this paper, together with their Standard Deviation (S. D.) of errors.

As the time taken to determine the Seasonal Indices by the various methods is important, I took the time series of Merchandise Imports for a period of ten years, and calculated the Seasonal Indices. Denoting the Link Relative method by R_1, Modified Link Relative (Log correction) by R_2, Interpolation (as given in this paper) by I, I found that as regards the time taken for one determination completely checked $R_1 : R_2 : I :: 11 : 8 : 7$

For the particular example taken, it took 1.75 hours to transcribe the material and to determine the Indices, completely checked, by the Interpolation method. For the Link Relative method, 7.75 hours were taken It is desirable, if possible, to have two independent determinations, and the above times would consequently have to be doubled. The Variate Difference method roughly takes the same time as the Link Relative method, when the trend has been removed from the time series

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